Standards

for
mathematics education

SLO/NVORWO

The NVORWO Commission on Standards for Mathematics Education

Frits Barth
Maarten Dolk
Henk Drayer
Willem Faes
Fred Goffree, Chair
Ronald Keijzer
Frank van Merwijk
Gert Muller
Jan Willem Oonk
Wil Oonk
Coen Schinkel
Marian Steverink

Editors: Fred Goffree & Maarten Dolk
Translation: Ruth Rainero
Copy editor: Betty Heijman

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Foreword

Standards for Mathematics Education is a translation of a portion of 'Proeve van een nationaal programma rekenen-wiskunde & didactiek op de pabo', a Dutch publication containing standards for the subject area of mathematics education at institutes for primary teacher education in the Netherlands. This handbook was developed by a group of twelve Dutch mathematics educators, with the support of the National Institute for Curriculum Development (SLO) and the Freudenthal Institute. The core of this handbook consists of eighteen standards which describe the mathematics education program at institutes for primary teacher education. The remaining chapters may be regarded as a reservoir of ideas for educators. The complete table of contents of the Dutch-language publication is printed at the end of this booklet, so that readers may find the original Dutch version of the sections that have been translated here.

The Dutch Standards differ from their American counterparts, partly due to the difference in prior circumstances and partly to the differences between Dutch and American culture. The following information on the Dutch educational system is intended to orient the reader with the educational situation in the Netherlands in general and the Dutch standards in particular.

Approximately15,000,000 people live in the Netherlands. The country contains 8,000 primary schools for children between the ages of 4 and 12. There are thirty-eight institutes for primary teacher education, which employ approximately 150 educators for the subject of mathematics education. Since 1968, all educators in mathematics education belong to a professional association. They meet one another at a number of annual conferences, and during workshops and courses. The continual development of instructional methods in mathematics education is the topic of discussion at these gatherings. Primary school teachers are educated at four-year institutes of 'higher vocational education' (as opposed to at universities). Every primary school teacher must be able to teach all subjects to students ranging in age from 4 to 12 years-old. At present, during the final year of the course, the student teachers choose a specialization, either in teaching the younger children (4 to 8 years-old) or the older children (8 to 12 years-old). The reform of primary school mathematics education in the Netherlands has a long history. Thanks to the influence of Hans Freudenthal, the (American) New Math movement did not take root in this country, and a different reform of mathematics education began to take place as far back as 1968. Mathematics curricula and sample lesson material were developed at the IOWO (now the Freudenthal Institute), which were then elaborated upon and incorporated into textbooks by teams of authors. A characteristic aspect of the implementation of realistic mathematics education in the Netherlands is the considerable influence exerted by textbooks on primary education. Six major realistic textbooks are published in The Netherlands: *De wereld in getallen*, *Rekenen & Wiskunde*, *Pluspunt*, *Rekenwerk*, *Operatoir Rekenen*' and *Naar Zelfstandig Rekenen*.

As is stated in the Standards for Mathematics Education, teacher education in the subject of mathematics is characterized by three pillars: reflection, construction and narration. These pillars were the deciding factor in the choice of sections to be included here. Sections 3 and 4 deal with reflection, and section 5 with narrative knowledge (narration) and section 9 with constructive analysis.

1 Introduction

Why, in fact, did a national plan arise for teacher education? What inspired the NVORWO[1] in 1990 to approach the National Institute for Curriculum Development (SLO) regarding a project of this nature? The reasons were twofold. On the one hand, primary schools, in which textbooks based on realistic mathematics education were increasingly being introduced, were in need of well-educated teachers who could put the implementation of these textbooks into practice. On the other hand, the numerous institutes for primary teacher education developing at that time were organized in varying ways - some quite experimentally. The intrinsic change demanded by realistic mathematics education of teachers on a national level made it imperative that a nationwide approach to education be established - with quality at all locations.

A nationwide network of primary school educators for mathematics education has existed in the Netherlands since 1968. The majority of these educators has been involved in the development of realistic mathematics education at the primary school level. From the very start, the content and design of the courses to be offered in this subject area also formed a topic of discussion and development work. The amount of time in the entire course load allocated to the subject of mathematics education received attention as well. During the nineteen-seventies, in addition to Wiskobas blocks, the theoretical aspects of teacher education and a standpoint regarding this education were also established. Stated succinctly: “the creation of learning strands between children’s subjective (informal) mathematics and objective (formal) mathematics is taught at the institutes for teacher education by using the students’ own mathematical learning processes, their reflection on these processes, and an acquired basis of didactic orientation enables the students to examine the children’s learning processes, to organize them, and thus to learn how to teach.” (Goffree, 1979, p. 313) During the nineteen-eighties, this concept was elaborated upon further and developed into a complete curriculum (Goffree, 1985, 1992, 1993, 1994). The existence of this nationwide network of primary school educators (which can now be regarded as an integral part of the more comprehensive NVORWO infrastructure comprising school consultants, researchers, assessment developers and primary school teachers) also facilitated a national consensus on the quality of the courses offered.

The acknowledgment of the necessity for a national curriculum for mathematics education at institutes for primary teacher education and the prospect of creating such a curriculum in consultation with all the educators concerned inspired the NVORWO to submit the abovementioned proposal. A development project was initiated comprising a core group of prominent primary education educators and supported by as many of their colleagues as possible (Wijdeveld, January 14, 1991).

Development work for vocational education

Curriculum development for institutes of primary teacher education has its own, specific history. Although attention to the requirements of the teaching profession has increased during the past hundred years, the dominant role has traditionally been played by the subject matter. This is not particularly surprising, considering the fact that primary school teachers must deal with a great amount of subject matter. And, of course, in order to be able to teach it, they must have mastered it themselves. An extra problem arises, however, for the subject of mathematics.

For some students (and educators), even learning primary school subject matter in this subject area requires considerable effort. At times it appears that the entire course of study

could be filled by acquisition of this basic knowledge.
During the nineteen-fifties and sixties in the Netherlands this was, in fact, not unusual. One may assume that the presence of so many subject areas at primary school[2] and the extensive amount of subject matter involved has made it impossible to raise the curriculum development for instructional theory at the institutes for teacher education above the subject matter itself. Little attention has ever been paid either to the profile of the ideal teacher or to insight into the development of his or her professional skill. In the area of mathematics education, however, a number of impulses in this direction did begin to crop up in the nineteen-seventies (see Goffree, 1992).
If one looks at how most curricula for vocational education are currently developed, one can see that, in most cases, the following steps are followed (Nijhof, Franssen, Hoeben and Wolbert, 1993): A professional profile is constructed, in which the necessary qualifications for practicing that profession are identified, and from which the attainment targets (i.e. knowledge, skills and attitudes) are derived for the course of study in question. The subject matter is then chosen, based partly on the students' own intellectual abilities at the start of the course. This subject matter is then organized along the lines of instructional theory and supplemented by potential educational activities. Finally, points of verification are introduced and tests, or guidelines for assessment, are designed.
This could be described as a rational approach. Regarded from a purely logical perspective, and viewed against the backdrop of traditional vocational education, this is indeed the way to design a vocational course of study. Upon closer examination, however, doubts may arise. A different development strategy can be seen in the manner in which problem-based learning was taken as the starting point for designing vocational education (for example, at the University of Limburg) (Schmidt, 1982). It is true that here, too, the point of departure was the study and analysis of the profession in question; the focus of investigation, however, was the core problems that would surface on the job, rather than the qualifications.

The curriculum was then designed around these core problems, and constant attention was devoted to the way the students, through working on the problems, could acquire the requisite professional skills. In the case of the course of study at the University of Limburg, the focus was not on the solutions to these problems but, rather, on the knowledge and skills that could be acquired through collaboration while searching for information with which to tackle the problems. And, of course, the goal was for students to develop a positive attitude with respect to problems and problem solving.
The rational model, described first, may indeed be suitable for fairly uncomplicated vocational profiles, that is, for professions in which instrumental knowledge is extensively used for conducting certain frequent (routine) activities. In such cases, the requisite knowledge is simple to map out and the necessary skills can easily be divided into sub-skills, each of which can fill a section of the curriculum.

