Teacher knowledge needed to teach ratio and proportion in secondary school mathematics: on using the ratio table

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Introduction
Ratio and proportion is a core topic in primary as well as secondary mathematics education. To many pupils, this topic causes different conceptual problems (see, for an overview, e.g. Tourniaire and Pulos 1985). As a tool for promoting understanding of ratio and proportion, in the Netherlands, the ratio table is used in primary and secondary mathematics education (Streefland 1985). The ratio table has also been promoted for solving proportion problems in the sciences (de Jong, 1995).

In the RDC2 of ATEE in Leipzig 1999, we agreed to work on ratio and proportion and to report on at the ATEE 2000 conference. As Dutch researchers co-operating in the BPS-project (Broekman and Van der Valk, 1999) we have focused upon the ratio table as it appeared to be a suited topic for discussion between science and mathematics teachers in the schools we co-operate with. As it appeared that many science teachers were not acquainted with the ratio table as it is introduced in mathematics, in the BPS-project, the question has arisen what knowledge teachers should have of the ratio tables. Being teacher educators as well, we have attacked this question with prospective mathematics teachers in mind. Van der Valk, Wijers and Frederik (2000) focused on science teachers and have described the science related aspects of the ratio table.

In various research reports about alternative approaches to teaching ratio/proportion because of the difficulties pupils meet, one can read – often between the lines – that teachers participating in the studies followed an in-service training, not only to get acquainted with the new approach, but also to study ratio and proportion again. Streefland (1985) reported as a main problem that teachers don’t realise what they are doing when they do calculations that involve ratio/proportion.

So teachers do meet obstacles in teaching ratio and proportion, one being a limited knowledge of the mathematics itself. Reviewing the literature we did not find any study to pre-service or in-service teachers’ knowledge of ratio and proportion. Thompson (1984) suggests that teachers’ beliefs about mathematics and how to teach mathematics are influenced by their own experiences with mathematics and schooling. This is confirmed by findings from studies by Zbiek (1998) and Tirosh (2000), saying that many pre-service as well as in-service mathematics teachers essentially struggle with the same problems as pupils do. Similar results were found with prospective science teachers (Aguirre 1990; Koballa et al. 2000; Frederik et al. 1999).

In addition to content knowledge, teachers need pedagogical content knowledge (PCK) (Shulman 1986): how to teach ratio and proportion; how does pupils’ learning of ratio and proportion looks like; how to deal with an educational tool like the ratio table?

This paper consists of two parts. In the first part first report some results from small-scale studies we did on prospective teachers’ knowledge of ratio and proportion and the ratio table. In the second part, reviewing literature and analysing the ratio table as a tool in Dutch Mathematics textbooks, we elaborate on the content knowledge and PCK that prospective teachers should learn during teacher education.
Part 1 Some small-scale studies to prospective teachers’ knowledge

Here we report three small scale studies to the knowledge of ratio and proportion, in particular of the ratio table, pre-service teachers use when preparing a lesson about the topic.

1.1 The first study: preparing a lesson about ratio/proportion

This study was done by a master student in mathematics education, Christine Spaanderman, guided by the first author. Spaanderman (1997) studied the knowledge of ratio and proportion with four volunteering prospective teachers (PTs), two being Master students in mathematics and two in physics. All were following a 2-months orienting teacher education course (Van der Valk, 1994) at Universiteit Utrecht. The topic of ratio and proportion was not dealt with in this course. Spaanderman asked them to individually prepare a lesson on ratio and proportion for grade 7 (age 12/13). Using the lesson preparation method (Van der Valk and Broekman, 1999), the following task was given:

Design a maths lesson of 40 minutes maximum about the concept of ratio and proportion to a group of 7th grade pupils. For this, you’ll get a full hour to prepare this lesson. Imagine you would give the lesson tomorrow.

In the preparation room, various kinds of maths manipulatives were present. However, there was no mathematics textbook, as textbooks were expected to have a great influence on the lesson prepared, thus veiling PTs’ knowledge of the topic.

After one hour of preparation, the pre-service teachers were interviewed. The notes they made preparing, were collected. The audiotaped interviews were reviewed and the most interesting parts were transcribed. The data were analysed, using as points of attention:

- Aspects of content knowledge on ratio and proportion
- Aspects of pedagogical content knowledge, such as the use of tools or representations (manipulatives, the ratio table) and knowledge of pupils’ difficulties with ratio and proportion.

Results of the first study

Here, results are summarised and illustrated, if possible with quotes from the prospective teachers.

