Find the shortest path?
Thanks to graph theory!

University of Luxembourg

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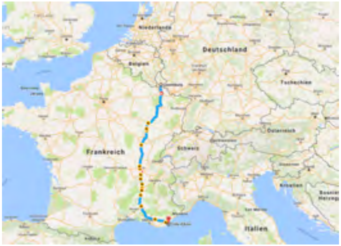
Some applications of graph theory in daily life



1) Eulerian graphs and the Königsberg bridge problem

The seven bridges of Königsberg

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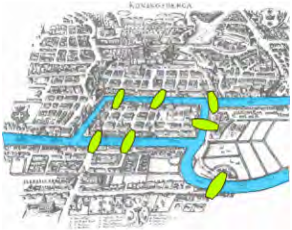

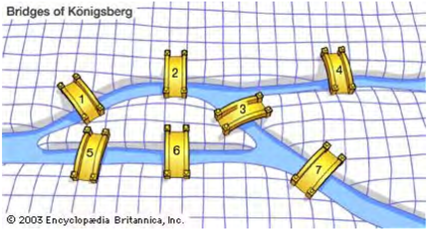

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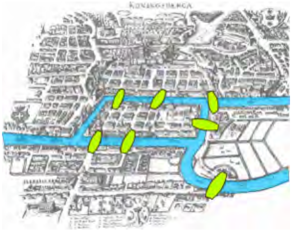

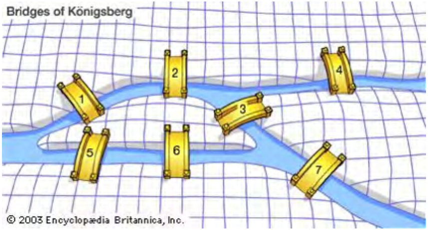

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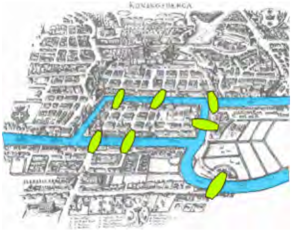

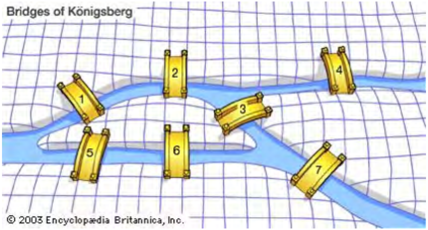

The Königsberg bridge problem

Is it possible to find a walk through Königsberg that passes exactly once over each of the seven bridges and returns to the starting point?

Graphs: definition and properties

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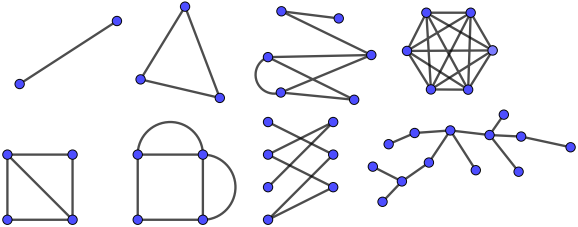

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$\Rightarrow$ Numerous problems can be modelled by graphs!

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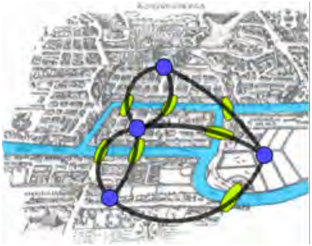

The graph of the Königsberg problem

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edges = bridges connecting the corresponding districts

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The Königsberg bridge problem (‘graph version’)

Is it possible to traverse the graph of Königsberg (starting at an arbitrary vertex) using each edge exactly once and return to the starting vertex?

Eulerian graphs, eulerian cycles

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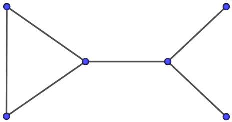

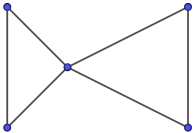

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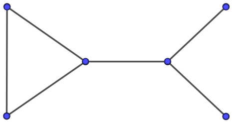

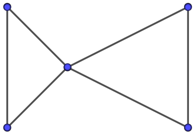

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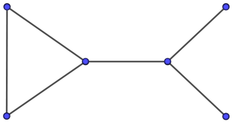

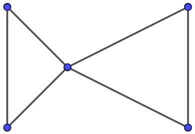

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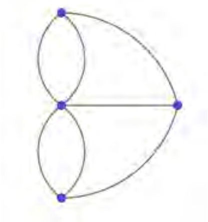

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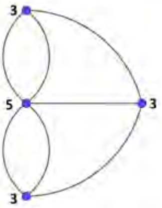

The graph of Königsberg has vertices of odd degree, hence it is not a eulerian graph and there is no eulerian cycle!