Development work for institutes of teacher education

This rational model is not satisfactory, however, when it comes to educating primary school teachers. For one thing, an approach based on profile and qualifications would merely lead to an instrumentally educated teacher, i.e., one who simply implements that which is available from, for instance, the educational publishers. But it is also unsatisfactory because of the fact that, in this approach, the student teacher remains entirely invisible. In contrast both to this top-down strategy according to the rational model and to the collection of core problems according to the second model, the project group (PUIK) posed an approach to development work inspired by realistic mathematics education.

In this approach, the students themselves and their professional development are the focus of attention. The concrete points of attention are subjective ideas, the students' own productions, reflection and interaction. The profile of the ideal teacher forms a desirable perspective for both student and educator. One advantage of this approach to development work is that both the institute as a research terrain and the educator/developer's practical knowledge in the area of mathematics education can be exploited. The project group thus chose an approach to curriculum development that lies close to the students, between the ideal profile of a beginning teacher and prevailing educational practice, and supported by an explicit viewpoint on how to educate primary school teachers in the subject area of mathematics.

An educational concept: three pillars

This viewpoint on educating teachers in the domain of mathematics, which arose in the development work of the nineteen-seventies and eighties, was further elaborated upon in the Puik-project (Puik, 1992, 1993, 1994). Currently, there are three pillars - reflection, construction and narration - upon which mathematics education can be built. The principles according to which realistic mathematics education is instructed at primary school form the foundation supporting these pillars (Treffers, De Moor & Feijs, 1989). This does not mean, of course, that instruction at institutes for teacher education is conducted in the same manner as at primary school; nevertheless, it is conducted according to the same principles and in the same spirit.

The similarity between primary school education and teacher education is the clearest in the area of reflection. Just as mathematics is learned by doing, so can the professional skills of the mathematics teacher be acquired through the performance of a great number of activities. These activities occur on three levels, namely, the level of the primary school subject matter, the level of teaching activities involving this subject matter, and the level of theoretical activity in the domain of the theory on mathematics education. At each level, activity will only lead to expertise if time is taken for reflection. Reflection on an activity, moreover, is an appropriate foothold for a course of study on a theoretical level. After all, a theory arises through the reflective ability of experts in the field and researchers. Throughout the course of study, a line of development is created that runs between practical activities, reflection on these activities, observation of the practical activities of others and examination of their reflections, examination of the theory as an extension of the reflections, and reflections on the theory behind one's own and others' activities (PUIK, 1993).

Reflection is a prerequisite for learning, particularly for learning from one's own activities. The skill of reflection, therefore, should be one of the attributes of a beginning mathematics teacher.

By the time they enter higher education, students have already gained a variety of learning and teaching experiences, including in the area of mathematics. Most students, therefore, as a result of their experiences involving subject matter, teachers and fellow students, already subscribe to a certain viewpoint, both with respect to mathematics itself and to learning and teaching math. This viewpoint is usually unintentional and almost always delivered implicitly with the subject matter. Neither the educators nor the students, themselves, are particularly conscious of the presence of this viewpoint. Educators can learn a great deal from reading students' essays on their own mathematical history. On the whole, students who were weak at school in mathematics usually have more to recount than the others, although one must read between the lines to find the viewpoint.

The students' personal assimilation and coloration of what is taught at the institute for teacher education is based on these experiences and viewpoints. Partly through new experiences gained both at their internship school and in their own course work, they will construct their own expertise and corresponding viewpoint. Should one, during the training course, wish to adjust and alter the existing preconceptions of beginning students in favor of a view on mathematics that fits the realistic approach, one should not commit the error of ignoring the students' initial circumstances. This is even important during the first phase of the course, in the section on basic numeracy. In this section teacher educators aim to improve student teachers own individual mathematical strategies and approaches. If one presents this course in a purely product-oriented manner and believes that new educational approaches for primary school children will be sufficient in themselves for remedying students' deficiencies, then one is simply reinforcing preconceptions that run counter to the principles of the available practice material. The necessary adjustments of the student teachers beliefs on mathematics education will then become even more difficult.

Not only does this constructivistic approach require special attention at the start of the course of study, but the final phase, too, must be regarded with more than the usual attention. One must at least ascertain whether the education has been successful in approximating the ideal profile of the beginning teacher and, if so, that this has not been exclusively in terms of knowledge and skills. The aspects of attitude and viewpoint are, after all, at least as important. The graduating students, too, must know what kind of mathematics teacher they have become. In order to guide such development along the right paths during their four years at the institute, students must frequently be offered the opportunity to express and discuss their individual viewpoints. This will provide them with important insights, as discussions bring up values and norms that affect both the subject and the teachers.

During the introduction of kindergarten math, a third pillar, that of narration, is added to those of reflection and construction. A great deal of knowledge of education of student teachers and teachers gathered from among the children in the classroom, has been notated in the form of anecdotes. The anecdotes of the educational pioneers Thijssen and Ligthart, for instance, in which children and teachers are described in the smallest detail, are well-known in the Netherlands. Researchers such as John Holt and Hans Freudenthal added reflections to their classroom observations, at times thereby creating a link with the underlying theory. And then there are also other Dutch development researchers, such as Van den Brink (1989), Streefland (1988) and Gravemeijer (1994). Their theoretical reflections are notated not infrequently in the form of anecdotes from the classroom. In other words, some primary school anecdotes have a theoretical charge and some do not. For teacher education, such narratives can play an important role in the area located between theory and practice, an area often mistakenly described in mathematics education as a gulf. Educators collect certain anecdotes as part of their repertoire and cherish the paradigms; these are the narratives that contain considerable exemplary properties. They represent numerous observations of a given phenomenon and are of a high theoretical caliber. Moreover, these narratives reveal essential aspects of (realistic) mathematics education (such as phases, intuitive ideas, levels and level advancement in learning processes) and are easy to remember. This is known as narrative knowledge (Puik, 1992).

Standards for quality

An investigation into the quality of the courses offered for mathematics education could commence with these three pillars. The focus of such an investigation is then on what the students construct for their education and what they do with it, their ability to reflect on their own activities in actual practice, and what they, as experts in this practice, have to report.

A researcher examining the quality actually looks at what the student is learning or has learned from the courses offered; more aptly stated, the researcher observes what the student is making or has made of the education. This is an excellent method of evaluation, albeit difficult methodologically and extremely labor intensive. Those who prefer more direct and cheaper research will lower their sights and simply observe what it is the students are being offered. The researcher will in the latter case focus attention on essential aspects of the learning environment, find the salient features of the course, and observe in a fair amount of detail what is taking place. In order to attach significance to these observations, they are placed in the encompassing framework of the philosophy of mathematics education and the concept of teacher education. This, then, initiates the moment of reflection (Inspectie van het onderwijs, 1989). These standards for a national plan for mathematics education at institutes for primary teacher education provide criteria for quality and a reference framework for examining the course of study. The assumption of the project group was that this investigation would be primarily conducted by the educators themselves, for the benefit of quality control and in order to improve their own courses in mathematics education.

2 Standards for mathematics education at institutes for primary teacher education

This chapter defines eighteen standards that are intended to be viewed as signposts. They describe where an educator should look to find quality at an institute for preservice primary teacher education. By formulating these criteria, a statement is thus made regarding the interpretation of the concept 'quality of the primary teacher education'. This is a concept that has not been precisely determined up to now, nor always described with clarity. What actually determines quality? Is the quality of the institute determined by the difficulty or ease with which students receive a diploma? Or by the degree to which students collaborate with one another? Or by the amount of support students receive from their teachers in obtaining their diploma? By the number of students that complete the course within four years? By the ease with which a new primary school teacher adapts to an already existing team of teachers? Or by the degree to which the institute formulates new developments in the field?
Discussions on quality have been based on criteria that are not necessarily explicit. We use these eighteen standards to elucidate our view of the quality of mathematics education at institutes for teacher education. The standards have been provided with subtle distinctions which offer the opportunity to focus attention on interesting details.