All four PTs seemed to have enough content knowledge to solve the proportional problems that are usually posed to grade-7 pupils. But they all had the problem that ratio and proportion is such a self-evident concept, that they found it difficult to put their– often vague – ideas about ratio/proportion1 into words. That was a surprise to them, too. It made Marianne unpleasant:

First, I want to explain why I found it unpleasant to do. I could not imagine anymore what can be explained about it, except a fraction of two numbers, actually. […] I cannot remember how I have learned it. Of course, I did learn it, but … it is so terribly stuck in my mind, that I don’t know anymore how it came into it.

In this, Marianne indicated what strategy she wanted to follow for designing the lesson: reflecting on the way she herself had learned as a pupil about ration/proportion. But she did not remember.

Some PTs expressed to know that ratio is not the same as fraction, but no one could verbalise the difference. E.g. Gerard:

1 We use the notation ‘ratio/proportion’ as in Dutch, only one word is used (‘verhouding’), not differentiating between ratio and proportion.
I wanted to call the words ‘ratio and proportion’ into the classroom. But I had to think about it for a longer while as I had expected, to bring it well to them. For you quickly go to fraction, but fraction and ‘verhouding’, that is something .. I don’t know whether or not they grasp it at once.

The notation of ratios appeared to be an issue for one of the PTs. She doubted about a, as she called it, ‘slash’ notation (a/b) or a colon notation. The other three chose without hesitation the colon notation a : b.

All PTs had the problem of how to guess what pupils’ pre-knowledge on ratio and proportion would be. Mark, at the start of the interview:

To start with, I did not know well anymore what ‘verhouding’ (ratio/proportion) is like. Yes, of course, I know, but I simply don’t know … well, what is their pre-knowledge or so, whether or not they already have heard of when you enter grade 7. I didn’t have an idea.

It is also interesting what the PTs did and did not include in their lessons. All PTs used some real-life contexts, e.g. the ingredients of a recipe or pocket money. Only one used the aspect of similarity of shapes. She was also the one who used visual support in the form of a table and a map. Other tools or representations like the number line, the ratio table, graphs and circle diagrams, were not used. Even the manipulatives, present in the preparation room, were hardly used.

Discussion on the first study results

The problem of not being able to say what ratio and proportion basically is, does not seem to us a problem of lack of content knowledge. In our view, our PTs had enough content knowledge of ratio and proportion, as can be expected from graduate maths and physics students. Going back to the basics of a concept in order to explain to pupils is difficult, not only for ratio and proportion, but for other concepts as well (Oldham et al., 1999; Frederik et al. 1999). To us, this seems to be related to the process of level reduction described by Van Hiele (1986). Prospective teachers have to reflect on their own concepts, re-discovering backward the levels they had gone through when they were a pupil. In our view, the knowledge of the very basics of a concept or topic, seemingly being content knowledge, in fact is an essential part of pedagogical content knowledge. This aspect is missing in the four categories of PCK Grossman (1990) has described:

a. overarching conception of teaching a subject
b. knowledge of instructional strategies and representations
c. knowledge of students’ understandings, thinking and learning a subject
d. knowledge of curriculum and curricular materials.

These four categories, however, can be used to interpret the other results of this study. The need for information about pupils pre-knowledge reflects a need for knowledge of the curriculum. We found a lack of knowledge of representations and of the use of manipulatives, as well as a lack of knowledge of pupils’ understandings. In comparable studies about other topics from mathematics (area, Oldham et al. 1999) and from physics (temperature and heat, Frederik et al. 1999) more knowledge of those categories was found. Maybe, the students do have more knowledge of the categories mentioned, but cannot show, as ratio and proportion is a too general abstract topic, asking form many more lessons than one.

1.2 The second study: preparing a lesson about scale

In the second study two prospective maths teachers, Ilona and Jaap, following the teacher orientation course, were interviewed in a way comparable to the first study. Again, the lesson preparation method was used.
In order to investigate whether a more concrete topic than used in the first study would result in us giving more information about representations and manipulatives, we chose ‘scale’ as the topic for the lesson. In the instruction, no reference to ratio and proportion was made whatsoever. Both interviews were recorded on audio-tape, transcribed and analysed in the same way as in the first study.
A main point of attention in the analysis was the relation between scale and ratio/proportion and the use of representations and manipulatives.

results of the second study
In the lessons prepared both PTs planned to use different kinds of tools and representations related to scale. They planned to relate map distances to real distances. Ilona related scale to enlargements and to reduction by a microscope. She appeared to have some knowledge of the maths curriculum, as she had taught some private maths lessons. That is why related proportion related to scale: I think it [i.e. scale] is a beginning of doing calculations with ratio/proportion. She did not use the ratio table in her lesson, but from the interview it appeared she knew that the ratio table is introduced later on in the textbook that is used by most schools [i.e. Moderne Wiskunde].
Jaap, however, found that proportion does not have to do with scale. When the interviewer suggested the possibility of using scale as an introduction to ratio/proportion, Jaap answered: Well, I think that proportion is something the pupils have dealt with much earlier [i.e. in the primary school].

