mathematics insight and meaning

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STELLINGEN

behorend bij
het proefschrift

Mathematics, Insight and Meaning

1. De constatering van Griffiths dat het toetsen van hogere denkvaardigheden verwaarloosd wordt en de mening van Rosen dat het toetsen van hogere vaardigheden vrijwel geen aandacht krijgt, dienen mede gezien te worden in het licht van de ervaringen met de examens wiskunde A.
   (Zie: Griffiths, H.B.: *Mathematics Education Among Other Contexts for Mathematics,* in: Steiner, H.G.: *The Education of Mathematics Teachers,* I.D.M., Bielefeld, 1979, p. 8-43.
   Rosen, L.: *Conference at UCLA on the Effects of Standardized Tests on Mathematics Education,* in Focus 6,5, 1986, pp. 1-7.)

2. De uitspraak van De Groot
   "Wat betreft de dingen die men niet op grond van ervaring zou verwachten staat één bevinding voorop: Zodra men beoordelaars (van toetsen) onafhankelijk laat werken, *valt de mate van overeenstemming altijd tegen!* Wat dat betreft zijn de ervaringen die men opdoet bij eigen onderzoek onvervangbaar"
   is voor wat het gecursiveerde betreft onjuist en overigens onbegrijpelijk.
   (Zie: Groot, A.D. de: *Vijven en Zessen,* Tjeenk Willink, Groningen, 1974, p. 125.)

3. De conclusie van het onderzoek van Van der Werf en Korf dat het Hewet-curriculum niet leidt tot een positievere attitude ten opzichte van wiskunde, is onvolledig en niet verantwoord uit methodologisch oogpunt.
   (Zie: Werf, M.P.C. van de en K. Korf: *Meisjes en Wiskunde; het Hewet-programma,* R.I.O.N., Groningen, 1985).

4. De bewering van Van Hiele dat de invoering van de 'moderne wiskunde' van didactisch standpunt moet worden toegejuicht en dat niet zelden gelijke resultaten worden bereikt met slechts de helft van de inspanningen, is grotendeels door de feiten achterhaald.
   (Zie: Hiele, P.M. van: *Begrip en Inzicht,* Muusses, Purmerend, 1973, p. 59-64.)

5. Het beleidsstreven naar objectief scoorbare toetsen staat haaks op het streven naar wiskunde voor allen.

6. De binnen de kring van de American Mathematical Society levende gedachte om, ter verbetering van het wiskunde-onderwijs, research-wiskundigen gedurende enige tijd in te schakelen bij het onderwijs op secondair niveau, zou voor die wiskundigen van veel nut kunnen zijn, maar niet voor leerlingen.
   (Zie: Jackson, A.: *An Interview with John Polking,* in: Notices of the A.M.S. 257, 1987, pp. 731-736.)

7. De veronderstelling van Krammer dat in de opvatting van leraren de werkvormen gesprek en zelfwerkzaamheid verbonden zijn met verschillende elkaar uitsluitende onderwijsstijlen is in z'n algemeenheid onjuist.
   (Zie: Krammer, H.P.M.: *Leerboek en Leraar,* S.V.O. reeks no. 82, Harlingen, 1984, pp. 195.)

8. Het feit dat er in de U.S.A. veel geld wordt besteed aan onderzoek van het wiskunde-onderwijs terwijl het niveau van het onderwijs zeer laag is (zoals o.a. blijkt uit het IEA-onderzoek), dient mede gezien te worden in het licht van de verzuchting van Kilpatrick waarom docenten het research-blad J.R.M.E. niet willen lezen.
   (Zie: Kilpatrick, J.: *Editorial,* J.R.M.E., 18.2 1987, pp. 82.)

9. In het algemeen, en meer in het bijzonder in dit geval, zou de laatste stelling bij proefschriften beter kunnen worden weggelaten.

10.

J. de Lange Jzn — Leiden, 14 oktober 1987.

Mathematics, Insight and Meaning

MATHEMATICS, INSIGHT AND MEANING

Teaching, Learning and Testing of Mathematics for the
Life and Social Sciences

Wiskunde, Inzicht en Betekenis
Leren, onderwijzen en toetsen van wiskunde voor de
Levens- en sociale wetenschappen
(met een samenvatting in het Nederlands)

Mathématiques, Compréhension et Pertinence
Enseigner, apprendre et tester la mathématique
pour les sciences humaines et sociales
(avec un résumé en français)

Matemáticas, Comprensión y Pertinencia
Enseñanza, estudio, prueba di matemáticas
para las ciencias humanes y sociales
(con un resumen en español)

Jan de Lange Jzn

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PREFACE

Mathematics education in the Netherlands is in a state of transition.
R. de Jong has studied this transition for primary education in his publication 'Wiskobas in textbooks' (1986). For secondary education a significant change took place with the introduction of new curricula for upper secondary education in 1985.

This publication can be seen as the end of an interesting period: the Hewet project. This project started in 1981, but years before prepatory activities had taken place.
The Hewet project was team-work. This team consisted for most of the period of M. Kindt, H. Verhage, E. Hanepen and myself. So this publication is the result of team-work, and in my opinion it was an excellent team.

During the project the idea for the present study arose. Guidance and stimulating ideas were offered by my promotor and co-promotor: F. van der Blij and A. Treffers.

A number of colleagues has offered their co-operation in preparing the final text. I'm very grateful to H. van Dongen, H. Freudenthal and S. Kemme whose comments were always to the point.

Most important in this study is the role and co-operation of teachers and students. The students of the Haarlem and Zevenaar school, the teachers of the two, ten and forty schools, they all contributed to this publication.

R. Rainero corrected my English, with valuable comments of H. Freudenthal.
E. Hanepen made this book as it is - also for her the final Hewet product.
I would like to thank her for all the organizational aspects of the Hewet project, and the beautiful booklets that were produced for the students.

Finally I would like to thank all colleagues from OW & OC for being fine colleagues and friends.

The Hewet project is over - I hope that all people mentioned above and others interested in math education find this publication a valuable one.

0 INTRODUCTION

In August 1985 a new curriculum was introduced in the Netherlands at upper secondary level: Mathematics A.
This curriculum was aiming at students that were preparing for a study at university in psychological, social and economical sciences. A curriculum that was considered by many as a revolution because it broke away from traditions in math instruction.

This study mainly deals with several aspects of Mathematics A.
Two main focussing points are distinguishable:

- What is the didactical methodology of the Math A curriculum? (The *teaching* and *learning* of Math A).
- How can we design proper achievement tests for the Math A curriculum? (*Testing* of Math A).

To place these two questions in a proper context we also pay attention to the mathematical *contents* of the program, the *project* that resulted in the new curriculum, the *experiences* of teachers and students during the experiments and a *framework* for a instruction theory. Furthermore we will find interesting differences in achievement testing between boys and girls.

The Mathematics A curriculum was developed between 1981 and 1985 as the main result of the *Hewet project*.
The *first* chapter describes the events leading to this project, taking as a starting point the Royaumont conference in 1959. Furthermore the basic scheme of this project is given, which seems interesting from the point of view of innovation.

The *second* chapter deals with *contents, methods* and *form* of Mathematics A, and gives a sketch of *a theoretical framework* for instruction as well.
To introduce the reader as soon as possible to the proper atmosphere the content description has been illustrated with examples from the experimental student material.

After the content-description an analysis of Mathematics A is given. That means: an analysis not only based on the experimental material, but also on reflection on years of classroom observations, discussions with students, teachers and teacher trainers.

This means an evaluation from the inside: the analysis is carried out by one of the developers of the Mathematics A curriculum. One of the goals of Mathematics A that is very important and present at all times is the ability to mathematize.
A general discussion about *Mathematization* is followed by a more specific one: Mathematization in Mathematics A.

Two ways of mathematization are being distinguished:
a. Mathematization as part of the problem solving process, which we might call: *Applied* Mathematization.
b. Mathematization as a way to introduce mathematical concepts, which will be referred to as *Conceptual* Mathematization.

The latter has a strong three-dimensional goal aspect: it is a didactical component of the learning process that contributes to a clarification of the process and product goals. (One dimension goal: general goals, two dimensions goal: behaviour and content component.) [1]
On mathematization we finally discuss some aspects of a theoretical framework for Mathematics A instruction.

A major role in the experimental student material is played by the *context*. After mathematization this is the second point for discussion in our analysis of Mathematics A.

The third and last point that will be treated in this chapter is clearly a one-dimensional goal: how to create a critical attitude?

Examples of mathematical material and work of students and teachers enable the author to support his views and the reader to acquaint himself with the kinds of experience of mathematics education in upper secondary school that were propagated by the team that carried out the project.

The way how students and teachers perceive Mathematics A will be discussed; comparing their views with our description gives remarkable results.

Mathematics A is clearly a curriculum and methodology that fits in directly with Realistic Mathematics Education. The shift from Mechanistic-Structuralistic Mathematics Education to Realistic Mathematics Education that takes place in the Netherlands is described and followed by an example that shows the sharp distinction between the two philosophies.

Emphasis on applications in Mathematics A makes it worthwhile to spend time on the continuing discussion about the dichotomy between pure and applied mathematics. This is taken care of in part 4 where we also discuss the sharp break-away from digital or analytical mathematics towards analog mathematics. Maybe another false dichotomy?

The *third* chapter gives the result of developmental research carried out among students and teachers of the experimenting schools.
Not only is a chapter on experiences a necessity when describing a project like the Hewet project but it also gives the opportunity for two interesting observations that are strongly related to other parts of this study.
A general observation is that most teachers that took part in the experiments experience the new Mathematics A curriculum as positive. And there is a noteworthy positive shift in their attitude. Both students and teachers gave in their comments considerable support for our views on Mathematics A as expressed in the previous chapter.
Finally teachers made clear that one of the problems with the Mathematics A curriculum lies in its achievement testing.

The problems mentioned and experiences by teachers when preparing tests for Mathematics A lead to the next phase of our study. In the *fourth* chapter we describe how teacher-made tests were collected and analyzed. This lead to the conclusion that timed tests were not operationalizing the goals of Mathematics A in a satisfactory way.

The analysis of Mathematics A, the results of experiences by teachers and students and the outcome of the research on the timed-tests all converge to the same conclusion: an exploratory study of the possibilities of achievement testing in an alternative way was needed. This study is described in next chapter.

In the *fifth* chapter the principles behind these tasks are first stated and discussed.
These principles are followed by a description of four kinds of alternative tasks:
- the two-stage task;
- the take-home task;
- the essay task;
- the oral task.

