Bijlage 15

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12 Maps and Models - Approaches to Vectors

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When students first meet vectors they can get lost in the complexities of Pythagoras and trigonometry, losing sight of the fundamental, geometric ideas. To help them with the detail they first need to develop intuitive ideas about the meaning of what they are doing.

Very often, vectors are introduced in the context of mechanics and the vectors involved – velocity, acceleration, force – are rather abstract. This abstraction presents an additional obstacle to students when they begin to grapple with mathematical detail. So how can we develop good physical insight – a 'feel' for the mathematics – alongside technical detail?

Two approaches are considered here. The first begins by taking a familiar example – a road map – and developing ideas about vector addition, invariance under rotation and resolution into components in a geometric way. By initially avoiding more abstract ideas about forces and motion, the formalism can be developed without too many new concepts being introduced together. The second involves the use of computer modelling to enable students to see what a vector is, what it means to add vectors together and to find components, by making abstract ideas more concrete.

The vector quantities that we encounter when we study maps are displacements

Motorists' handbooks often give tables of the distances between towns. What the table does not tell us is the direction from one town to another. Curiously, it is possible to find the relative directions just from the distances. Finding out how to do this can help students to understand how vector quantities combine, indeed what a vector quantity actually is.

Consider one simple case. The distance as the crow flies from Calais to Marseille is 879 km. The distance from Calais to Lyon is 612 km, and the distance from Lyon to Marseille is 270 km. Lyon should lie very nearly on a straight line from Calais to Marseille and you should be able to see why! This situation approximates very closely to the sort of scalar addition that students are used to – but not quite.

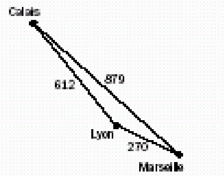

Figure 1 Calais, Lyon and Marseille must lie nearly on a straight line

Marseille to Bordeaux is 498 km, and Lyon to Bordeaux is 426 km. We already know that Lyon to Marseille is 270 km. These distances do not add up! The paths must be angled.

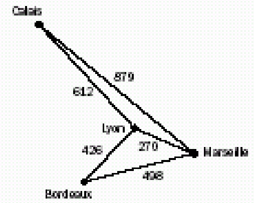

Figure 2 Lyon, Marseille and Calais are NOT in a straight line

- Using the table of distances in a road atlas, a map showing the relative positions of towns can be built up and the geometric meaning of vector addition emphasised
- At a more advanced level, one might then go on to make points about writing the laws of physics in an invariant way

To get at a possible layout of these towns, lines proportional to 426 km and to 498 km can be anchored on Lyon and Marseille, and rotated until they meet. (It should be clear that the location of Bordeaux is not the only possible one – the lines would meet just as well above and to the right of Lyon and Marseille.) From this example, students can begin to get an idea of how vector addition works: vectors are added geometrically by placing them tip to tail and the magnitude of the vector sum is not the same as the sum of the magnitudes.

Using this example, we can introduce students to another important piece of mathematics. The triangle Bordeaux–Lyon–Marseille looks as if the angle at Lyon is nearly a right angle. This can be tested, using Pythagoras' theorem: (Bordeaux–Lyon distance/km)^2 + (Lyon–Marseille distance/km)^2 = 254 000

The square root of this is 504 km. It is the distance it would be from Marseille to Bordeaux if the angle were a right angle. The actual distance, 498 km, is quite close.

By using a road atlas table of distances, a map showing the relative positions of towns can be built up and the geometric meaning of vector addition emphasised. However, it is quite clear that we have no way of knowing which way up the map should go. North might be in any direction. Figure 3 shows the map rotated so that North is, in fact, vertical.

What this tells us is that the coordinate system we use does not alter the relationship between vectors, nor the vectors themselves. They are geometric objects. In particular, their lengths are invariant under rotations, and similar conclusions can be drawn about reflections. At a more advanced level than we are discussing here, one might then want to go on to make points about writing the laws of physics in an invariant way. A consideration of such invariants, even at A level, could be used to make a link with special relativity, where the space–time interval between two events is an invariant, the 'length' of a four dimensional vector.

The resolution of vectors into particular components can be considered, once a particular coordinate system has been established. We could ask the question, 'If we drive from Marseille to Bordeaux, how far towards Calais have we got?' One answer is that since we were not heading the right way, we have gone 498 km and have 696 km left to go in a different direction. This is the answer the driver of the car would have to give to the not-too-pleased passengers.

Another answer is that we have gone about 200 km along the true direction, Marseille–Calais. That does not sound very sensible, since at Bordeaux we are nowhere near Lyon, having gone more than 400 km West as well. It sounds a bit more sensible if we ask how far North we have gone from Marseille. Then (if Calais is due North of Marseille, which is nearly true) the answer is about 200 km due North. This is shown in figure 3.

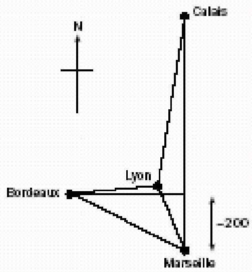

Figure 3 How far towards Calais do we get by driving from Marseille to Bordeaux?