Discussion on the second study results
In contrast to ratio and proportion, because of being more concrete, the concept of scale seems to cause less difficulties to PTs. As was intended, this seems to give more room for using tools and representations, like maps, mathematical shapes. However, PTs seem to be urged less to think about the concept of scale in depth. The relation between scale and ratio/proportion is not self-evident to PTs. The knowledge of the curriculum seems to add to clarity about the relation: is ‘scale’ used as an introduction to ratio/proportion or is it a topic of its own, just needing some knowledge of ratio and proportion from primary school?

1.3 The third study: teaching ratio tables from a textbook
In the third study, Jaap and Ilona were interviewed another time, one and a half year later. Then, after having decided to become a teacher, they had followed teacher education and were interviewed at the end.
During their teacher education course, the students had attended a discussion meeting about the use of ratio tables in mathematics and in the sciences curricula. In their teaching practice, they had met the ratio table in the textbook they used in the classroom.
In the interview, the lesson preparation method was used in an altered form. As a preparation to the interview, the PTs were asked to discuss together about teaching a lesson (7th grade) about ratio tables. In order not to promote copying lessons from the textbook, they were supplied of three pupil textbooks having chapters about ratio and proportion. Jaap and Ilona entered into a deep discussion, comparing the use of ratio tables in the textbooks. The interview had the character of a continued discussion, the interviewer’s role being asking for a deeper reflection on arguments they used. During the discussion, the interviewer made notes that were used as data.

results of the third study
From this interview the same prospective teacher difficulties appeared as from the first study interviews:
Being at the end of their teacher education, Ilona met the difficulties the members of the first study had met at the start of their teacher education: it is difficult to put under words what ratio and proportion exactly is like. In fact, Jaap had the same problem but did not care. He thought he knew what ratio is like: it is a fraction. More important to him, he felt secure how to calculate proportionally using the ratio table, as can be seen in the following excerpt.

Interv.: What do you mean with ‘ratio’/ proportion.
Jaap: In mathematics? There it is a fraction.
Ilona: Oh, that is strange. For me it is not a fraction. It is more – how to say it? I do not know how to put it in words ……… …… but it is more a kind of relation, qualitative, that you can put in numbers and than you can manipulate with the numbers. But I cannot say what it is.
Jaap: But I don't need to know what it is because with this tool – ratio table – I can do everything I have to do with these numbers.
Ilona: [after some thinking] For me this ratio table is a nice tool because everything is in it, but I realise now that I can ‘see it’ but that I can’t say what that everything is and that worries me. It makes it so difficult, because…. If I don’t know what it is it will be difficult to explain it to kids who can not follow the text in the textbook, or who can not do the exercise.
Jaap: I think, for me ratio and proportion has mostly to do with making mixtures, like lemonades.

From this excerpt it appears that Jaap knew how to calculate in the ratio table, but felt no need to say why the methods used are valid, probably because they are self-evident to him. He even did not want to thing about the ‘why’-question. See the following excerpt.

Interviewer: What is essential in that [i.e. in the ratio table]?
Jaap: That you can always go to one, or can manipulate in different directions if you put the numbers in a table…….. But don’t ask me ‘why’?
Ilona: Yes, but I want to be able to answer that ‘why?’ question, otherwise I feel very uncertain when I go into a classroom……..

In this excerpt, Ilona expressed a main point: she needed to be sure about the mathematics for the sake of giving a good explanation. This, she explained further in the following discussion that was started by the educator’s question what pupil difficulties they would expect:

Jaap: It is very difficult for me to have an idea about what the children find difficult. So I always try to start with an example that is clear for me. Entering the classroom with some lemonade syrup and cans of water is such a thing I often did in my teaching practice.

Ilona: Because I have sometimes difficulties myself I stay close to the textbook. But also I try to find out what is behind the text and that is my problem; this is so much time consuming.