In the discussion of the two-stage task we also pay considerable attention to

one of the weak spots of such tasks: the (im)possibility of objective scoring.
The question to be discussed here is: Do the benefits of the alternative tasks outweigh the lack of objective scores?
A question of a quite different order: What can be said of the differences between boys and girls when measuring achievement?

Again examples of tasks and student material are included to give the reader the opportunity to experience the quality of the tasks and to verify statements made by teachers, students and the author.

The *last* chapter offers a summary of our results. First we discuss the shortcomings of the Hewet-project as such.
Secondly we compare the contents and methods of Math A to its achievement testing, resulting in recommandations for the future, in order to let Math A survive in its intended form, and do justice to students and teachers as well.

I THE HEWET PROJECT

1 FROM ROYAUMONT TO HEWET

"Each country will have its own way of making the reform of introducing new material, of organizing the sequential study and of experimenting with possible programs.
Channels should be provided for communicating the results of these programs and experiments between all our countries, so as to enable us to use the best thinking of all countries in stimulating new ideas.
The aim of all these programs is twofold:
Firstly, to provide a better preparation for university study;
secondly, to give to all pupils an instrument for use in daily life."

The above is the final recommendation of the Royaumont Seminar of 1959 held by the Organization for European Economic Cooperation. [1]
These recommendations had widespread effect all over Europe, in many cases leading to completely new curricula.
In the Netherlands one of its effects was the appointment of a Commission on the Modernization of the Mathematics Curriculum (C.M.L.W.); by resolution of the 12th of June 1961.
This Commission was assigned the task to study the modernization of mathematical instruction and, more in particular, to report on:

1. The subjects which should be tested in experimental classes under guidance of certain institutions, and the modifications implied by these experiments in the programs and examinations.
2. The measures that should enable teachers of secondary schools to be better informed about recent developments in mathematics.
3. The problem of the program for those children who show extra-ordinary gifts for mathematics.

Five years later the C.M.L.W. published a report with recommendations for mathematics curricula, to be introduced in 1968. The whole structure of secondary education was to undergo a drastic change in 1968, and it was felt that the introduction of the new mathematics curriculum should coincide with the introduction of the new structure.

The Dutch Secondary School system is based on six (or seven) years primary school. At that time lower secondary education covered three to four years and upper secondary education two years.

When the Commission started its activities, upper secondary education in the Netherlands was divided (at most schools) into two streams: an A-stream preparing for university studies in the humanities and social sciences and a B-stream preparing for natural sciences, medicine and technology.
For the B-stream the Commission proposed a compulsory program: Mathematics I (Mathematics One).
Math I consisted of calculus, probability and statistics. Furthermore there was an enrichment program, Math II (vector algebra).
For the A-stream a three hours per week course in the prefinal year was considered. It should cover subjects relevant for use in society.

It is noteworthy that both the Royaumont Conference and the Commission mentioned 'relevant to society' and 'everyday life mathematics'.
Afterwards, however, the C.M.L.W. withdrew their proposals for the A-stream, mainly because there was no room for three different mathematics curricula in upper secondary education - a decision that had far-reaching effects in the decade to come.

Mathematics I became very popular, not only with B-students, for whom it was compulsory, but also with A-students. In fact, the universities insisted that students of social sciences and economics needed Mathematics I in their continued study. So Mathematics I became a means of selection.
Thus many A-stream students were obliged to choose Mathematics I, which caused big problems for both students and teachers. Indeed, Math I had not been designed for A-students: it was too theoretical, it lacked relevance and - even in statistics - it had no proper applications.

In April 1973 the C.M.L.W. installed a 'Subcommission for Special Subjects', later to be called the 'Upper Secondary Commission'.
In fact, in July 1973 it had been decided that the first task of this Subcommission should be to study the problem just mentioned. In December 1973 the Subcommission published its first report. [2]
After four years experiences with Math I and Math II they concluded that "rather far-reaching changes in structure and content were inevitable."
The task of carrying out these changes was considered very urgent.
The changes proposed were:
- Abolishing Math II.
- Creating a curriculum Mathematics B in the way that 'Probability and Statistics' is replaced by 'Solid Geometry'.
- Creating a brand-new curriculum Mathematics A intended for students who will continue in social sciences and economics.

It was proposed that in the new Mathematics A curriculum the applications should be interwoven with the mathematical contents. Little, if any, attention should be paid to 'concepts of mathematical structures'. With this statement the Subcommission gave short shrift to the rather 'Bourbakist' recommendations of the Royaumont Seminar.

In the same month when the Subcommission's report was published, another important event took place: a 'Working Group Mathematics I - Social Sciences' was installed. This Working Group had the task of tackling the problems that had arisen when the 'Academic Council' had advised making Mathematics I compulsory for those wishing to study social sciences; and of what to do with students with hardly any mathematical background who were nevertheless eligible to enter a social sciences study (via a social sciences polytechnic).
This Working Group published their first report in March 1974. [3] It contained, in the first place, recommendations for universities on how to solve the problem of deficient students.
Furthermore, the report stated that these recommendations were only to bridge a present gap, and that more structural changes had to follow.
A second report appeared in June 1975. [4]
It starts with the conclusion that a new mathematics curriculum was needed. This curriculum should be appropriate for those who use mathematics as a tool. It should emphasize the role of applications. And, in line with the Subcommission's report, the Working Group emphasized that a less isolated position of mathematics would certainly do no harm to the interest in mathematics of those students who are not inclined to formal and abstract mathematics.
Just before this report, the Subcommission published a second report in the Journal of the Dutch Association of Teachers of Mathematics (Euclides), which actually was based on the work of the Working Group. [5]
This report did not differ considerably from the first report and the reports of the Working Group.

While the Ministry of Education did not react to the reports, other activities are worth mentioning.
Because of the large amount of work to be carried out, the C.M.L.W. had been in need of an Institute. As early as 1971 this Institute for the Development of Mathematics Education (I.O.W.O.) had been created, under the directorship of Professor Freudenthal.
This I.O.W.O. reacted positively to the reports by preparing a plan to develop new Mathematics A and -B programs, as desired.

Dating from the end of 1975, it proposed experiments at two schools concerning the Mathematics A curriculum to start in August 1977 and to be followed by experiments on the Mathematics B curriculum at the same two schools starting in August 1979 [6].
After two years, the first round with two schools only was to be followed by a second group of more schools from August 1980 onwards and an even larger group from 1981 onwards.

Another activity carried out by I.O.W.O. was a survey (among the 450 schools with an upper secondary department) concerning the problems the schools encountered with the 'A-students' who had to choose Mathematics I. About 50% of the schools responded and more than one hundred had taken measures to counter the problem. In most cases extra lessons were given to the A-students. Developments such as the C.M.L.W. had in mind in 1966 - a special Mathematics A program - could be reported. Among them we may mention a project in Amsterdam where a contract was signed between a number of Amsterdam high schools and Amsterdam University. This contract involved an agreement to allow students to enter the Social Studies Department at the University provided they had followed a special Mathematics A program in high school. [7]

A third activity worth mentioning carried out by I.O.W.O. is the development of student material intended for use in the 10th grade[1] (secondary school; 16 years olds) which brought into practice what in the philosophy of I.O.W.O. will later be described as the 'realistic' approach. (See Chapter II.3.1). This activity preluded on I.O.W.O.'s proposals for experiments leading to Mathematics A and Mathematics B. [8]

In May 1978 the Ministry of Education itself installed a 'Working Group on Reshuffling Mathematics I and II', for short called the HEWET Commission after their initials in Dutch (*H*erverkaveling *E*indexamenprogramma's *W*iskunde *E*en en *T*wee).
Its task was:
- To study which mathematics subjects were requested by the respective disciplines at the Universities.
- To study the consequences of the above for the mathematics final-examination program.
- To study the consequences for the H.A.V.O.[2] program.
- To advise on the desirability of in-service teacher training.
- To advise on the year in which the new programs should be introduced.

The letter of installment did not refer in any way to the earlier mentioned reports, nor did it mention the possibility or desirability of experiments.
The Hewet Commission published its first report in February 1979. [9]
This report was mailed to all schools and the 'Academic Council', as well as to other interested parties.
The report consisted of:

- A plea for the introduction of two new curricula: Mathematics A and Mathematics B, along the line of the recommendations of the report of the Subcommission.
- A list of subjects for Mathematics A and Mathematics B:

  Mathematics A:
  - applied linear algebra;
  - applied calculus;
  - probability and statistics;
  - automatic data processing.

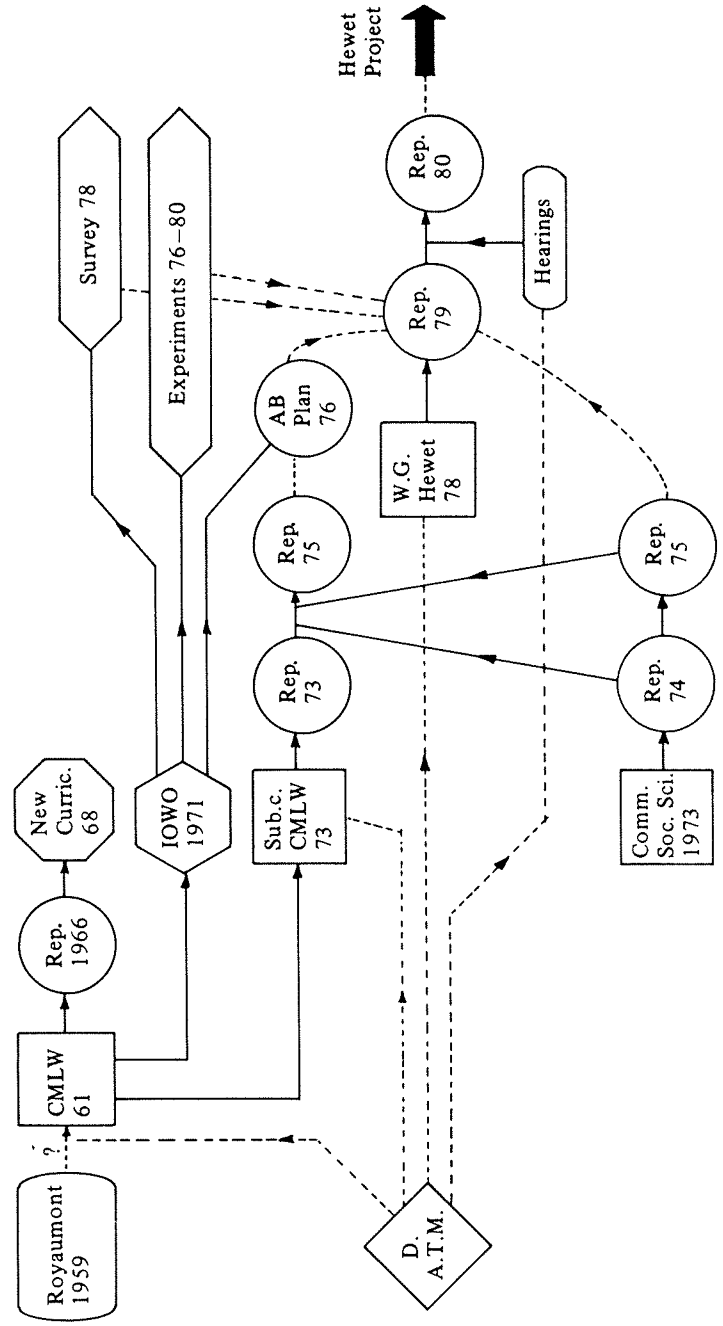
  Mathematics B:
  - calculus;
  - solid geometry.