The Standards as Spotlight

The eighteen standards can also be viewed as:

1 A spotlight to be focused on certain aspects of the educational environment.
This spotlight function can enable the educator and student to view the quality of a given area at each point in the education. When all eighteen spotlights are 'on', the entire educational process becomes brightly illuminated.
2 Reference framework for educators
The educators can use the standards as a frame of reference for examining the courses available.
3 Gauge for educators.
Educators can use the standards as a gauge for measuring the education of their own institute and as indicators for reflective observance of their own instruction. They can thereby examine whether their instruction sufficiently contributes to the development of the student's professionalism. Is the education broad enough, for instance, and are all areas being dealt with sufficiently? The standards can inspire the educator to reconsider the content of the institute's courses and the educational concept. Should the education prove at any point insufficient, the standards can then serve as indicators for designing new areas of the curriculum.
4 Hallmark for external evaluators.
Others, too, can use these standards to assess the education and the educators. For external evaluators, the standards can serve to illuminate vital areas of the curriculum.

Eighteen Standards for Mathematics & Didactics

1 The mathematics education for Pabo[3] students reveals characteristics of realistic mathematics education.
1.1 Mathematics education is characterized by a positive atmosphere of collaboration and enjoyment.
1.2 Interaction in the mathematics classes at the institute are characterized by the fol-

lowing: articulating one's thought processes, listening to the solutions and explanations of others, negotiating, convincing and being convinced.

1.3 Students are problem-oriented, involved, motivated, and take responsibility for their work.

1.4 Mathematics is carried out in recognizable situations and familiar contexts. The students therefore have a natural approach that utilizes their common sense; they take time to construct a plan and pay attention to how it is organized.

1.5 The educator is aware that the self-esteem of many female students is undeservedly low in the area of mathematics, whereby they have ceased to use their common sense.

1.6 When doing mathematics, the teacher pays special attention to those students who have low self-esteem, an inappropriate faith in rules and who experience a sense of security in indiscriminately copying the educator or fellow students.

1.7 When working on mathematics and studying instructional theory, the students devote attention to crystallizing their ideas, sketching situations, designing diagrams, making and using models. (In specific situations, students develop a model of that situation. This model can then be generalized to apply to many other situations. The character of the model of the first, specific, situation thereby changes, and becomes a general model for many situations, forming the foundation of the development of a more formal mathematical knowledge.)

1.8 In mathematics education, the students are offered the opportunity to construct situations, reflect upon them, and elucidate them.

1.9 Students acquire insight into their own repertoire of strategies and approaches to mathematics; they take into account their own strong and weak points and those of others, and are constantly involved in their own development.

1.10 Mathematics education at the institute is characterized by an accessibility to the educators' and students' available knowledge and expertise.

1.11 Help is promptly given to slow students and attention is also paid to quick students by, among other things, involving them all in the educational process.

1.12 Education in teaching the subject of arithmetic is characterized by clarity with respect to the expectations of the educator.

1.13 In mathematics education, students work both independently and under the guidance of the educator.

1.14 Mathematics education does not stand in isolation at the institute; other primary school subjects and areas of development are linked to the subject of mathematics education.

2 *The mathematical subject matter is studied on one's own level and thereby placed in mathematical educational perspective.*

2.1 By working on one's own basic numeracy during the first phase of the course, a natural approach and reflective capabilities are revealed and stimulated.

2.2 Various areas of instructional theory are often introduced through mathematical work at the students' own level or by focusing on collaboration with fellow students.

2.3 The characteristic construction of realistic learning strands (for, for instance, counting, adding and subtracting to 100, multiplication tables, long division and fractions) is also presented by using problems on the student's own level.

2.4 Devising an explanation to certain problems, whether for children or fellow students, generally begins by reflectively solving the problem on one's own level.

2.5 Designing education for specific subject matter is often initiated by one's own mathematical activities based on this same material.

3 *Students acquire insight into children's learning processes in the area of mathematics.*

- 3.1 Children's mathematical activities (whether written, verbal or on videotape) are analyzed from various perspectives.
- 3.2 Students develop activities themselves in order to acquire insight into children's learning processes.
- 3.3 The students regularly talk with individual children (in clinical interviews) about specific problems and their solutions.
- 3.4 Students study a given method (such as the method of Kwantiwijzer) introducing diagnostic interviews with children and then hold interviews in accordance with it.
- 3.5 Learning processes in the area of mathematics are a frequent topic of lectures, small group work and reading assignments.
- 3.6 How to increase the level of understanding of both children and students is a topic of mathematical educational research.
- 3.7 Children's own mathematical productions provide study material for small group work on mathematics education and also serve as illustrations of knowledge transfer.

4 *Students acquire theoretical knowledge of mathematics education through actual practice.*

- 4.1 Theoretical opinions are always illustrated with examples from actual practice.
- 4.2 Students' narratives on education (from internships) are placed in a theoretical framework.
- 4.3 Students' experiences - both in primary school instruction and in their own education - are subjected to a theoretical examination.
- 4.4 The educator chooses his narratives on education on the basis of personal experience, but also on the basis of the theoretical content.
- 4.5 Students acquire a great deal of their theoretical knowledge through paradigmatic examples taken from educational practice (narratives).
- 4.6 Students personalize the instructional theory through theoretical reflections on their own experiences and by applying the theory to actual practice.

5 *Students develop a rich repertoire of mathematics instructional theory*

- 5.1 Material available in this area from the educational publishers can be perused at the institute for preservice primary teacher education.
- 5.2 Mathematics textbooks are viewed as a source of inspiration and are studied in order to develop strong ideas on mathematics education.
- 5.3 Students are familiar with the major long-term learning strands for mathematics in terms of exploratory contexts, concept formation, introductory problems, core concepts, use of models, educational goals, stages, and potential final levels.
- 5.4 Students become familiar with promising and respected inventions for mathematics education (such as, for instance, the empty number line and they gain some experience with these inventions in the primary school classroom and are able to critically evaluate them.
- 5.5 Students have a collection of highlights from the primary school curriculum at their disposal.
- 5.6 Students view videotapes and read or hear reports of prototypical interactive lessons. At both the primary school and the institute they practice their own variations of such prototypical interactive lessons.

5.7 Certain tried and tested educational projects, form the canvas for the students' own designs.
5.8 Different ways of explaining, posing questions and guided discovery are supplied with actual examples.

6 *Students develop a broadly applicable diagnostic repertoire*
6.1 Aid to individual children are recorded as case studies.
6.2 Clinical interviews and diagnostic discussions are given the necessary attention with respect to design and depth.
6.3 Special attention is paid to the educational perspective in diagnostic interviews.
6.4 The special help available to children at the internship schools is a topic of study.
6.5 Tests and test lessons that accompany a specific textbook are used by the students to develop their own testing material.
6.6 Familiar sticking points in mathematics education are afforded both theoretical and practical attention.
6.7 Remedial textbooks, are available for further analysis.
6.8 When at all possible, students work with a remedial teacher at the primary school.
6.9 The students are included in designing a plan of treatment for a child who is falling behind.
6.10 The phenomenon of (extremely) gifted children does not escape attention.

7 *The students become familiar with the realistic mathematics textbooks now available.*
7.1 Evidence of realistic mathematics education found in the textbooks is discussed.
7.2 The most recent textbooks are compared with older ones within the framework of the development of instructional theory.
7.3 Textbooks are studied in the light of desired goals.
7.4 Successful aspects of teacher's guides to textbooks are examined in order to enrich the diagnostic repertoire and the repertoire of teaching strategies.
7.5 Textbooks and their accompanying teacher's guides are analyzed with respect to the perspective on realistic mathematics education propounded by the authors in the teacher's guide.
7.6 Small segments of certain textbooks are earmarked for constructive analysis, that is, they are used during the preparation of education for the internship school.
7.7 When possible, review of a textbook by the staff at the internship school is attended by the students or simulated at the institute.
7.8 Sticking points in the textbook are inventoried by the students in collaboration with their mentor at the internship school.
7.9 Textbooks form a rich field of investigation for the study of learning strands, explanations, concept formation, differentiation, working self-reliantly, spotting problems, assessment, etc.
7.10 Students articulate a personal evaluation of a textbook of their choice.

8 *Knowledge of pedagogy, educational and developmental psychology and general instructional theory is applied to the field of mathematics education.*
8.1 Various approaches are explored and practiced during mathematics lessons.
8.2 Use of manipulatives plays a role in realistic mathematics education against the backdrop of activity psychology.
8.3 A foundation for the five fundamental educational tenets of realistic mathematics, such as the relation between construction and accommodation, is found in general educational psychology.