Here, Ilona explained the difficulty not understanding aspect of mathematics had for her in dealing with the textbook. Jaap did not had that problem, because he did not felt a need for understanding the mathematics so deeply.

Jaap again referred to the lemonade syrup and water situation he already used in the first excerpt of this study. He told the interviewer that he had liked the way his primary teacher used to bring in water and lemonade syrup to make mixtures and how you can do the same with paint to change a colour. We conclude from this that the mixture situation forms an important representation to him.

However, to probe his understanding, the interviewer asked about the mixture of two different mixtures. He presented the task: One volume of a 20% mixture is put together with double the amount of a 30% mixture, what kind of a mixture do you get?

Doing the task, Jaap impulsively tried to manipulate the numbers as if he could just ‘add up’ to 50 and halve it to 25%. Feeling he made a mistake, he completely lost the way. Connecting
this with Jaap’s idea of ratio being a fraction, we conclude that he lacked some content knowledge, without being not aware of that.

discussion of the third study results
It seems to us that, compared to the first interview, Jaap and Ilona had learned how to use the ratio table from the textbook in the classroom, but had not succeeded in understanding ratio and proportion more deeply. The fact that they, nearly being certificated, in fact struggled with the same problem as the starting PTs from the first study, on the one hand stresses the importance of knowledge of the basic understanding of ratio and proportion, on the other it indicates that our teacher education did not meet the need for getting this knowledge, a need we see as teacher educators and felt by Ilona as well. Ilona aptly verbalised that this basic knowledge is a part of PCK, as that include, in the words of Shulman (1986), “the ways of representing and formulating the subject that make it comprehensible to others”. Jaap did not feel that need, probably because of his mixing-lemonade-syrup-and-water representation of ratio and proportion.
The importance of this representation can be seen in the light of the ‘primitive-models’ theory (Fischbein et al. 1985). It says that each basic operation of arithmetic generally remains linked to an “implicit, unconscious, and primitive intuitive model” (p.4). These models comprise the original meanings that were assigned to the operations and are thus behavioural in nature and meaningful to learners. The models are assumed to exert an unconscious influence on the learner’s problem solving efforts, long after a concept has been formalised and thus appear largely responsible for the difficulties encountered when dealing with situations involving arithmetic operations. If this is extended to ratio/proportion situations one could say: Jaap’s primitive model may imply that the ratio between lemonade and water keeps the same if you take half of the mixture. This model is insufficient for ‘mixtures of mixtures’ of the kind that the educator presented to him.
The PTs did not seem to have any idea of specific pupil difficulties on ratio and proportion, they just were busy with their problem of teaching the topic, not with understanding the learning of pupils.

1.4 Conclusions
Teaching ratio and proportion asks for content knowledge as well as pedagogical content knowledge of prospective teachers. In our particular situation, PTs being graduated mathematicians, mathematics teacher education should provide PTs with pedagogical content knowledge. In our three small scale studies we found found that Grossman’s four categories of PCK are suited for describing PCK but that it is needed to add one category being ‘basic knowledge of the topic’, in the case studied: of ratio and proportion. In fact, PTs have mathematics knowledge based on those basics, but because of ‘level reduction’ (Van Hiele 1986) they are not aware anymore of the basics. For explaining the topic to pupils and for understanding tools (like the ratio table) and strategies textbooks use, it is needed to reflect on the basics again.
We found indications that the following knowledge is needed and that our mathematics education did not provide it sufficiently.
- Knowledge of the basics of ratio and proportion. For its content, see section 2.1 of this paper.
- Knowledge of using the ratio table as a representation and an instructional strategy, connected to the place of ratio and proportion in the curriculum (Grossman’s categories 2 and 4). In section 2.2 this knowledge is explained, by analysis of the use of the ratio table in a textbook.
Knowledge of pupils’ difficulties in learning ratio and proportion and the ways the use of the ratio table can contribute to solving those difficulties (Grossman’s category 3). In section 3.3 the results of a review of the literature about those difficulties are reported.

From the studies, no need for reflection on the overarching conception of the use of ratio tables (Grossman’s category 1) came to the fore, but we think it would be useful too. This overarching conception is the conception of Realistic Mathematics Education. However this is a more general topic, we do not deal with it in part 2 of this paper, but we refer to Streefland (1984, 1985) and Treffers (1985).