- A detailed sketch of the both new programs.
- A plan for experiments at *three*, *ten* and *forty* schools.
- A plan for in-service training.
- A proposal relating Mathematics A and B to university studies for which they would be required.
- A proposal for an adjustment of the curriculum for the years preceding the Math A and Math B programs.

This first report gave rise to some 120 written reactions - from schools, teachers, Academic Council, and many others.
The vast majority of these reactions (more than 90%) was in general favourable to the report.
Furthermore three hearings of math teachers were held. They were organized by the Dutch Association of Teachers of Mathematics.
Many suggestions were given and they were used in the final report that appeared in February 1980. [10]
With the publication of the report the activities of the Hewet Commission were terminated. The next step would be the reaction of the Ministry of Education.

On the next page we have summarized the developments leading to the

Hewet-project in a schema (fig. I.1).



The pre-Hewet-Project period: from Royaumont to Hewet

fig. I.1

2 THE HEWET PROJECT

Early in 1981 the Ministry of Education gave the 'green light' to start the Hewet project.
As far as the schools were concerned there were two important pillars on which the project was based:
- Experiments at schools.
- In-service teacher training.

The organization of the project was also based on two pillars:
The Hewet team; to be assisted, advised, counselled, supervised by the Hewet Advisory Commission.

The Hewet team consisted of collaborators of the former I.O.W.O. This institute had to cease its activities on December 31, 1980, for political reasons. But part of its activities were to be continued by the 'Research Group on Mathematics Education and Educational Computer Centre' (OW & OC), established at Utrecht University on January 1st, 1981.
The Team operated at OW & OC, and consisted of about two to three scientific workers and one secretary.[3]
The task of the Team was to carry out the project, leading to the introduction of the new curricula in August 1985.
The Advisory Commission[4] was installed in April 1981 with the following tasks:
- to advise the Ministry on which schools should participate in the experiments;
- to advise the Team on the experiments and teacher training;
- to advise the Ministry on the organization of teacher training.

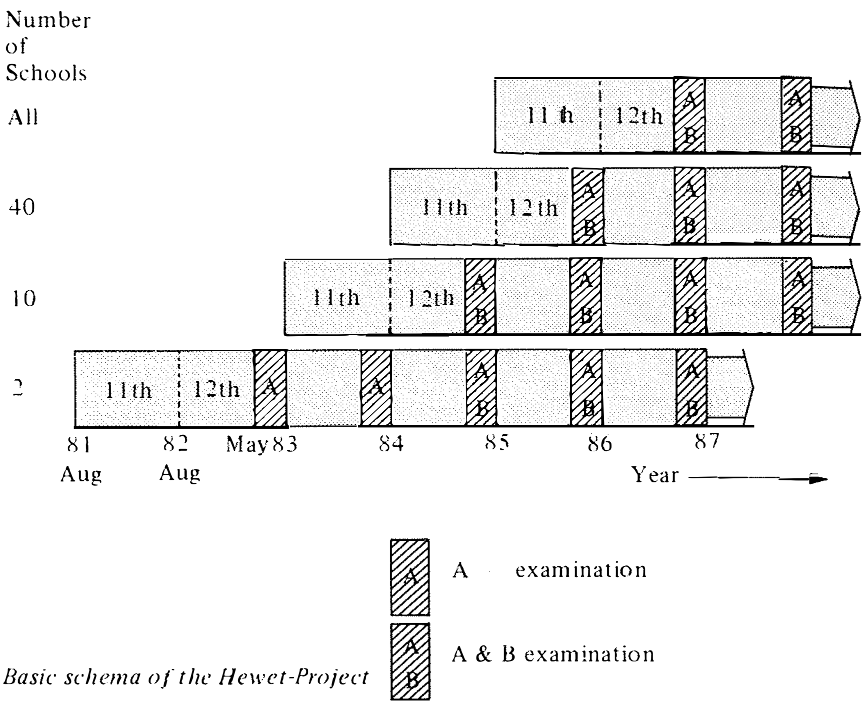
The Commission met at a rate of about four times per year and it consisted of seven persons representing Secondary Education, Tertiary Education, Inspectors and the Ministry of Education.

2.1 The basic schema

Experiments at two[5] schools started in August 1981, in the 11th grades. These experiments were to be continued in the 12th grade and concluded by an examination in May 1983.

While the two schools continued they were joined by ten more schools in August 1983.
In May 1985 these twelve schools underwent examinations. The exams were

concerned with the new Math A curriculum only.
In August 1984 forty more schools joined the experiments, in this stage concerned with both the new A- and the new B-programs.
As planned, in August 1985 the new programs were introduced at all schools.
This leaves us with the following schema for the experiments (fig. I.2):



Basic schema of the Hewet-Project

fig. I.2

For in-service teacher training courses to be fruitful two things were considered to be essential:

- Teachers were to be trained as much as possible *before* they were actually using the new programs.
- Much attention should be paid to the experiences acquired from the experimental schools.

This led to the following structure:

- The teachers at the first two schools had no advance training, but the Team itself attended almost all lessons at school.
- The teachers at the next ten schools had to be trained during the year 1981-1982, using the experiences of the first two schools. This course was

given by the Team.
Regular teacher trainers attended this course in order to be prepared for the courses later on.

- The teachers at the forty schools had to be trained during the year 1982-1983 using the experiences of the twelve schools.
These courses were given by regular teacher trainers.
- The remaining teachers had to be trained from 1983 onwards, with much emphasis laid on the year 1983-1984.

This gives the following schema (fig. I.3):

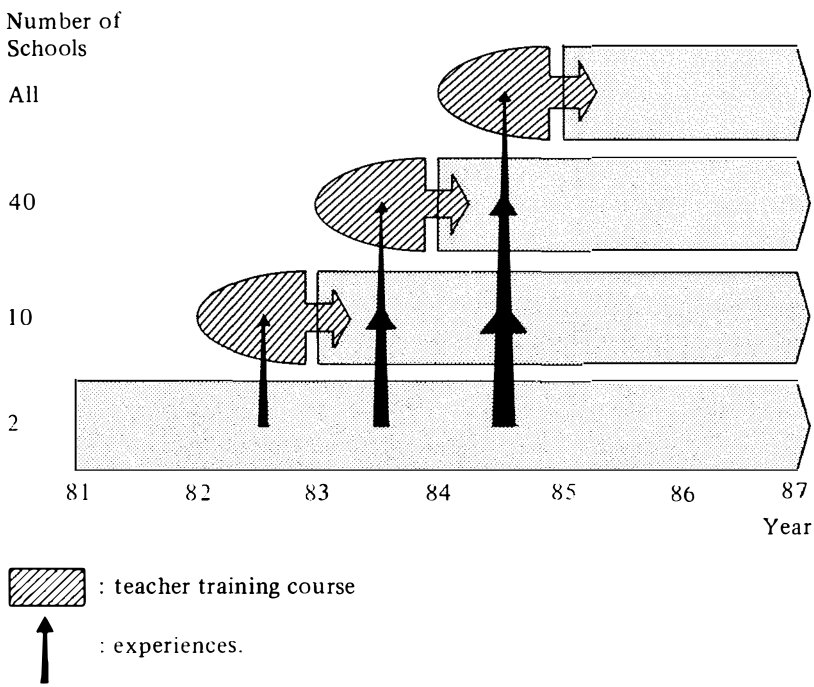


fig. I.3

With this basic schema in mind we will now describe the development of the project through the years.

2.2 The first year 1981-1982

In January 1981 the Team started to develop experimental student material to be used at the first two schools from August 1981 onwards. During their I.O.W.O.-time the Team members had experimented with Math A-like material. (See schema I.1.) The experimental Hewet materials, however, had to diverge in a sense from those I.O.W.O. materials. Indeed, they were to be used also by teachers not familiar with the I.O.W.O. philosophy.

The two schools were selected by the Team. The main criterion was the insider knowledge of Team members about the schools.
One of the schools was a typical city school (Haarlem), the other was situated in a rural area (Zevenaar).

The development of experimental texts was a very time-consuming task. It often happened that materials were completed the day before they had to be used. One Team member attended lessons in Haarlem, the other one did so in Zevenaar. They made class observations, interviewed students and teachers, in order to find out what level could be reached and what level was desirable, and what changes should be made in the material.
Together with the teachers the first restricted-time written tests (timed tests) were designed. At this moment the problems concerning the testing of Math A became clear, eventually resulting in this study.
The experiences and observations led to a series of articles, mainly for mathematics teachers, but also for counsellors, teacher trainers and others. [11] As indicated in our basic schema, the first experimental teacher training course for the ten schools was given, again by the team members, and occasionally assisted by the teachers and students of the two schools.
Points to be discussed during this first course were:

- how the schools should inform parents;
- consequences for the 10^{th} grade;
- to work on 'capita selecta' from the student material; (on Math A and Math B);
- to discuss experiences from the two schools;
- mathematical background information;
- how to make achievement tests;
- the use of the computer.

The teacher training courses for the forty schools would be given by University teacher trainers. These teacher trainers also took part in this first course and, together with the Hewet team, a scenario was developed for future teacher training courses. Each course would consist of sixteen sessions of

three hours each.
During this first course a new problem arose: most teachers agreed that the program of the 10th grade should be 'adjusted' in a 'Hewet-like' way.
Some teachers were already using some older I.O.W.O. publications but it became clear that there was a need for more material. The Team decided to solve this problem by developing additional booklets. If desired the twelve schools could use exclusively these 'Hewet-like' materials in the 10th grade. It turned out that six schools actually did use all these materials, while all others used one or more of these booklets.