8.4 Considerable attention is paid to the pedagogic relationship with the children and to the pedagogic climate in the mathematics lesson.
8.5 Realistic mathematics education for young children (K-2) is linked to other theoretical orientations, such as fundamental development and experiential learning.
8.6 In addition to the considerable attention devoted to cognitive aspects of the learning process, affective aspects also receive attention.
8.7 Familiar topics from the theoretical side of developmental psychology (such as Piaget's phases of development and criticism of phenomena such as seriation and conservation), are discussed in relation to research in the area of mathematics.
8.8 In certain circumstances, material and methods such as those of Maria Montessori, Peter Petersen, Celestine Freinet and Rudolf Steiner may provide the impulse for a comparative study.

9 Students acquire skill and take pleasure in designing education and educational materials for mathematics.

9.1 Students design their own education based on their own work on a rich variety of problems; core concepts here are inspiration and reflection.
9.2 The designing of education offers the opportunity to contemplate local theories in the domain of application.
9.3 When designing education, students begin to view existing textbooks, theories and general educational insights and skills from a new perspective.
9.4 Designing educational material is seen as one's own educational production, necessitating reflection upon what one has learned.
9.5 Teaching something one has designed oneself has the nature of an educational experiment; the focus is therefore not only on the instruction.
9.6 The student's own design process, too, deserves further consideration; a logbook or design book can play a crucial role here.
9.7 Design products of professional developers in this field are presented as examples and provide motivation.

10 Links are created with other primary school subjects and their corresponding instructional theories.

10.1 Small, clear projects are undertaken, in which the students deal with a number of subjects in relation to one another.
10.2 Links with other educational areas are sought in existing mathematics textbooks.
10.3 Mathematical activities in other subjects can be inventoried, such as, for instance, in geography, crafts and physical education.
10.4 Integrated activities are emphasized, particularly in mathematics for kindergartners.
10.5 At the end of the preservice education course, time is reserved for comparing the instructional theories of various subjects.

11 Collaboration between students is stimulated and rewarded.

11.1 Students work together to solve mathematical problems and discuss approaches and solutions.
11.2 The preparation of internships, as well as the actual instruction and reflection, often takes place in small groups.
11.3 In the small group sessions on mathematics education, students are encouraged to work on the problems together and to come to an agreement.

11.4 The students themselves are partly responsible for the progress of the group.
11.5 Preparation of final projects may include workshops and presentations, so that the students can take advantage of the expertise available in the group.
11.6 Advanced students help beginning students acquire numeracy.

12 *The institute ensures that a great variety of situations and instances are available in which students can optimally develop as individuals in the professional sphere.*
12.1 Students experiment and practice at primary schools.
12.2 Students attend staff meetings at their internship school.
12.3 Students engage in discussions with future colleagues and school consultants.
12.4 Students attend a parents' evening and, if possible, provide a contribution.
12.5 Students aid, stimulate and assess the children.
12.6 'If requested by school staff, students conduct a special study assignment; for instance, on comments made about a given mathematics textbook or on extra material for certain students.
12.7 Students are challenged to contribute their own interests, knowledge and skills and to expand these by, for example: collaborating on a mathematics project for a certain group or for a number of groups simultaneously; designing a mathematical treasure hunt for the upcoming school field trip; writing an article on mathematics for the school newspaper; setting up a work corner in the workshop center for beginning students, who will soon be starting internships in the lower grades of primary school.
12.8 Students draft an educational 'contract-with-oneself', in order to anticipate the potential of the institute's fertile environment, and in order to take optimal advantage of this environment in a personal manner and according to individual interest.

13 *Students feel included at the institute and take personal responsibility for their own development in the area of mathematics education.*
13.1 Students are regarded from the very start as future colleagues and are treated as such.
13.2 Students are stimulated to draft a 'contract-with-oneself' for their particular specialization at the end of the course.
13.3 Students' projects and own creations are regularly exhibited at the institute.
13.4 The logbook plays an important role in assimilating the education.
13.5 In their graduation projects the students reveal their ability to apply their knowledge and, on the basis of these projects, can expand their theoretical knowledge.

14 *Students develop as reflective practitioners.*
14.1 Time is regularly allotted for making reflective solutions, taking reflective notes and reflecting on the theories.
14.2 Reflection increases as the level rises. It develops from the simple notation of events to analytical commentary grounded in a theoretical background and a developing perspective.
14.3 During interactive lessons at the institute, students regularly articulate both their solution procedures and their thoughts on the activities.
14.4 Reflection on actual practice increasingly becomes a point of assessment.
14.5 Logbook notes are regarded as confidential communication between student and educator.
14.6 Reflection forms an important aspect of the educational 'contract-with-oneself'.
14.7 Students also reflect upon their own individual learning style and transport this

style to a higher level.

15 *The image of the primary school is constantly present in a variety of ways.*

15.1 Both during lectures and in small group work, primary education is regularly portrayed through narratives, student projects, textbooks and videotapes.

15.2 Time is reserved for presenting the students' essential experiences in actual teaching practice.

15.3 Experiences gained during the internship are set against the backdrop of realistic mathematics education.

15.4 Changing exhibitions in which students portray primary education are presented at the institute, for instance: a poster of children's work, a photographic essay of a lesson, a slide show with sound on diagnostic work.

15.5 Interesting events in the area of mathematics (ranging from a successful explanation to an arithmetic project involving the entire school) stimulate students to engage in their own creative productions.

15.6 Students make so-called situation analyses within the framework of educational projects, in which the image of the practice school is evoked on a number of essential points.

16 *The profile of the ideal primary school mathematics teacher functions at crucial moments of the course as a beacon and point of standardization.*

16.1 One can see in the mathematics education program how work on the professionalization of the future teacher is successively undertaken throughout the various segments of the course.

16.2 At a certain point in the course, a checklist of elements for creating an ideal mathematics teacher is presented so that the list can fulfill a functional role in the course from that point on.

16.3 When drafting an educational contract-with-oneself, the profile of both the ideal mathematics teacher and the ideal teacher of other subjects is taken into account.

16.4 Certain aspects of the profile require personal interpretation by individual students; time is allotted for the reflection necessary here.

16.5 The educator's own choice of content and design for the profile reveals the rationale of the educational program as well.

16.6 The profile of the ideal mathematics teacher provides a good foothold for collaboration with colleagues from education courses in other subjects so that, when useful, integration of the courses offered is possible.

17 *Working on the students' own numeracy is the focus of attention during the entire course.*

17.1 Conclusion of the first phase of the course with a test on mathematical skills is regarded as an important milestone on the way to professionalization.

17.2 Mathematical skills and numeracy lie on one line; mathematical skills end with the 6th grade worksheets, while numeracy is always regarded from the perspective of theory on mathematics education.

17.3 Beyond the lectures and group work, various signs presuming numeracy are also found in society at large.

17.4 Concrete problems are used when presenting the problem of innumeracy.

17.5 Students are called upon to interpret signals from society at large in a numerate manner.

18 *Students are given the opportunity to become familiar with software in the field of mathematics.*

18.1 Students evaluate educational software and also construct evaluation criteria for these programs.
18.2 When possible, the learning processes intended by certain software programs are studied by the students at the internship school.
18.3 Students investigate and develop the use of software in certain courses, thereby subjecting both the course and the software to a fundamental analysis.
18.4 Interesting software acquires a special place in the course.
18.5 Suitable mathematics software is always available for perusal at the workshop center.
18.6 The use of educational software in primary school is subject to discussion both within a general framework and within the framework of instructional theory.
18.7 Concise and meaningful software segments are used at the institute for introducing an educational topic (for instance, mental arithmetic and estimation).

3 Reflection

Reflection, as a technique in teacher education, was stimulated by Donald Schön, who introduced the term ‘reflective practitioner’ (1983). Schön pointed out that the application of theoretical knowledge takes place much less explicitly than is often assumed. The reflective practitioner - the true expert in the field - is described by Schön as someone who is both engaged in practice and, at the same time, can articulate exactly what (s)he is doing and thinking. The reflective practitioner is someone who is able to draw conclusions from his or her reflections for the benefit of the activities that follow. Schön calls this ‘reflection-in-action’ and, in the same context, speaks of ‘theory-in-action’ and ‘theory-on-action’. This can be understood to mean the theoretical reflections engaged in by educators. With professionals, this involves personally acquired and assimilated knowledge (theory) rather than theory that has been acquired on the level of reproductive knowledge.
The following section was inspired by this idea of the reflective practitioner who, in this case, is a reflective teacher.