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G is eulerian $\Leftrightarrow$ all the degrees of G are even.

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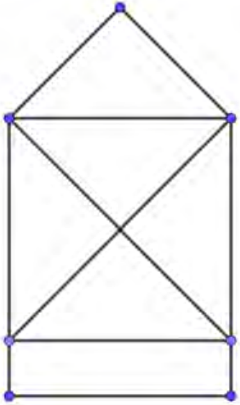

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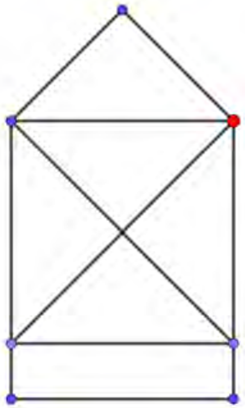

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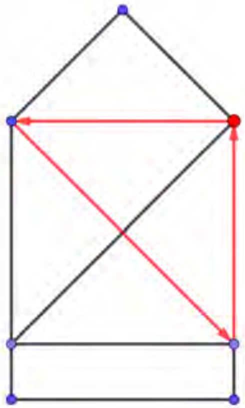

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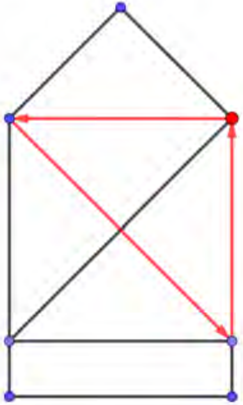

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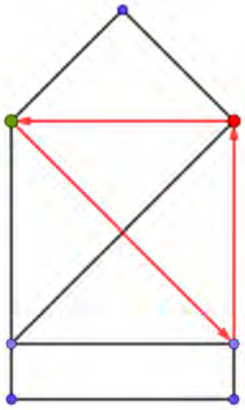

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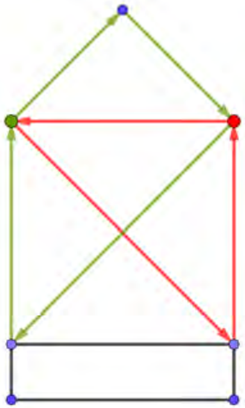

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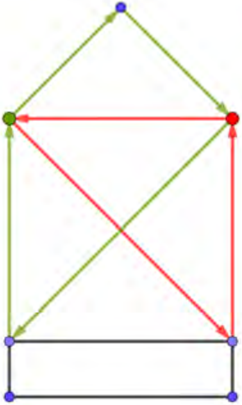

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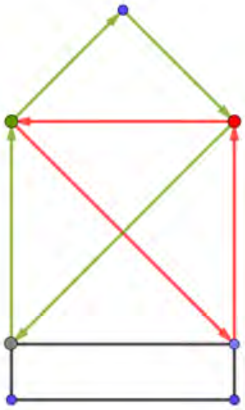

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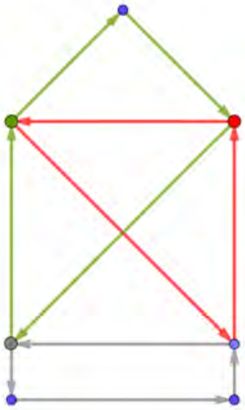

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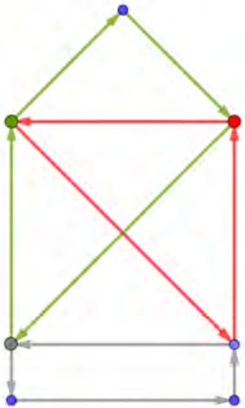

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7) Merge the different cycles, that is:

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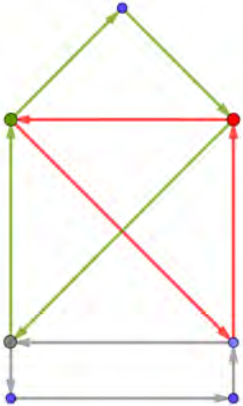

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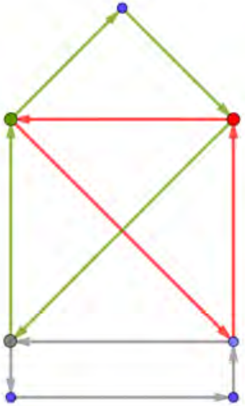

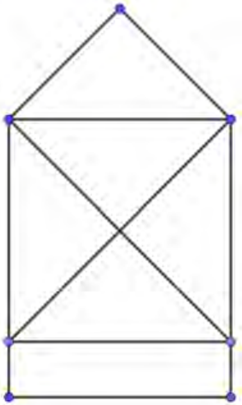

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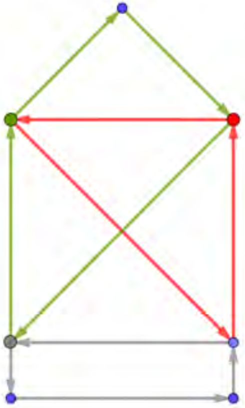

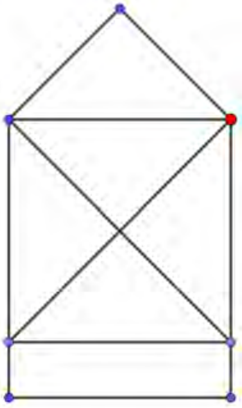

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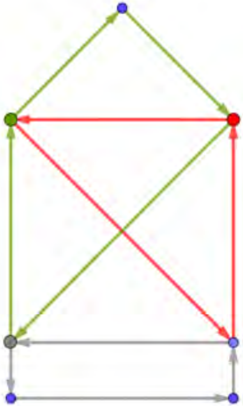

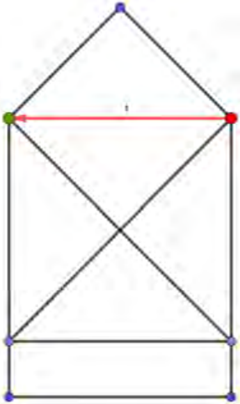

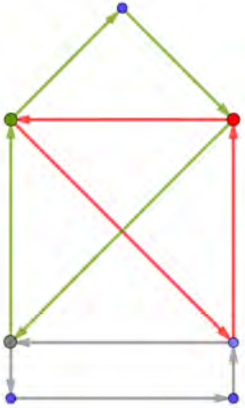

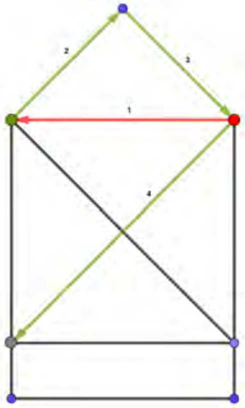
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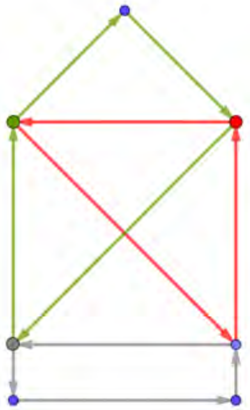

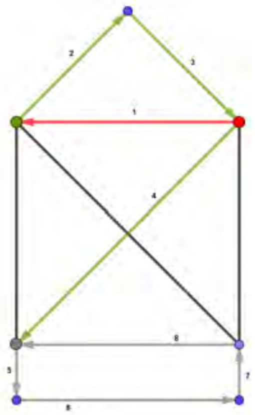

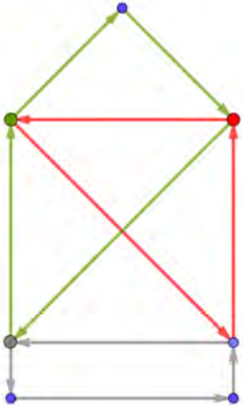

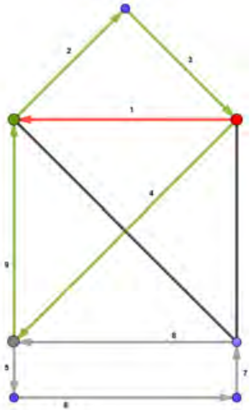
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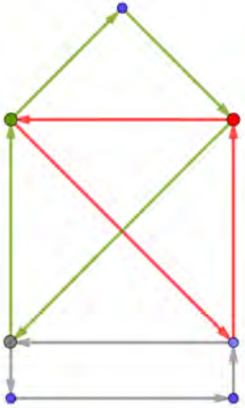