The ten schools that took part in the teacher training course were selected by the Advisory Commission from a number of schools that had shown interest. Selection criteria were geographical spread, spread in school size and denomination (protestant, catholic, neutral, or state).
For the selection of the forty schools an official invitation to participate in the experiments was sent to all schools. Some 120 schools applied so that the Commission had the task to select forty schools. Again the above mentioned criteria were used to select the schools. In May 1982 the Commission made up a proposal to the Ministry of Education. This list was accepted and published in March 1983. [12]

2.3 The second year

The teacher training course for the teachers of the ten schools continued during this year, as did the development of experimental material for the twelfth grade of the two schools. The new aspects of this year were: the design of school internal examinations and of the national final examination and the development of software in view of the automatic data processing part of the curriculum.
Again, at this stage, the teachers and students as well as the Team members were confronted with the restrictions of timed-tests: the internal examinations usually consist of a series of three timed-tests of about two to three hours, and the final examination is also a written timed-test.
The materials for the 11th grade of the two schools were rewritten, revised and adjusted.
A new aspect was added to the dissemination of information: informing authors. It was the intention that authors should use the experimental Hewet material as a source of inspiration for their own books to be published before August 1985, when the new curricula were expected to take effect. They were invited to discuss matters with the members of the Team.

One of the issues to be discussed was the content of the 'Solid Geometry' in the Math B program. Officially, the experiments were restricted to the Math A curriculum. However, small scale experiments were carried out as part of the Math II lessons. The fact that 'Solid Geometry' didn't get much attention in this phase gave rise to some difficulties in a later stage. These problems were solved in discussions between the Advisory Commissions, Team and Inspection in close cooperation with teams of authors.
This year's highlight was the first examination[6] at the two schools and a discussion between the students who had taken it and the teachers from the group of ten schools, in May 1983.
Concern among teachers and the Team about the examination again became clear although the examination itself did not cause surprises; the results were satisfactory. The discussion between students and teachers will be reported in Chapter III.1.2.

2.4 The third year

The third year was important because of the fact that the ten schools joined the experiments with the two new curricula in Math A and Math B. Although the Team members could not possibly visit the schools, there was close and frequent contact with the teachers in order to discuss progress at the schools.
Information was obtained on a number of aspects:

- number of students participating in the new programs as compared with the old ones (report in Chapter III.2);
- time spent on the various booklets in order to prevent overloading;
- restricted-time written tests (character, results and experiences; report in Chapter IV).

In Chapter III it will be reported how the reactions developed during the two and a half years of contact with the twelve schools.

The reactions concerning the problems of designing appropriate tests resulted initially in the above mentioned restricted-time written test survey. But soon it became clear that this was just a starting point for our present study:

"What are the goals of Mathematics A and
what are the consequences for achievement testing?"

As a result, the first alternative tasks were designed and experiments were carried out with essay tasks, take-home tasks and an oral examination (report in Chapter V).

The first publication in the series *Hewet and Tests* appeared in January 1984. [13]
Although the Team, joined by some teachers, continued to publish articles on the experiments, a wave of signals proved that many teachers felt themselves not sufficiently informed on the new program and its consequences. [14] For this reason the Team, supported by the Commission, decided to convoke eight regional information meetings. The meetings took place on afternoons - after school hours - and were attended by more than 800 teachers. Again it appeared that the teachers felt uneasy and were not satisfied with the support and information they had got so far.

This year the in-service teacher training consisted of nine parallel courses to be given by the university teacher trainers. As noticed before, these teachers had on several occasions attended the first experimental teacher training course, and had met with the Team to discuss a scenario for these courses. Both the members of the Team and the teachers of the first twelve schools were invited as guest speakers at the teacher training courses, aimed at teachers of the forty schools.

This year was as it were the last opportunity to adjust and rewrite the experimental material.
Among the more structural changes made during the years, we mention:
- more summaries, in order to make clear what has to be learned from a mathematical point of view;
- more exercises.

In May 1984 the second examination took place for the students of the two schools. The results looked satisfactory, although only after having been upgraded by the authorities. Several factors had contributed to the failure of this exam: it was too much, the first exercise was too difficult and not clear enough, several items contained parts that were very formal - and thus not suited for A-students. Clearly, the design of the exam was unsatisfactory. Once more the problem of achievement testing became evident: the Team felt frustrated by the restrictions of the restricted-time written final examination and by the fact that the view of the Central Exam Commission was at variance with the Team's.
For years to come (during the experiment), tension between the view of the Hewet team and of the Central Commission for Examinations persisted: Questions that tested higher goals like interpretation, reflection and creativity were omitted from the examination and replaced by questions like:
"Compute, How large" and so on.

2.5 The fourth year 1984-1985

Although the Team members did not visit lessons at the ten schools there was an intense communication via the training course, given by the Team to the teachers of the ten schools, which continued with a couple of meetings during the examination year.
With respect to the forty schools the picture was completely different. Direct communication between the Team and the teachers came close to being non-existent. Phone calls to the members of the Team were the only and quite frequently used opportunity. But nothing of a more structural character was foreseen for these forty schools. An increasing stream of signals convinced the Team that more attention should be paid to the teachers of the forty schools. It was decided to organize a one-and-a-half day conference in February 1985 for the teachers of the fifty-two schools.
This conference provided the teachers with considerable support: they had an ample opportunity to discuss experiences, to exchange ideas on how to solve certain problems, and to learn about the problems of achievement testing. At the request of the teachers another conference was held at the end of 1985.

The development of alternative tasks was discussed at these conferences - particularly the so called two-stage task, which will be discussed in detail in Chapter V.
Alternative tasks were favoured by many teachers, especially when such tests were readily available. But the tensions between the goals of education and the preparation for the examination became again palpable.

Finally the conference added to the convictions that the program was not overloaded, although some schools continued to have problems (see also Chapter III).

The experimental texts had now obtained their final shape. Several publishers were interested in publishing the experimental booklets in a more definitive form. This commercial version of the material was available from 1984 onwards. [15]
The last booklet was published in October 1986. Teacher training, which had now spread to thirty-two parallel groups of twenty teachers each, was given all over the country. Because of financial limitations only 650 teachers were allowed to follow these courses. This meant that only one to two teachers per school had had the opportunity to follow the course *before* the introduction of the new curriculum. New courses were offered the next year

but this could not satisfy all teachers.

Increasing interest for the Hewet project was noticeable in other countries. Presentations about the project were given in Australia, Argentina, Belgium, Brazil, Germany (FRG), Italy, Spain, the United Kingdom, Zimbabwe, and articles appeared in international magazines and Proceedings of Conferences. [16]

In May 1985 the examinations were taken by the students of the twelve schools. The results were satisfactory both to teachers and students, as far as Math A was concerned. For Math B however, the picture was completely different. Teachers felt that in no way did the items for 'Solid Geometry' reflect the contents of the experimental booklet. As we noticed before, the lack of attention paid by the Team to this part of the project may have contributed to this problem.

2.6 The final year 1985-1986

Officially the project terminated on January 1st, 1986. This seems strange as the first examination for the forty schools falls outside the period of the project. This was also felt by the Advisory Commission who sought successfully to continue their activities up to June 1987 - that is until the first nationwide examination.

All schools started with the new curricula in August 1985. The teachers could make their choice of what materials to use: four textbook series - including the commercialized Hewet materials - were available. But the teachers had little if any opportunity to choose: most textbooks appeared only a few weeks or months before they were to be used in the schools.
In the planning no guidance for the schools had been foreseen. But once more the Team took action, in this case together with the university teacher trainers.
A series of meetings was arranged in the fall of 1986 to inform teachers about the possibilities and preparations for the internal examinations and the national examination. Again, teachers showed much interest: between 40 and 120 persons per meeting attended the eight meetings.

Another important meeting took place in October 1986. This meeting - an initiative of the Advisory Commission - was intended to inform University teachers about the new curricula. Some fifty persons attended the meeting, which will have a successor in 1988 when the teachers will have assembled more experiences with students who meanwhile will have followed the new programs.

The examination - the last directly influenced by the Team - was regarded by many as a momentous occasion because it involved a large number of students (about 1000) and schools (51).
The results of the examination were again satisfactory - an average score of more than 6,5 on a 10-point scale. But earlier mentioned problems surfaced once more; some teachers and students complained about the examination. The items were too narrowly closed and did not properly reflect the goals of Mathematics A. [17]

The project has ended - at least officially. But activities continue. As a matter of fact, the really interesting part of the project is still to come: how will the contents and the teaching of the Math A curriculum develop in the future years?

Until now the project has up hardly been evaluated. Apart from the final report delivered by the Team [18] and the present study, only one study with evaluative aspects has been carried out: that undertaken by the Educational Research Institute (R.I.O.N.) in Groningen. The latter has not been published at this time.
The lack of any support in the near future is quite a serious problem. But this aspect will be discussed in our Epilogue (Chapter VI), when we will discuss briefly the Hewet project.
The influence of the final examinations on the teaching and learning of Mathematics A is another matter of concern and will be discussed from Chapter III onwards.

2.7 An overview

Although the basic schema of the Hewet project as presented in 2.1. of this Chapter is rather straightforward, the activities carried out by the Team were of a very complex nature.
The stepwise increasing number of schools - from two to twelve to fifty-two to all schools - is clear and represented by Row 5 in our schema (fig. I.4).