The reflective teacher

What kind of person is a reflective teacher? How does she distinguish herself from others? To answer this, let us first try and draw a picture of the reflective teacher we have in mind. In our opinion, this is someone who is able to:

– articulate her own instruction in an expert manner, both when speaking with colleagues and when conversing with lay persons (parents, for instance);
– learn from her own practice;
– resolve problems in a personal and creative manner;
– apply the theory in a meaningful and appropriate way;
– introduce classroom experiences in the form of stories (narratives), upon which she has reflected, for the benefit of her own education.

This is no small feat. In order to arrive at such a level, an educator (or a student) must have a number of important tools at her disposal, for example:

– She must in the first place possess a language in which she can express her reflections. In other words, she must be able to describe her own experiences. This requires a journalistic level of language use; take, for instance, the oral and written narratives from the internship. She must also be able to articulate more profound experiences and thought processes; a knowledge of professional terminology is necessary for this.
– In addition, it is important that she feel challenged to think over what she hears, sees and thinks whether this originates with the children or herself. This is called ‘developing reflection’, meaning that she learns from her own history by involving the (near) future as closely as possible.

The student’s or the educator’s knowledge of the theory will become increasingly apparent in these reflections. It should be clear, however, that one can only learn and teach reflection by doing it regularly. By remaining alert to the reactions of fellow students, educators, mentors and children, a student acquires a wealth of reflection material and also learns to reflect her own thoughts and statements in those of others. In the same way, the reflective educator also has an instrument to help her continue in her own further development.

Developing reflection

The following quote is from a journal written by a beginning teacher, which was included in a report by two American researchers, Gipe and Richards (1992).

> "Guess what? I have had my eyes opened and I still want to teach. I love it. I guess sometimes we get so involved with just teaching that we don't stop to think clearly. When I think in this journal I think how much I've learned about teaching. I have a long way to go, but now I know that some teachers and some schools don't treat kids like real people. The worst part is I don't think anyone cares."

This teacher clearly shows how much she enjoys teaching. She is aware of this, but does not accept it without criticism. The above quote also reveals that she is aware of the value of reflection. She takes pedagogical and moral considerations into account when determining her standpoints. Since the introduction of a new curriculum for mathematics education, in the nineteen-eighties, the major role played by reflection in education is undisputed. Actual practice reveals, however, that reflection really takes place only on occasion. What should, in fact, occur, is that the student should be confronted by a learning environment in which she is often invited to reflect on an increasingly higher level. We shall now use a few examples to illustrate these level changes in the (concentric) development of learning to reflect.

A report by a lay person

> "I began this (mental arithmetic) lesson by writing 6 x 25 on the board. Many children raised their hands to give the answer. Anouk said she knew it right away and that it was 400. Then I let Debbie give the answer. She said:... I asked the other children if this was correct. Then I told them to do the three rows of problems on the stencil - all in their head. The children worked quietly; they also got the right answers. It was a fun lesson!"

The above report is by Thessa, a first-year student. Although you could entitle her report 'Reflections on a mental arithmetic lesson', it actually shows an extremely low level of reflection. Anyone, cognizant or not of the situation, could have written this report. In fact, it is no more than a report by a lay person.

Writing reports in response to experiences with educational material and personal preferences

When given the opportunity to think, speak and write about one's own (lack of) cleverness in solving mathematical problems, the reports appear to gain some depth. In answer to the question: 'What did you learn from Discovering Octavania', Lucienne, a classmate of Thessa's, answered:

> "It was hard for me to picture anything with the numbers. I couldn't get into the abstraction. Until I started working with the Oct-material. For instance, I couldn't do anything at first with the problem 111-33. Using the material to picture it helped. Especially the number-line really helped with the additions and subtractions. I also learned not to say 'but it's obvious' when I have to explain something to someone who doesn't get it. You can explain something by asking yourself how you figured it out yourself. But even better you can figure out what the other person is thinking exactly. You can't explain something from your own arithmetic past. That's too long ago."

Lucienne clearly shows that she is aware of the essential role played by the manipulatives in her learning process. She is able to produce a good report, possibly because she applies her personal preferences and can state pedagogical considerations.

Relationships between one's own thought processes and activities and those of children

Francoise, another first-year student, reacted as follows to the question, 'what is the connection between how young children learn arithmetic and your own work on the problems from 'Discovering Octavania'.

> “First you have to know the names of the numbers. Then you have to be able to count and so you start to see some structure. You also begin to realize that you’re not really calculating much, but that you just know a lot by heart. With children, you have to be really aware that they do have to calculate everything, just like we have to do with 6 + 7 in Discovering Octavania. Actually, we also worked by using the building blocks, I’m kind of aware of that. But I did find that what we already knew got in the way. When you convert back into your own number system it’s very confusing.”

In this critical reflection, Francoise makes the connection between her own thought processes and activities and those of primary school children. The reflection thus acquires a theoretical-pedagogical significance.

Theoretical reflections

Monique, a fourth-year student, is explaining her final project:

> “As far as my approach in general is concerned, I prefer a development oriented approach. On the one hand, because I don’t think I’m suited to a pure child oriented approach: I just can’t help asking questions, stimulating reflection, or bringing something up for discussion. On the other hand, I think it’s important for children - especially for immigrant children - to be offered something regularly. In many cases, I don’t think it’s enough just to create a rich environment.”

Monique is able to weigh the pros and cons of certain theoretical and pedagogical principles and to arrive at a personal evaluation. Making the reflective report undoubtedly inspired her to do this. She knows how to make the connection between theory and practice and also how to use one thing and another on behalf of a personal standpoint (vision).

Concluding remarks

Every educator would like the students, in the course of their education, to naturally and increasingly be able to reflect on their own thought processes and activities and those of the children. But this is not such a simple matter in daily educational practice. Most students require guidelines and stimulation in order to arrive at good quality, meaningful reactions.

4 Theoretical reflections

A theoretical reflection is more than theory alone. The word ‘reflection’ indicates the presence of a situation, which is then observed, recognized, pondered and analyzed with the help of one’s own experiences and knowledge. In a theoretical reflection, the educator demonstrates how the theory can help one understand the practice. Sometimes the educator is even able to make predictions after observing a phenomenon.

> Jorie (4;9) is crazy about marbles. She has just brought some marbles home from school and has laid eight of them on the table. She counts the marbles one by one. But they’re lying rather close together which makes it difficult for her to synchronize the counting: her finger moves a little more quickly than she counts. At first she counts seven, but then she begins to doubt this. To be quite sure, she counts the marbles once more, but this time in a special way: as she counts, she picks up the marbles one by one and moves them away. Now there is no question in her mind: there are definitely eight of them (Goffree, 1993, p.185).

The observer of Jorie’s counting (Buys, 1991) has the following comments:

> “It is not clear why she began to doubt her first result - seven. Perhaps she already had some experience with incorrect results due to unsynchronized counting. In any case, she knows how she can be sure. She organizes the synchronized counting perfectly. In this way she demonstrates how important it is to organize arithmetic work.”

Here we see an example of an observation followed by a (theoretical) reflection. Without the reflection, this observational report would still have been fascinating, but it would have gives less food for thought in terms of learning and development processes. Goffree (1993, p.181) has the following comments about observation:

> “It is inconceivable that observation be disconnected from theory. Those who have little theoretical insight will not observe much in educational practice. As a consequence, not much will come of developmental support and stimulation.”

The following example of theoretical reflection is taken from a class at an institute for primary teacher education. A group of second-year students is busy solving a problem involving tile setting (Goffree, 1992, p.72).

Groups of students are working in various ways, trying to find out how large a square table can be if the tile setter has 1250 tiles. Graph paper and material are being used, but diagrams and bare problems appear as well. The educator walks around the classroom, offering a hint now and then and taking notes on what he sees. After a quarter of an hour he calls the group together and reviews some of the solutions. With each solution he reflects briefly on what he, himself, has observed: trial and error, systematic activities or notations, help or confusion provided by the materials, how articulating the problem to one another plays a role, unusual solutions (repeated addition instead of repeated subtraction), and more. Lastly, he sums up the essential characteristics of the learning process of acquiring an algorithm.

Both examples show how theoretical reflections can be grounded in theory as well as in practice.