Part 2: aspects of the knowledge to be dealt with in teacher education

2.1. Basic mathematics of ratio and proportion and of the ratio table

Many articles about ratio and proportion in mathematics education take the definition of ratio and proportion for granted. However, lack of agreement about the definition gives rise to inconsistencies within and between different articles. By some authors ratio is defined by proportion and proportion by ratio. E.g. by Confrey and Smith (1995) saying:

ratios are never singular instances of a relationship between magnitudes but are constructed by objectifying and naming that which is the same across proportions. To recognise ratio is to recognise the homogeneity of ratio across more than one instance.

(p. 74)

And by the same authors, proportion is explained as ‘equal numeric ratios’.

To escape this, some authors define ratio as the quotient of two numbers, a definition that can be found in Mathematics Dictionary (1976):

[Ratio:] The quotient of two numbers (or quantities); the relative sizes of two numbers (or quantities).

However, there are several clear differences between the concepts of ratio and quotient:

- in a quotient, the two terms are numbers; in ratio, other terms can be used, like vectors, complex numbers or undetermined quantities (e.g. ‘the length and width of an A4 sheet of paper have the same ratio as the length and width of an A3 sheet of paper’).
- in a quotient, the terms consist of a pair of numbers; a ratio can consist of three or more terms, e.g. 3 : 4 : 5.

Even if we limit ourselves to ratios with numbers and only two terms, clear differences can be pointed at:

- every quotient has a place on the real number line; a ratio does not have;
- negative numbers don’t appear in a ratio; in a quotient, one or both terms may be negative numbers
- in a quotient, there is a numerator and a denominator. In ratio, there is no such discrimination
- in ratios, comparing of two situation or aspects of a context is central. If the ratio of dollars to apples is 1 : 4, it is important to know that the 1 regards the money and the 4 regards the apples that can be bought by the money. In a quotient, the two values can be changed for one value, in this case ¼ or 0.25. Behind the numbers of a ratio, its context is hidden!

Tourniaire and Pulos (1985) make a distinction between internal and external ratios. If quantities of the same nature are compared (e.g. the length of an A3 and A4 sheet of paper),
the ratio is internal. If quantities of different natures are compared (e.g. dollars and apples), the ratio is external. There are two different, but equivalent definitions of proportion: the scalar definition using internal ratios and the functional definition using external ratios. Both are described below for three term ratios, following Spaanderman (1997).

The scalar definition says:

two ratios \((a, b, c)\) and \((x, y, z)\) are equivalent if some real, non-zero number \(\alpha\) can be found, to which the following equalities apply:

\[
\begin{align*}
x &= \alpha \cdot a \\
y &= \alpha \cdot b \\
z &= \alpha \cdot c
\end{align*}
\]

The functional definition says:

two ratios \((a, b, c)\) and \((x, y, z)\) are equivalent if two terms \(\beta\) and \(\gamma\) can be found for which the following equalities apply:

\[
\begin{align*}
b &= \beta \cdot a \quad \text{and} \\
\gamma \cdot a \\
z &= \gamma \cdot x
\end{align*}
\]

It is worthwhile to note that if \(a\) and \(b\) are quantities of different dimensions, \(\beta\) has to be valued as a quantity as well. Even so, if \(a\) and \(c\) are quantities, \(\gamma\) is a quantity. However, as \(a\) and \(x\) are of the same nature, \(\alpha\) is a number and can be valued as an enlargement or increment (or, being smaller than 1, a reduction factor). These properties are very important to the sciences, as the ratios of several quantities have been given a meaning as new quantities, and the like. The ratio between the mass and the volume of a substance is the density of the substance; the ratio between the distance a car has driven and the time spent is the speed of the car.

From the above, it can be inferred that ratio and proportion are no simple, ‘unitary’ (Tourniaire and Pulos 1985, p.200) constructs, but multiple ones. Therefore, one should not conceive of a linear teaching sequence (Vergnaud, 1999). An important tool promoting a non-linear teaching sequence is the ratio table.

### 2.2. Using the ratio table as an instructional strategy

The Realistic Maths Education has adopted the ratio table and developed it as a tool for promoting pupils’ understanding of ratio and proportion. It provides children with experiences a strong ratio and proportion conception can be build upon (Streefland 1985).

#### Ratio and proportion in Dutch lower secondary education

In this paper, we sketch how the ratio table is being introduced in grade 7 of Dutch HAVO/VWO schools (higher ability students). This tool for structuring ratio and proportion tasks is in a simplified form already used in Dutch primary school mathematics. A ratio table is a two-row table, each row having a label indicating the meaning of the numbers in the row. The columns are filled with pairs of numbers that have the same ratio. Therefore it is called ratio table.