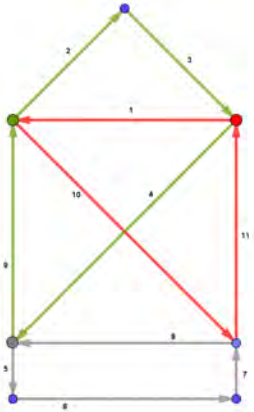

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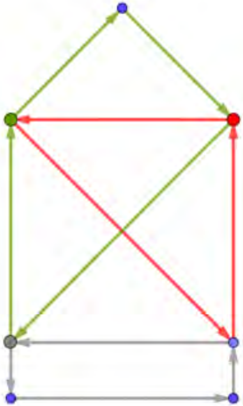

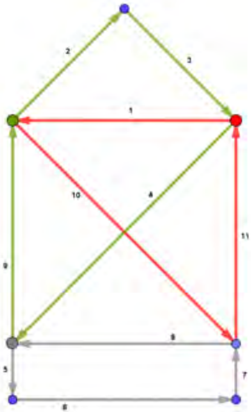

$\Rightarrow$ the resulting cycle is a eulerian cycle!

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$\rightarrow$ Activity

One more time: The Königsberg problem

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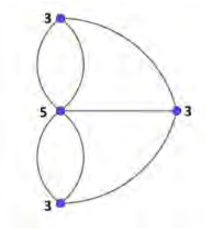

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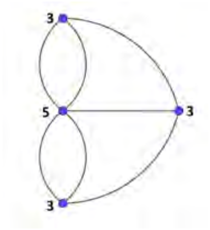

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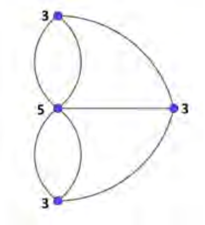

Euler's theorem implies that the Königsberg bridge problem has no solution.

Question:

How many bridges must be added (at least) to Königsberg (and where), in order that the problem has a solution?

Semi-eulerian graphs, eulerian paths

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Variation of the Königsberg problem:

Semi-eulerian graphs, eulerian paths

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Is it possible to find a walk through Königsberg that passes exactly once over each of the seven bridges without returning to the starting point?

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Equivalent: Is it possible to traverse the graph of Königsberg (starting at an arbitrary vertex) using each edge exactly once without returning to the starting vertex?

A graph that may be traversed in this way is called a semi-eulerian graph and such a passage is called eulerian path.

Semi-eulerian graphs, eulerian paths

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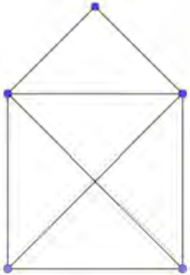

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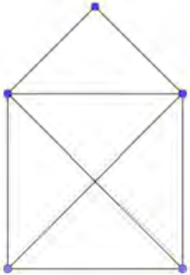

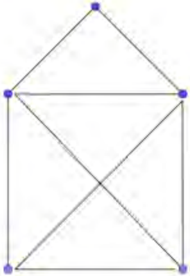

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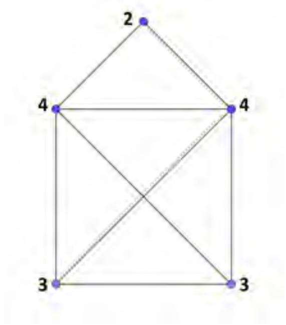
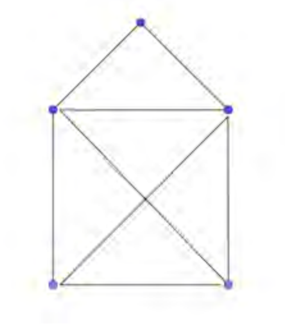
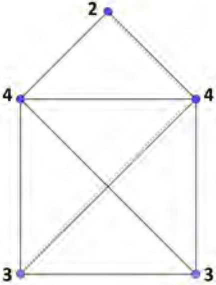
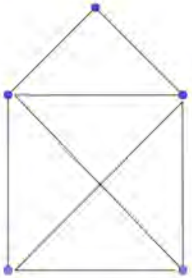