Making theoretical reflections is an essential element of an educator’s work. Many things are used by the educator to present knowledge, including, for instance, students’ internship experiences, primary school situations and anecdotes, textbook examples and events during a student work group. The educator uses a theoretical reflection to show how theory can illuminate practice. The educator’s specific expertise is here especially apparent and distinguishes itself from, for instance, that of a primary school counselor, whose primary experience is that of actual school practice.

The value of a theoretical reflection can mainly be found in the educator's ability to refer back and forth between theory and practice. This interaction between theory and practice adds depth to both components and can lead the student to a higher level. And, no less important: the educator's theoretical reflections provide the students with insight into their own learning and development processes and into those of the children!

Each country has its own set of (famous) examples of theoretical reflections in the literature. In the Netherlands there are for example articles by Freudenthal (1975) with his grandchild Bastiaan. In the US literature the book by John Holt (1965) 'How children fail' is such an example.

5 Presenting paradigms

Those who work in education usually have a number of anecdotes ready at hand. Often these describe someone's humorous or otherwise notable experience with a student or a colleague. The majority of these anecdotes do not pretend to be anything more than their face value. This changes, however, when they carry or evoke a message. Such anecdotes can then acquire a considerable value. Professor Hans Freudenthal was a world-renowned storyteller. Do any of us not know a few of his fine stories about his children or grandchildren? Take, for instance, the following anecdote about his grandson Bastiaan (Goffree, 1994).

> "We're sitting at the dinner table. Bastiaan opposite his younger sister, father opposite mother, grandmother opposite grandfather. Suddenly, during dessert, Bastiaan says, holding up a spoon containing six blueberries, "we're this many". "Why?", I ask. "That's how I see it", says Bastiaan, continuing: "two children, two grown-ups and two grandpa and grandma." (Perhaps the berries in the spoon were lying in the same die pattern as our seats around the table, but that I didn't see.)

This proved to be no coincidence. The next day, holding four strawberries in the palm of his hand, Bastiaan said, "This many live in our house." At that time, Bastiaan was still uncertain of how to use numbers and he absolutely refused to count, which is unusual at that age. Rather than having an understanding of numbers, in this story he displays something of a geometrical comprehension, which is perhaps normal at this age." Bastiaan (4;3) didn't count, but he recognized similarities between quantities by their geometrical structure. This is one of the many examples of the way in which Freudenthal, through an anecdote, demonstrated insights into children's learning processes. Anyone who has once heard or read such an anecdote will not soon forget either the story or its purpose. This is the strength of such stories: they are theoretically charged and they carry with them the kernel of ideas on learning theory and education. And this was precisely Freudenthal's objective with these anecdotes: to describe observations of unique learning instances that provide insight into learning processes - particularly into learning leaps or level increases.
Such characteristic 'classic examples' or paradigms can be of great import for those who are involved in education and who wish to continue to learn from their own practice. As Freudenthal (1980) states:

> "One can learn more from a single paradigmatic instance than from a hundred irrelevant ones (...) Such an opportunity should be taken advantage of."

Freudenthal himself took optimal advantage of such opportunities, especially with respect to forming a link between theory and practice. And this particular link can be formed extremely well through concise, theoretically charged anecdotes. Paradigms have a natural connection to 'narrative ways of knowing'. They recount quality observations and provide them with a reflective note and a strong theoretical charge.
These anecdotes offer lucid insight into an exceptional phenomenon. Through this 'narrative way of knowing' (Gudmundsdottir, 1991), such anecdotes are seen as an excellent way of closing the gap between theory and practice. Anecdotes can help one understand how theoretical elements can be used in practice and they can also aid one in recognizing the theoretical elements in this same practice. The paradigms described here are such anecdotes.

Four examples

David and his teacher.
David: 15's odd and $\frac{1}{2}$'s even.
Teacher: 15's odd and $\frac{1}{2}$'s even? Is it?

David: Yes!
Teacher: Why is $\frac{1}{2}$ even?
David: Because erm, $\frac{1}{4}$'s odd and $\frac{1}{2}$ must be even.
Teacher: Why is $\frac{1}{4}$ odd?
David: Because it's only 3.
Teacher: What's only 3?
David: A $\frac{1}{4}$.
Teacher: A $\frac{1}{4}$'s only three?
David: That's what I did it in my division.

Anyone reading this for the first time will probably not have the faintest idea what is going on. But once the idea of models for fractions is introduced, the 'clock model' will spring to mind. This anecdote is used in Mathematics Education classes at institutes for teacher education in the Netherlands as a way of revealing in a nutshell how children form their own concepts, in this case with respect to fractions. It shows how children will sometimes design a 'mental model' all on their own to aid them in providing numbers with meaning or solving problems. Educators who carry such anecdotes in their theoretical baggage will be able to situate and guide children's learning processes in increasing breadth and depth.
An educator will use such a paradigm with a group of student teachers and purposefully allow the tension of incomprehension to build. The dénouement will be revealed either when one of the students suddenly calls out, or when the educator draws a clock or a circle on the board.

Steven

Steven (5) had drawn a pond with a few ducks swimming in a line. The teacher said, "I see you have five ducks in your pond." Steven looked at his drawing in some confusion and replied, "That's not five, because there isn't one in the middle."

This anecdote clearly demonstrates how Steven's concept of the quantity 'five' was limited to the image of the (geometrical) structure of a die. It reveals - as did the story about Bastiaan - a significant facet of how young children develop the numerical concept. The anecdote shows how a misunderstanding can arise that may disturb further learning either temporarily or for a longer period of time. Thanks to certain examples, Steven had developed an image of the concept 'five' in which he was focusing on the wrong aspects. This anecdote, too, is often recounted. It is seen through more quickly, but evokes nonetheless a theoretical context. What does a kindergarten teacher do to develop such an image, and what can be done to prevent this? Directly linked to this example is the question of how such images arise. How does a child develop this and what can a teacher do to help?

Els

Els was an average student in arithmetic, and she could solve bare problems pretty well. One day she was presented with the following problem: 'Next door to me live a family with a father, mother and son. The son is fourteen years old. His father is four times as old as he is. How old is the father?' Els drew (fig.1):

figure 1

and she said: "that's added up twice."

And then (fig.2):

figure 2:

and she said: “and so that’s four times.”

This anecdote about Els illustrates the problems models sometimes can cause children, and shows how an ostensibly insightful model can prove much more mechanistic for some children then one would expect. (The case in question involved the intersection model, where the idea of addition lies in the background.) At the same time, it demonstrates how progressive schematization can be a natural approach. The educator can turn this problem into an educational one: what exactly are the educational implications of this anecdote? Based on this observation, what would you now decide to do and why?

Paul and Necmiye

In an interview situation, Paul and Necmiye were asked certain multiplication products. Paul was rather slow and didn’t seem to know all the products by heart. Sometimes he could be seen counting on his fingers and would later report a strategy that did not seem to correspond to his behavior, although it was a good strategy. Necmiye knew all the multiplication products she was asked. Sometimes she repeated the multiplication: “oh, yeah, that’s 6 times 3, um, that’s...” and then gave the answer. She mentioned no strategies; “I just know it”, she stated frequently. The last problem asked was 12×6. Necmiye didn’t know this one, nor was she able to figure it out. She remembered 11×6, but 12×6 proved too difficult. For Paul, on the other hand, this problem was no different from any of the others. This anecdote clearly displays the power of strategies and their advantage over mindless memorization. The educator can first play the part of the ‘Nieuwe Media’ videotape of this interview (Van Galen et al., 1989) in which Necmiye and Paul are first asked a number of products, and then Necmiye is asked the product of 12×6. This is a fine moment to stop the tape and have the students think about memorization and use of strategies. The discussion will change once the students have seen Paul solve 12×6.

What an educator can do with paradigms at an institute for teacher education

- Paradigms are illustrations of a theory. They help one better understand and situate the theory.
- Paradigms also help one situate one’s own experiences with respect to certain theoretical ideas. They clarify one’s own experience in practice.
- Paradigms reinforce theoretical ideas. Such anecdotes aid one in remembering and recalling the theory.
- Paradigms are therefore also a label for a theoretical idea. Thanks to the paradigm, we know exactly what is being discussed.
- Paradigms can also provide the occasion for a small investigation at one’s own internship

school. Do some kindergarten children here also believe that five always has to have a dot in the middle? Such investigations produce new anecdotes that can once again be used to comprehend portions of the theory.
– Paradigms can also be included on a test as a way of asking about certain parts of the theory.