As an example, we describe the structure used in the most popular textbook of Dutch secondary mathematics, *Moderne Wiskunde (MW)*. MW starts focussing on the ‘vertical aspect’, the ratio number.

3 Marlou goes from door to door with ‘kinderzegels’ (stamps sold by children to benefit children). One sheet of stamps costs dfl. 7,50.
Marlou does not want to calculate again and again how much is to be paid. So, she has a small list with her. It is shown below.

<table>
<thead>
<tr>
<th>number of sheets</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>guilders</td>
<td>7.5</td>
<td>15</td>
<td></td>
<td></td>
<td>..</td>
</tr>
</tbody>
</table>

x ...

a. Copy the list and fill in the numbers
b. What number has to be inserted at the arrow?
c. Copy and fill in: the ratio 4 over 30 is the same as 2 over ...
d. Write down another two ratios that are in the table.

figure 1 Introduction of the ratio number in MW, volume 1a (for grade 7)

Attention is paid to the recognition of the proportionality of the table. For, not all the tables in the maths book having the same shape are ratio tables.
Initially, the tasks are about simple ratios that can easily be recognised. The number of 25 cent coins in a number of guilders, minutes in an hour, photographs per reel, etc. In this way, by calculating by heart and by reasoning, pupils can discover all kinds of ways to calculate in the table, maintaining the proportions. Such as by multiplying and dividing, in particular by doubling, halving, and dividing by 10 and other exponents of 10. They are stimulated to indicate, by placing arrows at numbers, what exactly they have done.

11 At Beert's bakery 350 grams of butter cookies cost dfl. 6.30. Jean-Paul wants to calculate what price he has to pay for 75 grams.

a. Why does Jean-Paul first calculates in the table beneath how much 50 grams and 25 grams cost?
b. Copy the table and complete it.
c. How much did Jean-Paul have to pay?
d. Can you imagine a faster way to calculate the costs?

<table>
<thead>
<tr>
<th>grams of cookies</th>
<th>350</th>
<th>50</th>
<th>25</th>
<th>75</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of cents</td>
<td>630</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

figure 2 Pupils can discover that they have to multiply or divide the top and bottom cells with the same number to keep the ratio constant.

Later on ‘proportional addition and subtraction’ is added. In the context of Dutch coins, ‘dubbeltjes’ (10 cent coins) and ‘kwartjes’ (25 cent coins), it works like this: 1 guilder added means 4 kwartjes added; another guilder, another 4 kwartjes, etc. In that way, the additive structure is discovered.

How to calculate with ratio tables?

**multiplication**

\[ \times 7 \]

<table>
<thead>
<tr>
<th>spring roll</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>28</th>
</tr>
</thead>
<tbody>
<tr>
<td>guilders</td>
<td>1.25</td>
<td>2.50</td>
<td>3.75</td>
<td>5.00</td>
<td>35.00</td>
</tr>
</tbody>
</table>

\[ \div 2 \]

<table>
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<th>14</th>
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<td>35.00</td>
<td>17.50</td>
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</tbody>
</table>

\[ \div 2 \]
addition

<table>
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<th>4</th>
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subtraction

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<th>4</th>
<th>10</th>
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<td>2.50</td>
<td>3.75</td>
<td>5.00</td>
<td>12.50</td>
<td>10.00</td>
</tr>
</tbody>
</table>

figure 3 Overview of the ways of calculating with ratio tables

As maths education continues, the tables gradually get more empty and instead of part of the task, they become a tool that can be used by choice, if the tasks are appropriate.

3 A supermarket has two bargains:
a 650 g. pot of mayonnaise for dfl 1.98
and a 450 g. tube of mayonnaise for dfl. 1.44.
Comparing the two, what bargain is the cheapest?

figure 4. This task requires that the students make two ratio tables that are compared.

Lastly the ratio table is used to work with fractions and percentages (through standardising to 100). Thus, the tables form a connecting element between ratio, proportion, fraction and percentage.

How do you calculate percentages with a ratio table?

A school has 950 pupils. 38 of them are ill. You want to know what the percentage is.

1. Write the number of sick pupils in the table and the total number of pupils underneath.
2. Then, in the bottom row, go by way of 1 to 100.
3. The percentage then appears in the box above 100.

So the percentage of ill pupils is 4.

eample

| number of ill pupils | 38 | ... | ... |
| total number         | 950 | ... | ... |

| number of ill pupils | 38 | ... | ... |
| total number         | 950 | 1 | 100 |

: 950 x 100

38 : 950 x 100 = 4

figure 4: Using a ratio table to calculate percentages.