By considering such ideas, students can begin to understand what a component actually is in a very concrete way, before the introduction of trigonometric techniques and before the resolution of more abstract vectors.

Figure 4 A Modellus model that can be used to develop intuitive ideas about the meaning of a velocity vector

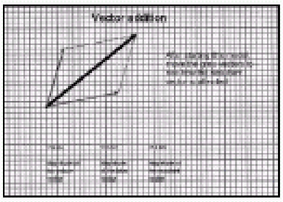

Figure 5 Vector addition in Modellus

So, the idea of maps can lead students towards an understanding of what a vector actually is, of how vector addition works, of the meaning of Pythagoras' theorem, of what components are and towards ideas about invariance and symmetry. All these ideas are developed in a familiar context and in a very concrete way. More abstract applications of the ideas can now be tackled with students confident in their understanding of the fundamental ideas.

Computer modelling provides another way of seeing directly what the mathematics means and of developing an intuitive feel for it

The models described here were written using 'Modellus', a system developed at the New University of Lisbon. The system allows you to define a mathematical model (using the same notation as one would on paper) and then represent the model graphically, as a tabulated set of values or as an animation. Figure 4 shows the animation of a very simple model that develops an idea of what the velocity vector means.

The software provides its own time variable, t, so that difference equations can be written to define two components of velocity. These are used in the animation window to attach a vector with these components to an object, in this case a ball. The position of the ball on the screen is given by the values of x and y. When the model is run, the user is able to grab the ball with the cursor and move it around the screen. Doing so changes the values of x and y and the software adds the velocity vector to the object. By dragging the ball around the screen, the student experiences directly what the velocity vector means: it tells her which way she is moving her mouse and how fast. The abstract becomes concrete and the mathematics is tied to experience.

The ability to grab hold of objects in the animation window and move them around provides a way of developing a good intuition for the mathematics. A second example showing how vectors are added illustrates this. In the model, a diagram of two vectors and their resultant is shown, with construction lines to show how the parallelogram rule is used. When the model is running, the student is able to grab hold of either of the vectors and move it around. The resultant changes accordingly.

The power of the program is difficult to demonstrate on paper, but when students can move vectors around and see the result of the vector addition instantly, their intuitive understanding surely has to grow. The complications of scale drawing are handled by the software, while the student gains direct experience of what vector addition means.

A third example concerns resolution of vectors into components. Again, Modellus allows direct manipulation of a vector and a visual resolution into components. The relevant window is shown in figure 6.

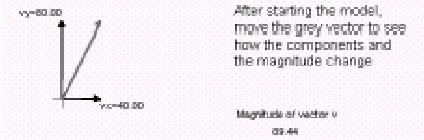

Figure 6 Modellus can be used to show how vectors are resolved into components, and the resolution can be animated by dragging the grey vector around on the screen

Again, the tip of the vector can be moved around on screen and the components follow. The mathematical model controlling this animation is a simple statement of Pythagoras' theorem, but it could equally well have been written using trigonometry. By setting up such a model for themselves, students are able to check that they have handled the mathematics correctly: any errors are immediately apparent on screen when the model is run.

As a final example, consider projectile motion. The parabolic path followed by a projectile is the result of vector

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- Students should ultimately be able to resolve and add vectors without the aid of the computer

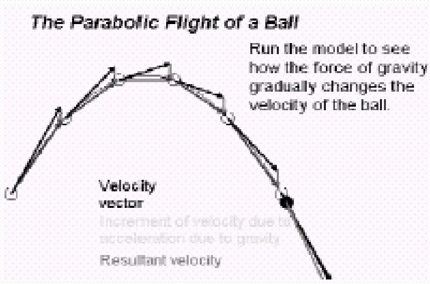

Figure 7 Animating parabolic motion and showing how vector addition leads to the projectile's path

addition. A small change in the velocity of the projectile is induced moment by moment by the acceleration due to gravity. This process can be modelled by calculating changes to the velocity over a finite time step. The new velocity is then used to compute the path over the next time step before the velocity is updated again. This whole process may be animated as, step by step, the projectile crosses the screen and vectors are added to the path. The animation window from Modellus in figure 7 shows this vector addition pictorially.

Perhaps, then, computer modelling can help at least some students gain an understanding of vectors through interacting with animations. In this way, abstract concepts can be made more concrete and the complex mathematics given meaning in the mind of the student. Of course, computer modelling is not the solution to everyone's problems and the examples given here are no more than a beginning. We would still expect our students ultimately to be able to resolve and add vectors without the aid of the computer, but the computer may provide a gentler introduction to some abstract ideas. Students might then be less intimidated and more successful when they encounter more abstract ideas.

DISCUSSION POINTS

- Defining vectors simply as quantities that have magnitude and direction is unhelpful. The key thing is knowing how they are added.
- Beginning with displacements that can be added 'tip to tail' builds on students' intuition and makes vector addition straightforward.
- The displacement vector is too directly descriptive a starting point. It is quite different from other vectors such as force.
- Notation between mathematics and physics courses needs to be standardised. There could be more two-way cross-fertilisation between mathematics and physics.