A personal collection of paradigms

A fine collection of paradigms enables one to acquire a grip on and insight into one's own teaching practice and also enables one to improve its quality. All educators have such a repertoire of personal anecdotes. Although it should be noted that everyone has personal preferences, some anecdotes simply make a stronger impression than others. In addition, anecdotes will gradually become one's personal property: something that was first read or heard will be combined with one's own experience and thereby be come a new, personal, story. Each educator should possess a variegated collection of anecdotes, particularly for sections of the theory which cause more difficulty or are less directly appealing to the students. In addition to orally recounted anecdotes, the educator's repertoire may also consist of audio or video fragments of class observations, fragments from the educator's of the students' journal, or passages from a book.

The collection

It is clear that many sources can be used to build a personal repertoire. Anyone in search of fine paradigms should take a look at:
– The primary school.
– Student journals.
– Anecdotes of colleagues.
– Professional literature.
– Research literature.
– Video clips Educational television programs.

Collecting paradigms is not all that easy. After all, not every educational anecdote is necessarily a paradigm. They only become paradigmatic when the students are able without difficulty to discover the theory within the story. Good paradigms reveal instantaneously what is going on, contain little superfluous material and are easily remembered. In order for a paradigm to become a label for a section of the theory, it must also stand for such an element and be able, later, to easily evoke the theory.

6 Designing education

Since the new institutes for teacher education were established in the late nineteen-seventies, a distinction has been made between the various professional roles of a primary school teacher. The following roles are listed in the Model of an Educational Curriculum (MOLP) (SLO, 1982): children's mentor, educator, developer, innovator, and discussion partner (MOLP, p. 39). Since the appearance of the MOLP, there has been the occasional tinkering and, occasionally, the role of educator has disappeared from view. But nowhere is a teacher regarded as a professional problem solver, and for good reason. Learning through problem solving is not an unfamiliar concept in mathematics education. Problem-oriented mathematics education is held in high regard and compares favorably with so-called 'task-oriented' mathematics education. Mathematics is knowledge (and skill and disposition!) that is pre-eminently suitable for solving certain problems. Even the manner used to solve problems has a mathematical character.

The process begins with problem identification: a problem is identified or a given situation is problematized, in order for one to get a handle on it. At the same time, certain knowledge is actualized and experiences from previous problem situations are recalled. This is followed by the phase of problem analysis: an attempt is made to structure the problem and ideas arise for an initial approach. Sometimes the situation is so transparent that a plan of approach can be designed directly. With complex problems, however, that admit no algorithmic solution, a heuristic (searching) approach will have to be taken. Extremely obstinate problems also exist, of course, which defy any and all repertoires of heuristics. These are called 'wicked problems', and here a search must be undertaken for an entirely different path of approach. Available knowledge and skills are used while tackling a problem, and it is necessary at times to expand one's collected knowledge, whether through the aid of others or through one's own discoveries. Here, the learning instances during the learning process are explicitly revealed. If the focus is primarily on learning, it is critical that one or more reflective moments be included in the process.

For certain professions or areas within these professions, it has proved possible to acquire expertise through solving problems. In its original form, this was termed 'problem guided' learning (Schmidt & Bouhuijs, 1980). More recently, 'Probleem Gestuurd Onderwijs' (PGO) or 'Problem Guided Education' has become a popular term in the Netherlands. Since the late nineteen-seventies, the University of Limburg has designed its medical studies according to this approach. The students learn the profession of medical practitioner through solving (medical and related) problems. This takes place in educational study groups of approximately twelve students, who work according to a set approach to problem solving. In this approach, the primary focus is on learning, while problem solving and the problem solving process are subordinate. In PGO circles, the high motivation of these students is spoken highly of, as well as the fact that they learn to apply their knowledge to problems in actual practice and that new knowledge is acquired in direct relation to practice. This method works wonderfully in the medical world, where diagnosis is at the heart of a doctor's work; diagnosing is, after all, solving problems. A similar approach can be seen in the law department at the University of Maastricht, where the education was also designed according to the PGO model.

How does this apply to the teaching profession?

A teacher's main function in the classroom is not that of problem solver: not when preparing the lesson, not when choosing a textbook, nor when helping a slow arithmetic student. Perhaps we could classify making a diagnosis as problem solving. But a teacher's approach - which involves catering to a wide range of educational needs - does not have the characteristics of the problem solving approach described above. Catering to a wide range of educational needs (and many other activities on behalf of students) is a matter of observing, inventing questions, designing problems and accompanying tips and explanations, thinking up ideas, talking with students, encouraging the students to reflect on their own activities, etcetera. In the case of adaptive education (in which the teacher takes advantage of the differences between students), a similar situation is found - only here it is not limited to a student. Evidently, a teacher is an educational designer rather than a problem solver. This should be kept in mind if one is planning to organize (sections) of an institute for teacher education according to the PGO model. This case therefore only concerns motivation and learning applications.

The focus in this chapter is on educational design. Student teachers must learn to design education themselves, and they can learn to do so in the following ways: by observing how their own education was designed for them; by - as professional practitioners of reflection - designing education together with an expert at their institute; by becoming familiar with what professional educational designers have created; by watching education being designed at their internship school and by questioning the designers; by trying out their own creations in educational practice.

7 Rich problems as building blocks for mathematics education

Student teachers are often confronted during their education with rich mathematical problems. Such rich problems both invite a diversity of mathematical activities and reveal sections of the instructional theory. Rich problems demonstrate that mathematics can be widely applied. Student teachers are amazed by, for instance, what they can do with their own mathematical knowledge and background. Solving rich problems encourages them to work together and stimulates the development of a mathematical disposition. Rich problems also play an important role in primary education. They act as beacons in subject matter domains and provide the opportunity to visualize, schematize and model reality. They stimulate interaction and collaboration by students in primary school. They reveal to children the applicability of mathematics. The point of departure for the primary school instructional theory is created by students' experiences with rich problems. Reflection on these experiences offers students the opportunity to work with rich problems themselves with the children at the primary schools (Goffree, 1979).

Walking to Marseille

On a map of Europe (scale: 1: 15,000,000), the distance from Amsterdam to Marseille is 7 cm. How many kilometers is this distance in reality? How can you turn this into a rich mathematical problem? I discuss this matter of instructional theory with my students. They do not need much time to think my questions over. Almost immediately, one cries out, "you have to situate it within the children's experiences." Another continues, "a vacation in Marseille." I write on the board: 'you're going on vacation to Marseille'. We're on our way. I ask the students whether they can think of any more ways to enrich this word problem. One of them suggests the children's own contributions: have them look up the distance between Amsterdam and Marseille in an atlas or on a road map. Another student follows a different train of thought: "you can have the children figure out how long it will take you to get there, or how much gas you'll need."
My response to these suggestions is that, although this would indeed enrich the situation, we were actually searching for an enrichment of the problem that would maintain the original mathematical situation - which was about distance. Fred has a bright idea: "what if I tell the children that I'm going to hike to Marseille? I have three weeks vacation. Will I make it?" Not all the students are directly aware of the beauty of Fred's suggestion. I ask Fred to explain it in more detail. Fred knows exactly what he means, and replies, "this way you get the children to concentrate on the distance between Amsterdam and Marseille." I had written a list of characteristics of a good text problem on the back of the board, as that was what we were busy doing.

What characterizes 'rich problems'?

- solving rich problems generally leads to mathematical activities;
- different approaches are permitted for solving rich problems;
- rich problems are formalized in recognizable situations, which are often taken from everyday life;
- rich problems are by nature open-ended;
- rich problems often share common ground with other disciplines;
- rich problems appear in a great variety of forms: puzzle, brief text, story, newspaper clipping, et cetera.

Why are rich problems so important?

- rich problems offer various possibilities for mathematization and didactization (Freudenthal, 1991);
- rich problems invite the students to enter into mathematics and its mathematics education;
- rich problems provide opportunities to cross borders, both in a mathematical and a theoretical sense;
- rich problems provide opportunities for visualization, modeling and schematization;
- rich problems expose the applicability of the mathematics;
- solving rich problems contributes to the development of a mathematical disposition;
- rich problems encourage collaboration and interaction;
- rich problems reveal the connection between different subject matter domains.