It is always possible to use different ways to calculate the figure asked for. The student is allowed to find his/her own way. That is one of the main advantages of working with a ratio table. We illustrate this below.

A photograph is made of Ellen when she taking her first steps as a baby. In reality, she is 76 cm long. On the photograph she is 8 cm long. Her 3-year-old brother Jan helps her with her first steps and he is shown, too. There, he is 10 cm long.

What is Jan’s real length?
For this task the following ratio table can be made (or given) to structure the task and the data.

<table>
<thead>
<tr>
<th>length on photograph</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>real length</td>
<td>76</td>
<td>?</td>
</tr>
</tbody>
</table>

The (initially) simplest way of doing the calculation is the ‘horizontal’ way. The first number in the upper row (in this case the 8) is processed at once or in some ‘easy’ steps until the desired number (in this case 10) is found. One possibility is to multiply 8 by 5 (result 40) and next to divide it by 4 (result 10). The first number in the lower row (76) now is processed in the same way, as shown below.

\[ \times 5 : 4 \]

<table>
<thead>
<tr>
<th>Length on photograph</th>
<th>8</th>
<th>40</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>real length</td>
<td>76</td>
<td>380</td>
<td>95</td>
</tr>
</tbody>
</table>

Another ‘horizontal’ way is reducing to 1. In this case e.g. by dividing 8 by 2, three times. The result is shown below.

\[ : 2 : 2 : 2 \times 10 \]

<table>
<thead>
<tr>
<th>Length on photograph</th>
<th>8</th>
<th>4</th>
<th>2</th>
<th>1</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>real length</td>
<td>76</td>
<td>38</td>
<td>19</td>
<td>9.5</td>
<td>95</td>
</tr>
</tbody>
</table>

In both cases, the answer can be read from the table: Jan is in reality 95 cm long. The reduction-to-1 method can provide the insight that the ratio between ‘photograph’ and ‘reality’ is 1 to 9.9. One can also say: the scale’ of the photograph is 1 : 9.5. Reduction to one can also be applied to the lower row if it this easier or if it has a meaning in the context.

For a ‘vertical’ way of working with the ratio table, insight into the ratio number is needed. This means an advanced understanding of ratio. In our case, a pupil must have reduced to one, as in the above, or must have seen ‘at once’ that 8 has to be multiplied by 9.5 to get 76 (or 76 has to be divided by 9.5 to get 8). Then the pupil can see that 10 has to be multiplied by 9.5 as well to get the number asked for (95).

A third way of working is to use cross-products. In this example a scheme can be used showing three numbers, as follows:

\[
\begin{array}{c|c|c}
8 & 10 \\
76 & ? \\
\end{array}
\]

Then, the cross product can be calculated:

\[ 8 \times ? = 76 \times 10 \]

so

\[ ? = (76 \times 10) / 8 \]

resulting in:

\[ ? = 95 \]

In the time before the ratio tables were introduced into Dutch mathematics, this way was often used as an algorithm for solving ratio and proportion problems. In fact, it does not fit into ratio table reasoning because, in contrast to the ‘horizontal’ and ‘vertical’ ways, it does not clarify the proportional way of reasoning. To many students, the cross-product way of working is a trick they don’t understand the background of. Some pupils will remember that they have to put the numbers into the scheme somehow …. but how? … and that they have to multiply and divide. But it remains unclear what precisely they have to do and why.
The ratio table can provide a meaningful and short alternative to the cross product algorithm in the following, reduced form, that can be used for all ‘non nice numbers’ and that closely links to the way the numbers are typed into the pocket calculator:

<table>
<thead>
<tr>
<th>Length on photograph</th>
<th>8</th>
<th>1</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>real length</td>
<td>76</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>:8</td>
<td>x4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Ratio tables: a forceful pedagogical content tool**

Now, we can characterise the ratio table with the following five characteristics:

1. the table consists of two (or more) rows and two or more columns, with numbers in the cells
2. the rows have a label, indicating the meaning of the numbers and specifying, if needed, the units used
3. there is no preference on what to choose as the upper or lower row
4. the ratio between the numbers in the columns is the same for all columns; this can be used to calculate an empty place in a column
5. so to get the numbers of a column, the numbers of another column can be multiplied or divided by a certain number; proportionately adding or subtracting are possible as well.

The ‘horizontal’ transformation in 4. corresponds with the scalar way of using proportion; the ‘vertical’ transformation in 5. corresponds with the functional way.