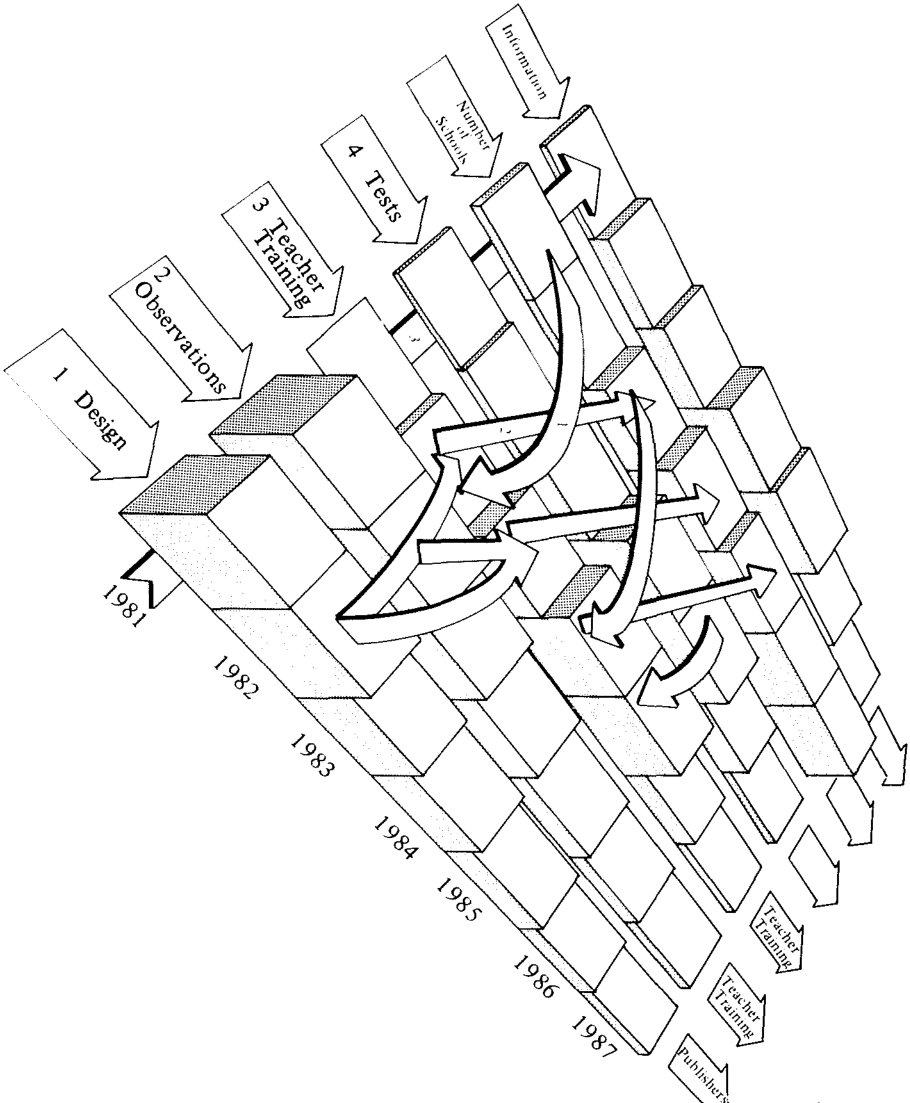


fig. I.4

This schema shows that designing and developing the experimental material was at its peak during the first years (Row 1). The same holds for guidance for the schools: the first two schools were guided very closely, from the ten schools onwards this was less and less the case. (Row 2).
Other activities, however, took over: teacher training activities (Row 3) and interest in the problems of achievement testing grew more and more to their peaks in 1984.

The arrows indicate the influence of one activity on another.
Arrow 1 indicates that experiences from the two schools played a major role

in teacher training for the ten schools (Arrow 2).
Arrow 3 indicates that all activities eventually found their way into one or more publications (Row 6).
The schema illustrates the fact that, although the Team consisted of only two to three persons, it covered all kinds of activities with continued shifts in time from one activity to another, and with strong interactions between all of these. This is a characteristic feature of the Hewet project.

Having shown the reader a bird's eye view of the overall project our next chapter will describe in detail Mathematics A.

II MATHEMATICS A

In this chapter on Mathematics A we will first describe the *contents* (from a mathematical point of view). Thereafter we will analyze in detail essential aspects of Mathematics A such as *mathematization*, the role of the *context* and the *critical attitude* of the students. This description will make clear that achievement testing by means of restricted-time written test only is not sufficient.
Furthermore we will place Mathematics A in the broader context of Realistic Math Education.
Finally, we will discuss some other aspects: the dichotomy between analog and digital thinking in relation to Mathematics A, and the ever ongoing discussion on pure and applied mathematics.
Math A is a curriculum where applications play a vital role, according to the Report. This will also be our starting point when describing the mathematical contents.

1 THE CONTENTS

1.1 Introduction

The Hewet report requires Mathematics A to aim at students who are expected to pursue studies at the university in disciplines where mathematics is needed only as a tool. This means that at all times the *usefulness* of mathematics should predominate. According to authors like Engel [1] and Pollak [2], usefulness is one of the main reasons for the society to support mathematics. Very few among the students who take the Math A program will become professional mathematics. Many of them, however, will specialize in economics, social sciences, medicine and will use mathematics as a tool.
Engel even argues that "if for some branch of mathematics there are no convincing applications, this branch of mathematics should not be in the curriculum."
Pollak claims that usefulness should be fundamental for teaching as much mathematics as we do. This means that we must exhibit, exercise, and emphasize this usefulness at every opportunity.
"The most important subjects from an applied point of view are: calculus, linear algebra and geometry, probability and statistics, computers. These topics should be treated extensively. They must be integrated into one

coherent course."
This was pronounced by Engel in 1967 and shared by Pollak.

It is remarkable how closely the Hewet report approached what after years of research turned out to be that kind and level of mathematics as used as a tool in the 'soft' disciplines. Although the report did not mention geometry explicitly, some geometry did find its way into the curriculum anyway, as will be seen shortly.

The Hewet report recognizes that at first sight the broad program may look confusing, but it argues that the unity of this curriculum is rooted in its applications.
In an even stronger way Klamkin [3] claims that one of the reasons why students have difficulty in applications is that most of mathematics is learned 'vertically', that is that its various subjects are taught separately, neglecting the cross-connections. In applications one usually needs more than just algebra alone or geometry alone. Consequently, courses should be designed 'horizontally', cutting across several different mathematical branches.

In Hilton's view, the relation between horizontal programming and applications is too restricted. He notices a new unification in mathematics as a whole. [4] Up until to ten years ago the most characteristic feature of mathematical research was the 'vertical' development of autonomous disciplines. This development, which according to most authors started in the last century, was severely attacked by Kline in several well-known publications. [5]
But, according to Hilton, it appears that the autonomous disciplines are being linked together in such a way that mathematics is being reunified. This horizontal development should have implications for education as well: "We must break down artificial barriers between mathematical topics throughout the student's mathematical education."

According to the Hewet report the unity in the Mathematics A program should be aspired at via its applications.
The next schema illustrates clearly the horizontal and vertical components of the Mathematics A program as concretized in the experimental material. For the sake of completeness we have included not only the material for grades 11 and 12 (the official experiment), but also that for the 10th grade. The schema does not claim precision, but rather gives a global picture of the contents of the Mathematics A program, its vertical structure in mathematical branches and its horizontal structure in connections between those branches.

The four main streams are: calculus, linear algebra, probability and statistics, and the use of computers. As indicated before, some geometry, and more precisely, spatial geometry plays a role as well.
The role of the computer has not been indicated in the schema, because of its integration into the other branches, as we will point out later on.
Next, we wiil have a closer look at the four streams (fig. II.1).

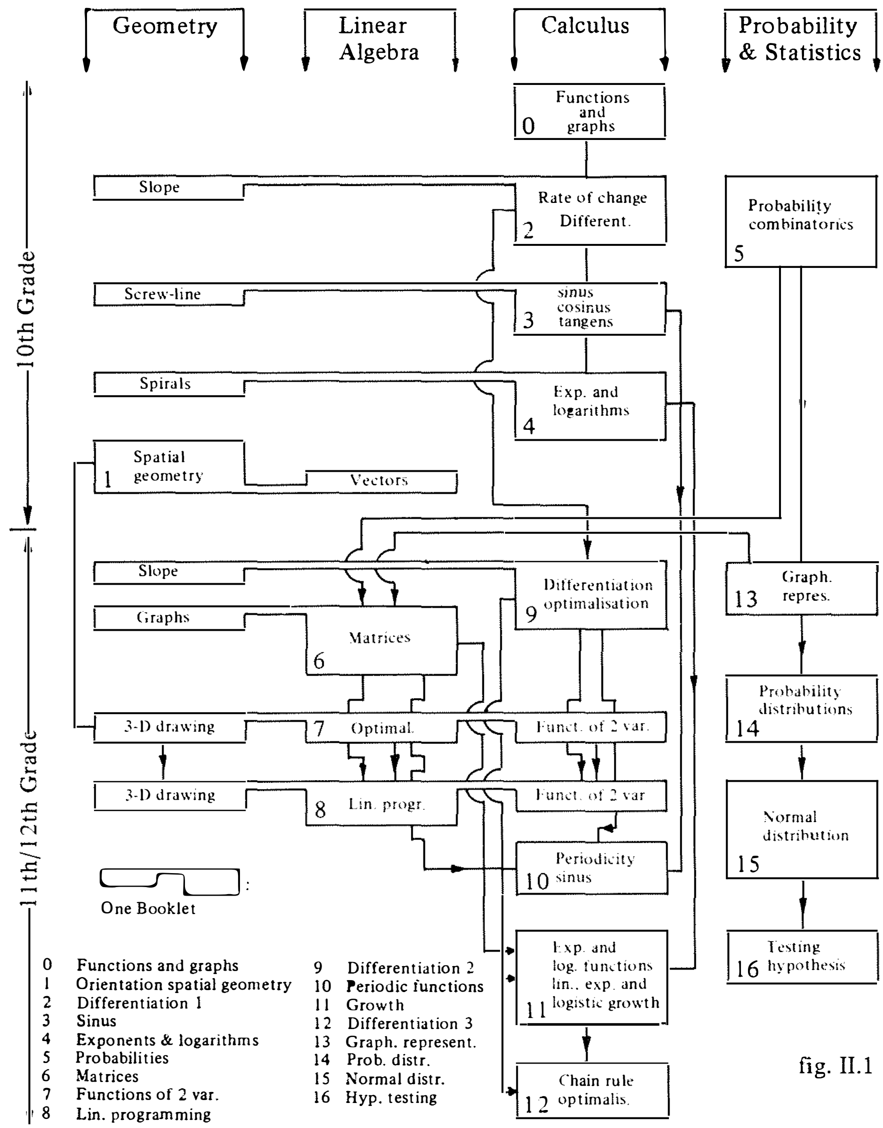


fig. II.1

1.2 Calculus

If need be, one can distinguish three substreams:
First *'differentiation of functions';* second *'periodic, or more specifically trigonometric functions'*, third *'exponential and logarithmic functions'*.
In our schema in 1.1 *'differentiation of functions'* is taken care of in 2, 9 and 12; *'periodic, or more specifically trigonometric functions'* in 3 and 10, and *'exponential and logarithmic functions'* in 4 and 11.

a. Differentiation of functions
Booklet 2 starts with studying rates of change in various phenomena.
Basic questions are:

"How can you see from a graph whether a magnitude grows or diminishes?"

"How can you see from a graph whether its rate of change is large or small?"

"Is it possible to quantify the rate of change?"