How can rich problems be introduced?

- rich problems can be introduced at the institute as a way of accessing subject matter components;
- rich problems can provide a foothold for long-term learning processes;
- rich problems can play a role in acquiring numeracy
- rich problems can be introduced to help students learn to reflect on their mathematical disposition;
- rich problems are excellent examples for use during internships;
- rich problems can be used to elicit reflection on important theoretical principles.

Getting down to work at the institute for teacher education

Rich problems play a crucial role throughout the course of study. Helping the students to develop a feeling for rich problems can serve as a significant aid in structuring the education. Students must develop a good eye for rich mathematical problems. And they - with their knowledge of so many subjects - may have the advantage here. It is the educator's responsibility to situate rich problems in a theoretical perspective. The following are a number of points for attention:

- colleagues with a background in a different discipline can be a source for rich problems. Take, for instance, biology, health and hygiene, geography, visual arts, physical education, history, music, and languages;
- special students can be a source of rich problems, thanks to their specific backgrounds;
- students can be sent to search for and investigate rich problems;
- the educator in mathematics education has a broad range of interests; (s)he reads newspaper and magazine articles to develop a good eye for rich problems;
- now and again, an educator seizes a discussion taken from the internship practice for educational enrichment.

8 Constructive analysis

Constructive analysis is, in the first place, analysis. It entails analyzing subject matter for primary school mathematics, which is subject matter that has been more or less theoretically processed. Why should primary school teachers need to analyze this? On the whole, they do not; perusal of new materials on the educational market often remains limited to superficial browsing, which leaves one with a subjectively tinted impression. It is quite a different matter, however, if one is planning to acquire new material for actual use in the classroom. In that case, the material is examined more meticulously, and an attempt is perhaps also made to imagine how it could be applied and how certain children would respond to it. Then, the material is analyzed from the perspective of 'to buy or not to buy'. The situation shifts yet again if the teacher is being obliged to use material in the classroom that is new (to him or her). Substitute teachers are often faced with this, as are teachers who are teaching a given grade for the first time. But even a teacher's decision to take interesting worksheets or suggestions for lessons from a familiar source requires more than a superficial perusal of the material. If the situation described here occurs with a mathematics textbook, then the teacher's guide will generally relieve the teacher of much of the preparatory work. At the same time, however, a (detailed) teacher's guide takes personal contributions and emphasis out of the teacher's hands. At the very least, this creates a situation in which a teacher, in a docile and subservient fashion, finds herself encouraging the students to take initiative, perform activities and engage in reflection. Is a contradiction in terms perhaps concealed here?

We now come to constructive analysis. The adjective 'constructive' indicates the teacher's active and reflective contribution to the analysis of material for classroom use. This teacher will both take the initiative and make grateful use of the ideas and elaborations of others. Regarded from this perspective, constructive analysis is a form of educational design and is part of one of the lines of development that lead to professionalization. Where primary school mathematics is concerned, constructive analysis forms a significant component of teacher education. If mathematics education is to remain 'realistic', one cannot restrict oneself year after year to what was included (long ago) in the textbooks.

Aerial view of a village: an example

We now turn to page 174 of the third edition of *Wiskunde & Didactiek*[4] (Goffree, 1989), (fig.3). This page shows a worksheet with an illustration of an aerial view of a village. The accompanying text states: 'The teacher's instructions for this worksheet are as follows: this is a village, seen from above. In fact, it is a kind of aerial view of a model made out of matchboxes.' The investigation concentrates on this village. The question is whether it can determined with any certainty how many people live here. This question may be posed indirectly by, for instance, asking the size of the primary school that is just visible behind the large church tower. The children's learning activities involve such things as systematic counting, the concept of 'average', family size, the (intuitive) idea of random sampling, population composition, and simple operations. The circumstances are important: only approximate calculation is possible here and more than one answer may well be correct, each supported by sound reasoning. The objective of this bit of education is not, therefore, to find one (unequivocal) answer. It goes farther: the objective is to find investigative activities, within which the above-mentioned concepts acquire real meaning, and through which students can gain the opportunity to sustain their own convincing arguments and also appreciate those of others.

figure 3

Furthermore, the students are guided through a kind of thought experiment, in which problems are spotted, formulated and solved, and brought to a conclusion. In the meantime, attention is devoted now and again to what primary school children might be able to do with this problem situation.

Steps

Wiskunde & Didactiek recommends this problem situation as a good starting point for a mathematics education project. The following steps are suggested in undertaking such a project:

1. First solve the problem yourself.
2. Pose yourself questions and find the answers.
3. Describe reflectively your own solution process.
4. Where possible, look across the borders of the mathematical subject matter.
5. Adopt a theoretical disposition and try to imagine how your students would react.
6. Important: how do you intend to introduce this problem situation? Make up introductory problems.
7. Think about which question you will pose first.
8. How do you think the children will respond to this?
9. Think up essential questions you can use to follow up; these provide structure to the education.
10. Difficult moments will certainly occur, which require explanation or some organization. How will you deal with this?
11. Have you reserved something special for particular sections?
12. The children must also be allowed time to think things over (reflective moments). Et cetera.

Student difficulties

Experiences with constructive analysis have shown that students do not find this easy. Their difficulties are perhaps a signal to the educator that this topic should be presented in sections.

- Students have difficulty choosing a topic on their own.
- Students usually think too quickly in terms of subject matter and educational theory, whereas they should first spend time becoming acquainted with the possibilities and the difficulties of the topic.
- Often only afterwards do students see the point of beginning on one's own level and solving all the problems and assignments oneself.
- Students must gradually acquire a sense of which material lends itself to the approach of constructive analysis.
- Students must regard this manner of lesson preparation as a component in their entire teacher education, and understand that the development of increasing independence (= self-sufficiency) is included in this.
- Particularly difficult - but proven to be extremely motivating - is the search for material in one's own environment (hobby, sport, part-time job, parents' occupation, et cetera).

Appropriate assignments

A search is therefore made for assignments which allow for constructive analysis. The earlier example of the aerial view of the village belongs to the category of 'rich problems', of which every educator has a favorite 'top ten'. For example for Dutch educators: the surface area of the Netherlands, Van Gogh, a billion seconds, the towers of Hanoi, grains of corn on a chess board, 18,000 babies, Szygmund, on the road to Paris, Egbert the giant, the Hans-problem. Less open-ended than these, and placing fewer demands on individual creativity and contributions are, for instance: a student worksheet, a project taken from the primary school textbook *De wereld in getallen*[5], a geometrical (visual) problem taken from the primary school textbook *Rekenen & Wiskunde*[5] or from *Rekenwerk*[5]. Students who use a journal well, will now and then note constructive analyses of bits of subject matter that come their way, for instance, while browsing through a mathematics textbook, reading the newspaper, or watching a children's television program.

Notes

1 Dutch acronym for National Association for the Development of Mathematics Education.
2 The following subjects always appear in the primary school curriculum in the Netherlands, if possible in an integrated form:
 – sensory and physical education
 – Dutch
 – arithmetic and mathematics
 – English
 – a number of factual subjects, including geography, history, science (biology), social structures (including civics) and religious movements
 – creative activities, including the use of language, drawing, music, handicrafts, play and movement
 – self-reliance, i.e. social and life skills, including road safety
 – health instruction

 Schools in the province of Friesland must also teach Frisian and may conduct some lessons in that language. In the case of children with a non-Dutch background, some lessons may likewise be conducted in their own native language.
 English is taught to the top two classes in primary schools.
 Curriculum content and teaching methods are not prescribed.
 There is, however, a National Institute for Curriculum Development (SLO) with responsibility for developing curricula and models or alternative models for school work and sections of work plans. The schools can make use of these if they wish.

3 Dutch institute for primary teacher education.
4 *Wiskunde & Didactiek* is a text book used at Dutch primary school teacher education institutes.
5 *De wereld in getallen, Rekenen & Wiskunde* and *Rekenwerk* are primary school text book series.

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Appendix

Table of contents of the Dutch publication *Proeve van een nationaal programma rekenen-wiskunde & didactiek op de pabo.*

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NVORWO, c/o Tiberdreef 4, 3561 GG, the Netherlands