We characterise the *ratio table instructional strategy* as follows. The ratio table is introduced using partly filled tables taken from a clear real-life context. By filling in, strategies to work with the table can be discovered and developed. Next, pupils start with an empty table, identify numbers from the given problem, to be inserted in the table, and choose suitable labels. Having written down the table, the pupil has structured the task and the calculating can be started. Pupils’ insight has to be increase so far that they can easily manipulate numbers and can calculate the number asked for in an efficient way.

Calculating ‘vertically’ with the ratio number is an example of an efficient strategy, as is the ‘horizontal’ reduction to 1 (normalising). Via normalising, every proportion problem can be solved in two steps. For the actual calculating, the pocket calculator can be used. This way of working provides a, hopefully more meaningful, alternative for the ‘mechanistic’ cross-product algorithm that is just a trick to many pupils.

Using ratio tables, pupils get a clear structure in which they can find an answer on their own ways:
- a pupil gets a certain freedom concerning filling in numbers into a next column; (s)he is allowed to choose one out of the steps that are admitted;
- the pupil is allowed to use as many steps as (s)he needs or finds useful
- however, the ratio between the numbers in the upper and lower row have to stay the same.

Dolk (1997) speaks in cases like this of students having ‘construction room’.

In fact, this ‘construction-room’ characteristic can be seen as the sixth characteristic of the ratio table in mathematics education.

Middleton and Van den Heuvel-Panhuizen (1995) described how the ratio table can be used in US middle school (grade 5-8).

### 2.3. Difficulties pupils have with ratio and proportion

Many research studies (among others Hart 1984, Clarkson 1990, Schliemann and Nunez 1990) have been done to the difficulties pupils meet when learning ratio and proportion in the traditional ways. Following Streefland (1984, 1985) we summarise some main difficulties
about ratio and proportion. We add some reflections about the ratio table solving or preventing the problems. However good the experiences with the ratio table are, we found few data about its effect on pupils’ conceptual difficulties (Middleton et al. 1995).

- Pupils are inclined to apply additive thinking and replace multiplication by repeated addition; thus, they meet a difficulty doing tasks that ask for multiplication with a fraction or for division (e.g. 6 apples cost 2 guilders; how many apples can you buy with 5 guilders?).

As is shown in section 2, in the ratio table, pupils can follow additive strategies that can serve as an intermediate step towards multiplicative ones.

- Pupils use context-bound solution methods related to ‘real-life’ problems in cases that ask for mathematical reasoning with particular sets of numbers without dimensions. As the ratio table is a strong and meaningful mathematical structure, it may provide an attractive alternative for context-bound solution methods.

- Pupils have difficulties in using different algorithms connected with \( \frac{a}{b} = \frac{c}{d} \), like the cross product (the French call this ‘une propriété fondamentale’), invertendo, alternando, etc. (Hart 1984, De Kock et al. 1956);

Using the ratio table, pupils are allowed freedom to decide themselves on the number of intermediate steps (columns) to use for reaching the answer. During maths education, reduction of this number is aimed at. Then, as is argued in section 2, the ratio table can provide pupils with meaningful alternatives to the cross product algorithm, like the reduction-to-1 (normalising) method and calculating ‘vertically’ with the ratio number.

The ratio table is a transparent tool that promotes structuring and can act as a visual pattern. By that, it can also contribute to the solution of more general problems that play an important role in ratio and proportion calculations:

- Pupils mix up part-part and part-whole comparing situations, in particular in percent problems (e.g. saying that 90% of 80% is 70%).

- Pupils avoid using fraction and follow the so called ‘nice number strategy’ (Harel et al. 1994);

- Pupils ignore part of the data in a problem (Tourniaire and Pulos, p. 185);

- Pupils have difficulties with calculations, springing from limited understanding of multiplication, division and fraction, as well as decimal concepts (Van den Brink and Streefland 1979).

Ratio and proportion play a role in geometrical situations that have to do with similarity and enlargement. The role of the ratio table in solving those problems will be limited.

- In enlarging-figures-tasks, children are often so engrossed in the method that they forget to check by comparing whether or not the resulting enlargement is of ‘the same’ shape as the original;

- Pupils have problems with the word ‘similar’. It seems to be much more difficult to compare two figures than to use ratio to share amounts between people ‘so that it is fair’.

References.

(some titles on ratio and proportion in this list are not referred to in the article, but are in it for sake of giving an overview of the literature)