After this intuitive phase the concept of slope is introduced in an intentionally geometrical way: tangents are drawn intuitively, thus creating the need for a more quantitative and precise way of measuring the slope. This is done in classical context of the free fall (fig. II.2).

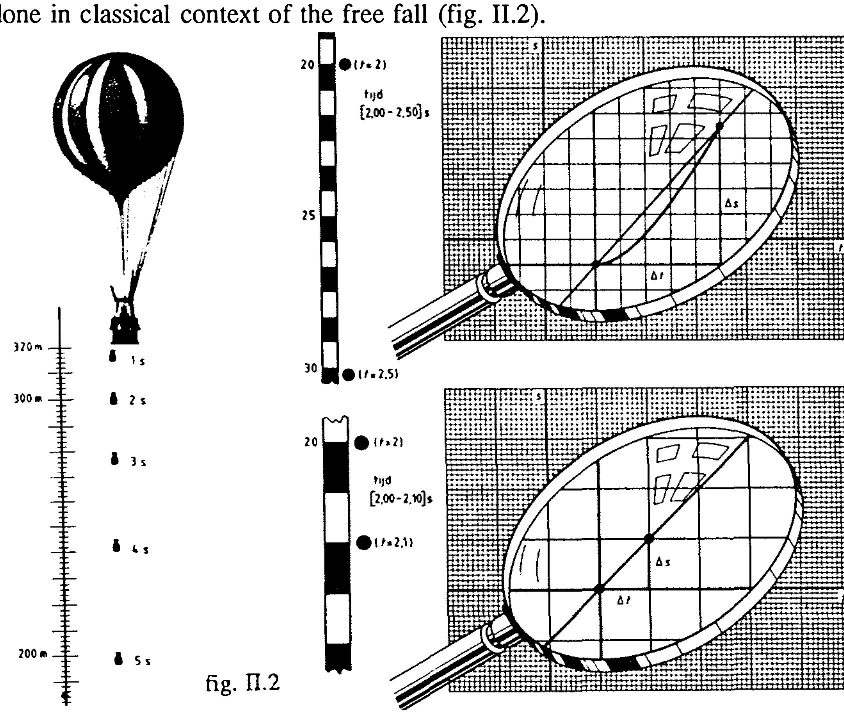


fig. II.2

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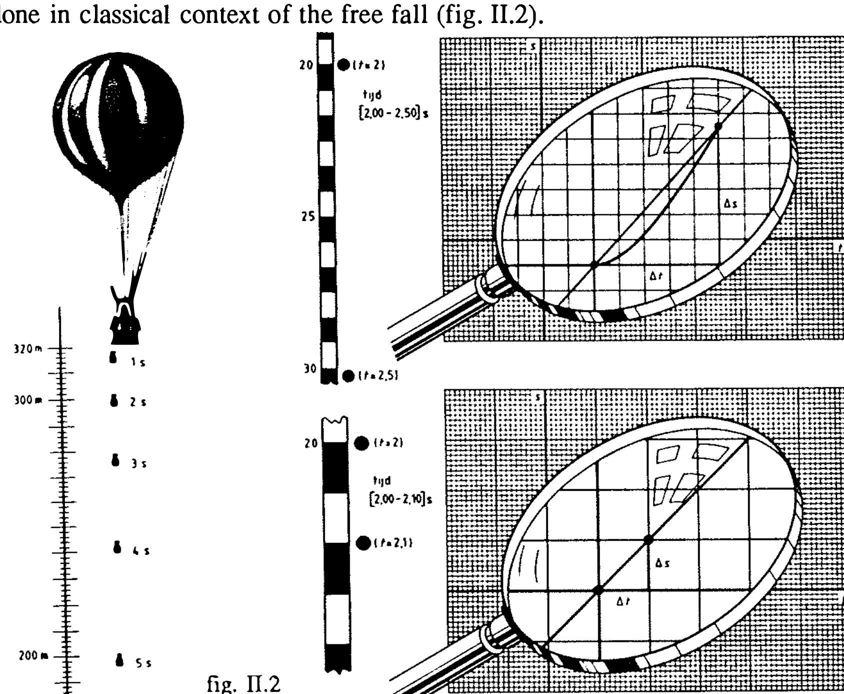


fig. II.2

Formalization then takes place and the rules for differentiating

$f(x) = c \cdot x^n$, $f(x) = g(x) + h(x)$ $f(x) = c$

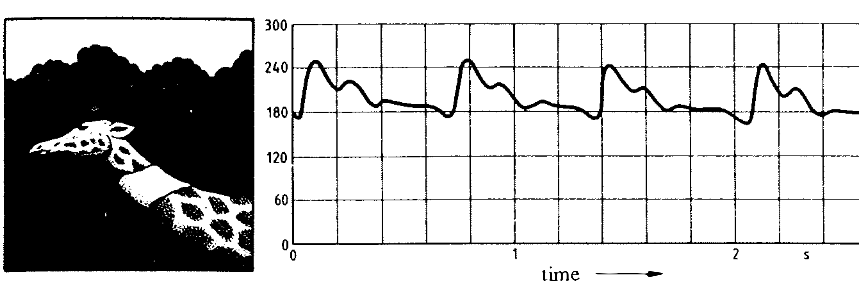
and their application to minima and maxima follow.
The next booklet (9) starts with recalling the basic concepts of (2) and introduces the concept of 'marginal changes' and $\frac{dy}{dx}$.
The computer is used to find the derivative of y = sin x, and the structure schemas (flow charts) are introduced as a didactical tool to help children to find and construct a formula or a function.
The product rule is next proven in a geometrical way, and the quotient rule concludes the rules of differentiation in this booklet. A number of applications is spread throughout the booklet, in this case optimalization problems.
The last book on differentiation (12) mainly contributes to the 'chain rule' and completes also the differentiation of exponential and logarithmic functions.
A more detailed description can be found in Chapter II, 2.2.3.

b. Periodic functions

The first booklet (3) starts with much emphasis on spatial geometry: the 'screw-line' is the starting point for introducing the graphs of y = sin x. This model not only gives the graphs of y = sin x and y = cos x but also delivers intuitively the transport of the sinus function by translation $y = \sin(x - \alpha)$, as well as the functions of the type $y = \sin \alpha x$ (fig. II.3).

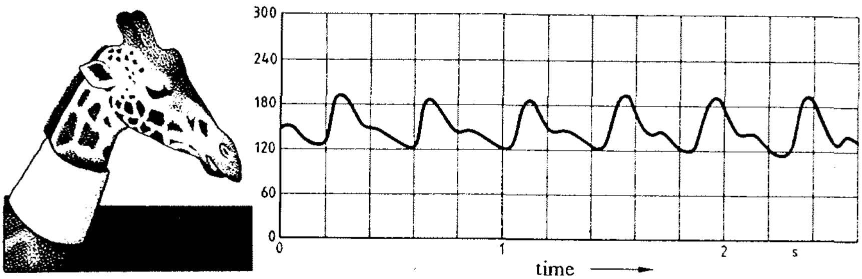
Much attention is paid to the construction of functions, both in a 'pure' way and in applications.
Finally, the graph of the tangens function is introduced with the glide ratio as a starting point.

The second booklet (10) starts with a series of phenomena with the common feature of periodicity (fig. II.4a,b).



Blood-pressure of a giraffe lying down.

fig. II.4a



Blood-pressure of a giraffe. fig. II.4b

Next periodicity is emphasized while the function y = sin t plays here still a minor role. The well-known prey-predator model is introduced, again in an intuitive way, though dominated by the use of trigonometric functions. Here students really need the knowledge acquired in the previous booklet (3).
After a discussion of multiple periodicity and internal and external biological clocks, the mathematical core of this booklet is reached by graphing functions like

$f(x) = a \sin b(x + c) + d$

and, equally important: how to approximate phenomena by means of such functions. Finally, the concept of 'trend' is treated in conjunction with functions of the form:

$f(x) = px + a \sin b(x + c) + d.$

c Exponential and logarithmic functions

The first booklet (4) has a rather lucid structure. It starts with a real world problem about growth in which students 'discover' exponential growth, and compare this with linear growth. Through a wide range of contexts the domain of the function $f(x) = 2^x$ is expanded from the natural to the rational numbers.
The concept of *growth factor* plays a vital role. This role will be discussed later on. Functions of the form $f : x \rightarrow a^x$ are extensively studied before the next step is taken: the introduction of logarithms. Basically, this is done in a quite straightforward way: the logarithmic function as the inverse of the exponential function. But the introduction leans heavily on intuitive tools by which the 'main property' becomes quite natural. Apart from the real world context used to introduce mathematical concepts, numerous applications conclude this booklet.