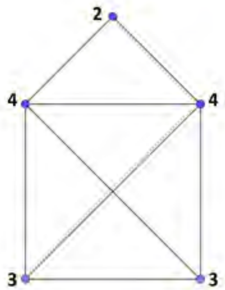


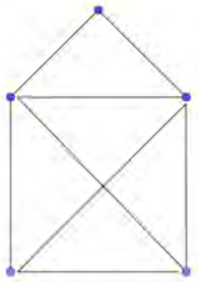




Theorem

Semi-eulerian graphs, eulerian paths





Theorem

A graph is semi-eulerian if and only if it has exactly two vertices of odd degree.

Semi-eulerian graphs, eulerian paths

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Remark:

The two vertices of odd degree are necessarily the starting and end points of every eulerian path!

Semi-eulerian graphs, eulerian paths

What about Königsberg?

Semi-eulerian graphs, eulerian paths

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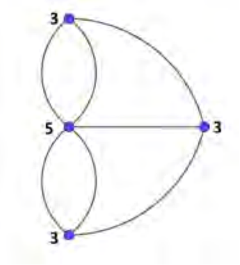

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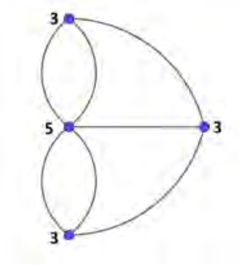
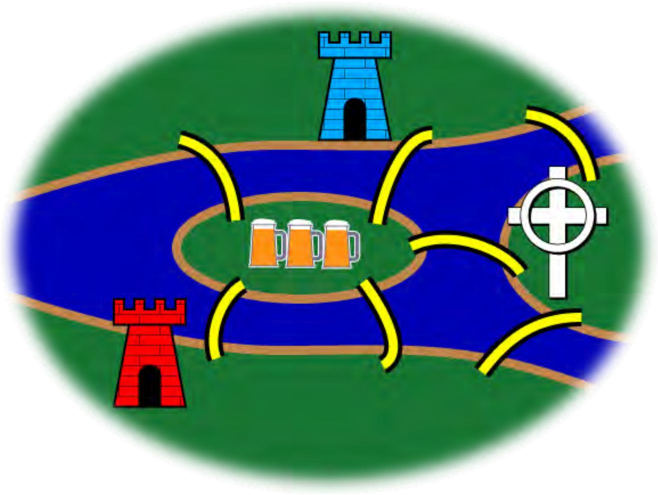


The graph of Königsberg has four vertices of odd degree, hence it is not a semi-eulerian graph and there is no eulerian path!

A variation of the Königsberg problem



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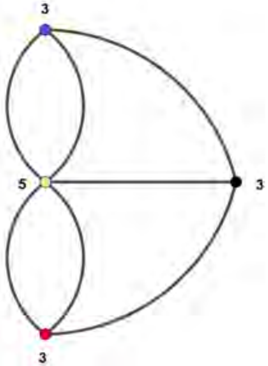

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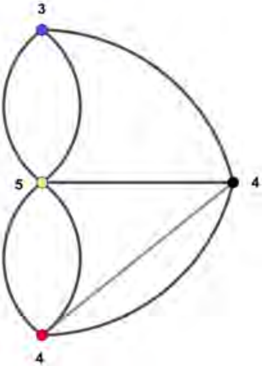

Le 8e pont

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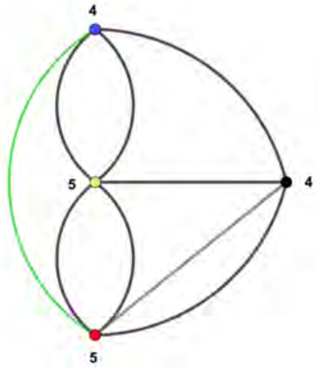