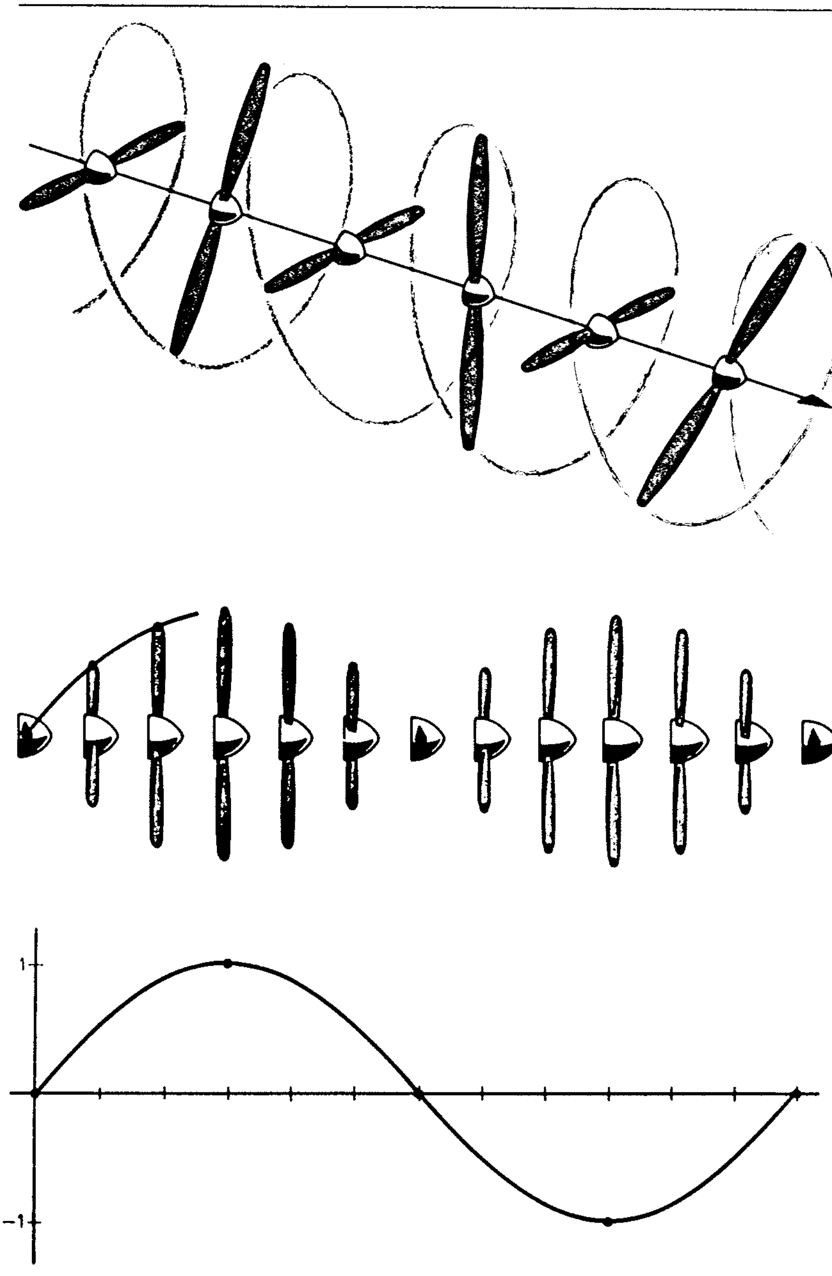


fig. II.3

Booklet (11) starts the same way as (4) to recall basic concepts such as growth factor and the analogy between the functions $y = b + ax$ and $y = b{\cdot}a^x$ for linear and exponential growth. The multiplicative character of the latter is stressed before the rate of growth is studied, again with the help of the computer. The students 'discover' that f'(x) is proportional to f(x): If $f(x) = 2^x$ than $f'(x) = c.2^x$.

The next chapter is an application where teachers as well as students may use all their acquired knowledge to solve a relatively simple problem. The chapter will be described in detail when we discuss the activity: *'mathematization'*. (Chapter III).

Logistic growth is introduced next, again in an informal way and even its simple differential equation is mentioned, which plays a role in the computer program which works with a comparable difference-equation.
After $\frac{da^x}{dx} = c_a{\cdot}a^x$ follows $\frac{de^x}{dx} = e^x$; and the relation between exponential growth and population-projection-matrices (Leslie) is made. Before turning to the use of logarithmic graph paper logarithms as treated in (4) are recalled.

The booklet shows clearly - according to teacher reactions - the 'horizontal' component in the curriculum: there are many cross-connections between the different mathematical branches.

1.3 Linear Algebra

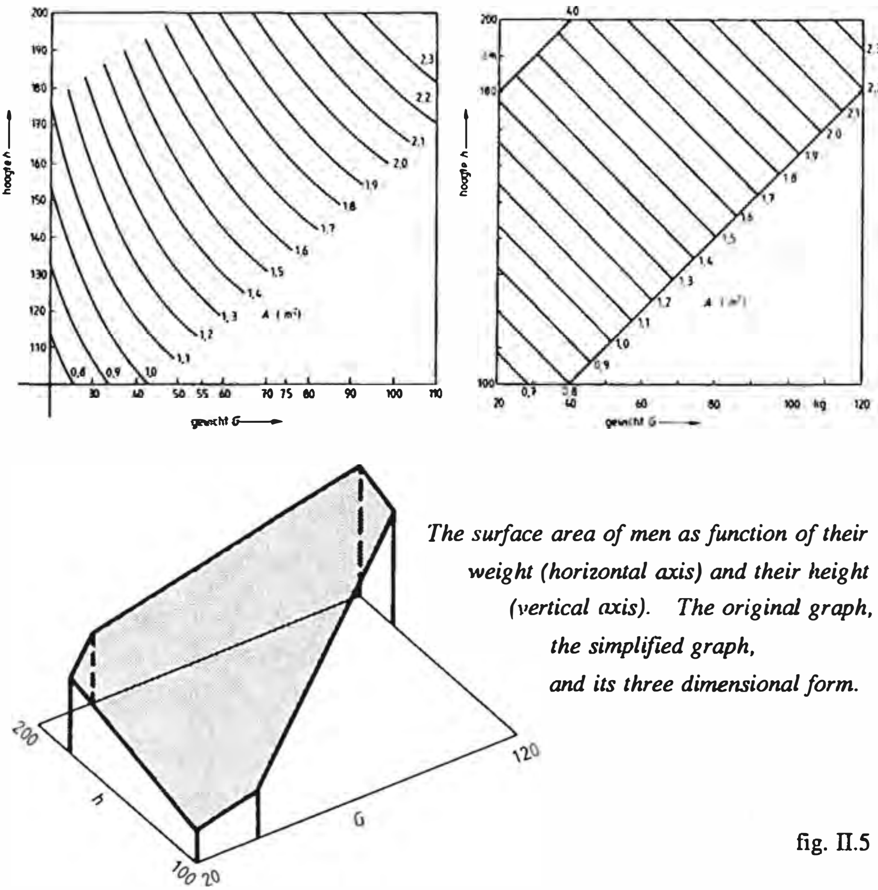
Basically, linear algebra is represented by two main subjects: *'Matrices'* (6) and *'Linear Programming'* (8). But since *Linear Programming* should also include the three dimensional graphic solution, a booklet on *'Functions of two variables'* (7) was indispensable which, as a side effect, may also improve the spatial intuition of the students. In this way the booklets (7) and (8) have strong spatial geometry components.

a. Matrices
The concept of a matrix is explored intuitively, in relation with combinatorial graphs. Next addition, (subtraction,) scalar multiplication and multiplication are introduced. All concepts are derived from real world situations. The 'unnatural' multiplication of matrices becomes intuitive by chosing an appropriate context. For a detailed description see Chapter II.3.2

After discussing a great many kinds of matrices the booklet ends with an actual research problem.

b. Functions of two variables

Placing this subject in the linear algebra stream would appear somewhat arbitrary since its aspects include calculus and geometry (fig. II.5).



The surface area of men as function of their weight (horizontal axis) and their height (vertical axis). The original graph, the simplified graph, and its three dimensional form.

fig. II.5

Isohypses and more specific contour lines introduce this subject, closely connected to the real world (Grand Canyon, see 5: Geometry). A spatial model of the Grand Canyon is drawn by the students in order to get used to a specific way of drawing three dimensional graphs of functions of the form

$z = ax + by + c$

The concept of functions of two variables is formalized later on before a number of optimalization problems is handled.

c. Linear Programming
Graphs and matrices are used to introduce the basic principle of linear programming. Before passing to the graphical treatment of three dimensional linear programming problem a special kind of linear programming problems is dealt with: transport problems are discussed, again while using combinatoric graphs and matrices.

Though not being part of the regular curriculum, the simplex method is treated in this booklet. This subject was included in the first national exam for the two experimental schools.

1.4 Probability and Statistics

Quite naturally this includes probability and combinatorics on the one hand and statistics and descriptive statistics on the other. The last subject is treated under 'Graphical Representation' (13).

a. Probability
The first third of this booklet (5) treats in an intuitive way the concept of 'chance' resulting in 'definitions' of *a priori chance* or *a posteriori chance.*
If one wants to quantify chances one has different (visual) representations at hand (fig. II.6):

- the lattice diagram;
- the tree diagram;
- the road diagram;
- the triangle of Pascal.

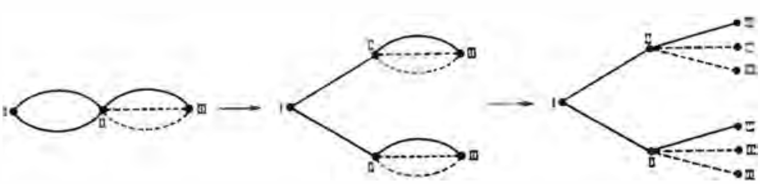


fig. II.6

Randy Walker walks every day from the 56th Street to the corner of 5th Avenue and 59th Street. (See crosses). He takes every day another route, but always a shortest one.
Randy's problem is: how many different routes are there (fig. II.7, II.8)?

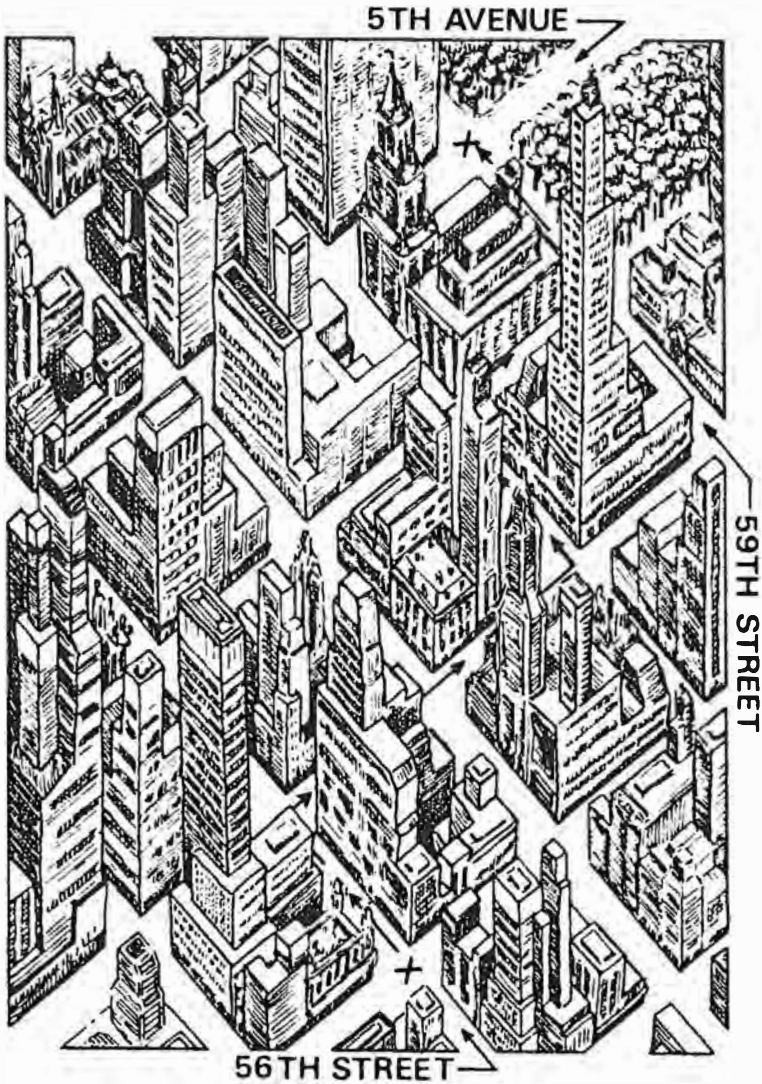


fig. II.7

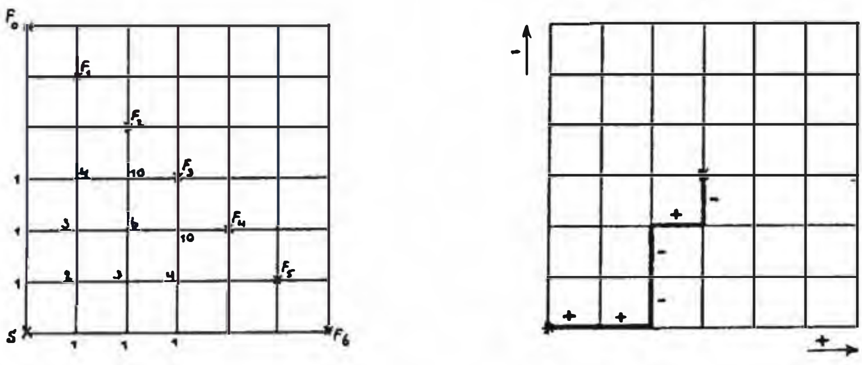


fig. II.8

Permutations and combinations, resulting in the binomium of Newton form the contents of the last chapter of this booklet.