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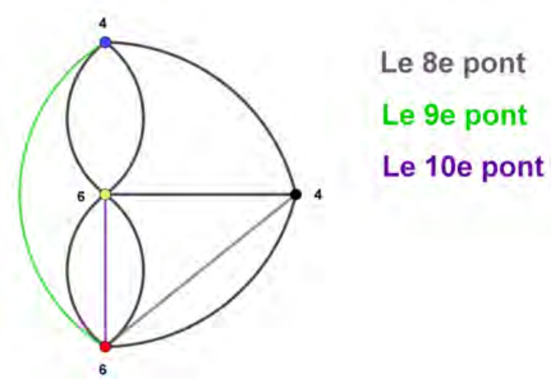

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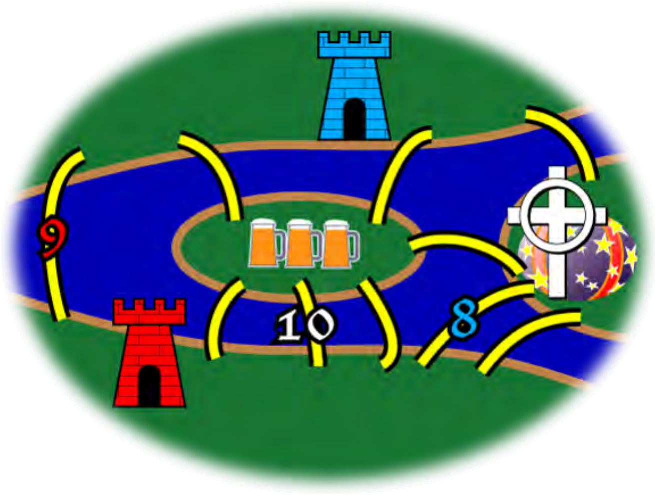

Application example: refuse collection

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Problem (1962):

Given a non-eulerian graph, find the shortest cycle that uses each edge at least once.

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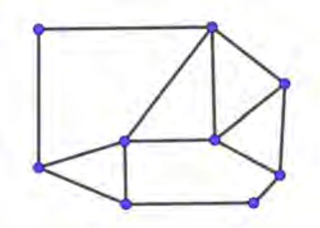

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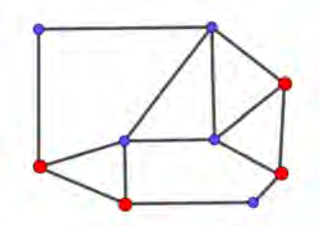

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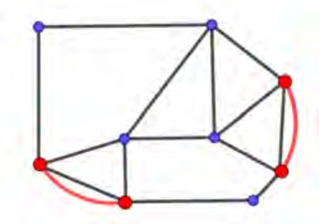

Application example: refuse collection

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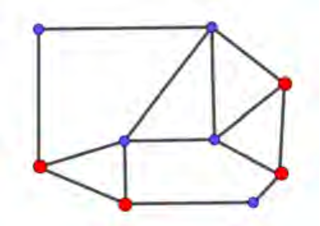

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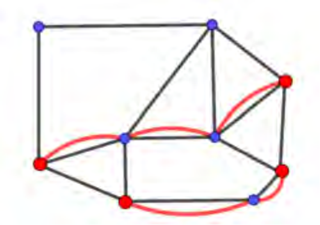

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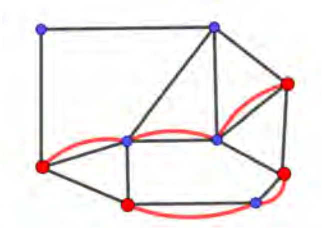

- Find a eulerian cycle in the resulting graph, this is the shortest route for the vehicle, as the total length of the streets it passes more than once is as short as possible!

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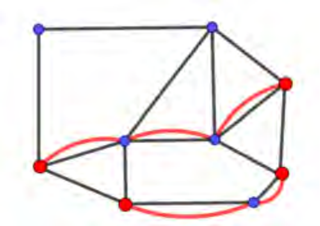

- Find a eulerian cycle in the resulting graph, this is the shortest route for the vehicle, as the total length of the streets it passes more than once is as short as possible!

Other examples: the postman ('Chinese postman problem'), route maintenance, winter road clearance,...

Picture credits

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