b. Statistics
Graphical representation, although the title of (13), is too narrow a description for the contents of this booklet. Of course pie charts, histograms, polygon graphs are treated extensively. And also a sober treatment of average, median, modus and standard deviation finds a place in the booklet. But considerable attention is paid to what one may call "looking critically at statistics".
More attention to this last subject is given in Chapter II.2.4.

The next booklet in this stream - a stream that, by the way, is rather isolated from the others - is *'Probability Distributions'*. The idea of a 'stochast' is introduced in the first chapter, followed by $E(X) = \sum_{i=1}^{n} p_i x_i$ in the following chapter.
The computer plays a role by simulating binomial distributions. The final result of this chapter is: $P(X=k) = \binom{n}{k} p^k q^{n-k} \quad (k = 1,...,n.)$
The hypergeometric distribution follows of course in relation to the binomial distribution. The formulas for the expectation for the binomial distribution conclude this booklet (14).

Booklet (15) starts with the study of a research report which deals exclusively with the Normal Distribution. The meaning of average and standard deviation is reintroduced (fig. II.9).

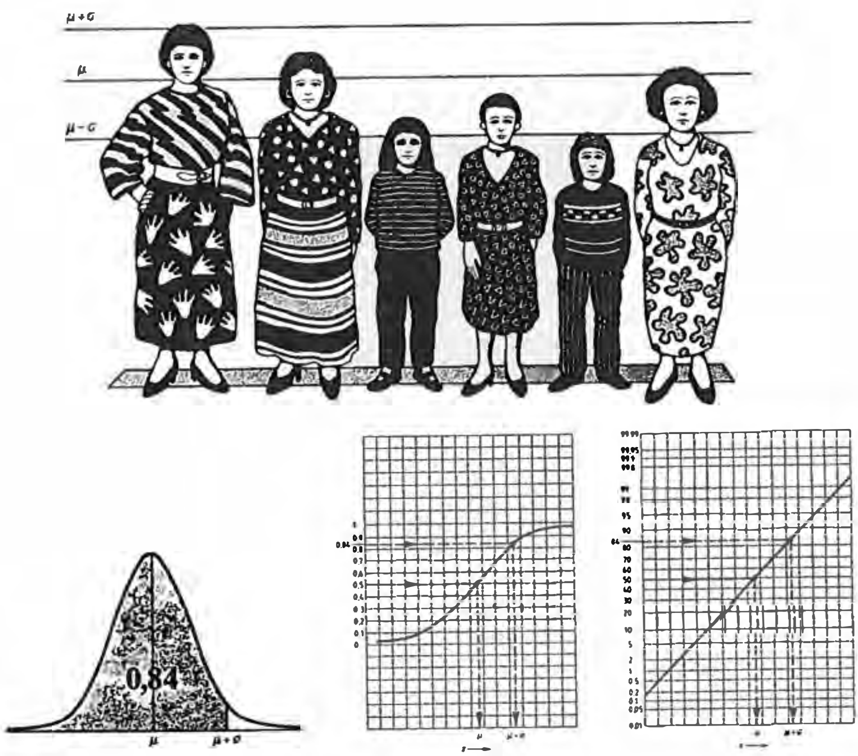


fig. II.9

The process of standardizing the Normal Distribution precedes the well-known 95%-rule of the normal distribution. The students get familiar with the use of normal distribution paper. References to log- and loglogpaper are made at this stage.
After the formulas $E(x) = np$ and $SD = \sqrt{npq}$ the central limit theorem and the approximation of a binomial distribution by the standard normal-distribution conclude this booklet.
The statistics stream ends with the treatment of testing hypothesis (16).

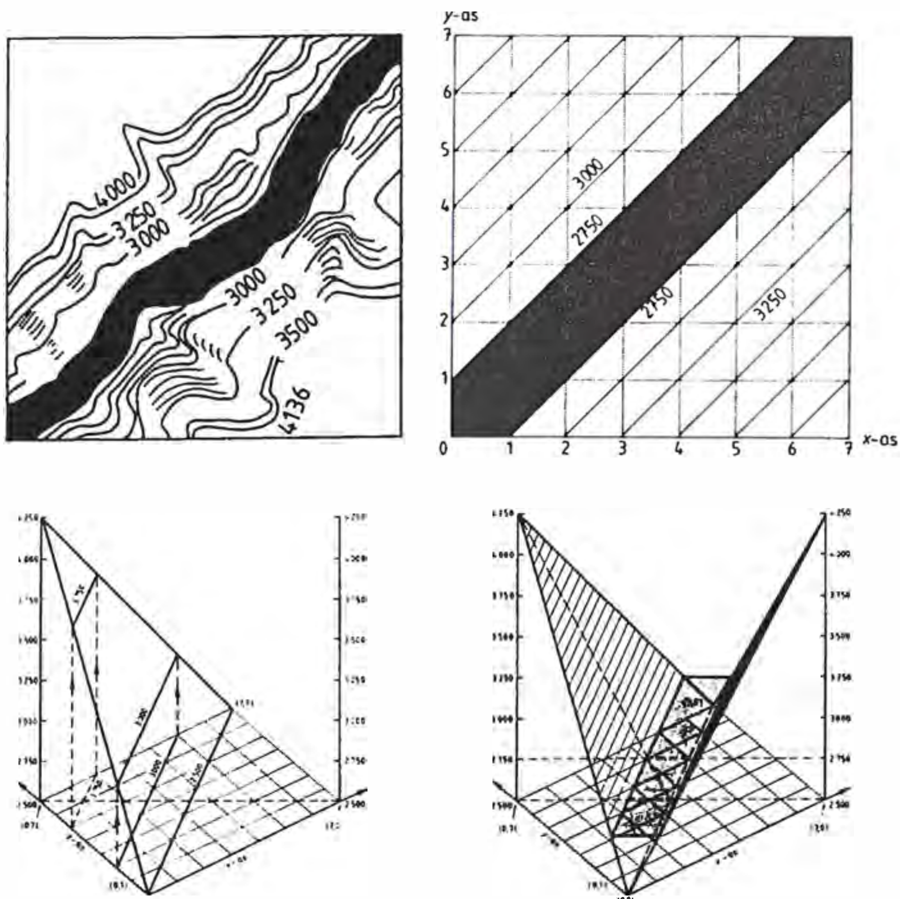
1.5 The computer

As we have seen, working with the computer is an integrated part of any of the other streams. Software was developed for use with different booklets. We mention:

Matrices, Linear Programming, Periodic functions, Growth,
Differentiation 2, Probability Distributions (Simulation),
Normal distribution.

Besides this aspect, we already mentioned the didactical role of structure-diagrams and computer output in *Differentiation 2* and *Growth.*
To make sure that all students have at least a small common background, a booklet on *Automatic Data Processing* was developed, containing chapters on Algorithms and Structure Diagrams and on Programming.

1.6 Geometry



Reconstruction of the Grand Canyon from a map.

fig. II.10

Our final remark concerns geometry. Although, strictly speaking, there should be no geometry in the Mathematics A curriculum according to the

Hewet report, a considerable amount is present in the experimental material. This geometry - usually three-dimensional - can be found in the subject *'Functions of two variables'* and *'Linear Programming'*, but the booklets on *'Differentiation'* also contain some geometry, as do *'Exponents and Logarithms'* in the applications on spirals. Finally, we mention the introduction of the sinus graph as a projection of the three-dimensional screw curve. [6]

2 ANALYSIS OF MATH A

2.1 Introduction

Analyzing the experimental material is a very complex task, but one can delineate a rough schema that represents the main aspects of the Math A curriculum as operationalized in the experimental material.

One thing is clear in all materials: the large role played by the *context*. This role will be discussed in detail in Chapter 2.3.
The role of the context is twofold: the start of any sub-curriculum takes place in some real world situation. This real world is not restricted to the physical and social world. The 'inner' reality of mathematics or the real world of the students imagination as well provides sources for developing mathematical concepts.
The context's second role is in the applications: they uncover reality as source and domain of application.

As we will show in 2.2.3 the real world situation or problem is explored in the first place intuitively, with the view on mathematizing it. This means organizing and structuring the problem, trying to identify the mathematical aspects of the problem, to discover regularities. This initial exploration with a strong intuitive component should lead to the development, discovery or (re)invention of mathematical concepts.

As our classroom observations made clear, by depending on such factors as interaction between students, between students and teachers, the social environment of the student and the ability to formalize and to abstract, the students will sooner or later extract the mathematical concepts from the real situation. This phase we like to refer to as *conceptual mathematization*.
At the same time reflection on the process of mathematization is essential.
The next phase recognizable in the material is the description of the desired and resulting mathematical concepts, followed by a more strict and formal

definition.

In the first stage of mathematization we developed our tools which, after formalization, we use in the second stage. And, by applying the concepts to new problems, one of the main results is reinforcement of the concepts and developing mathematization skills.
Finally, on the other hand problems solved will influence the student's view of the real world.

This introduction offers only a global sketch of the activities and essentials in the math A curriculum, based on an analysis of the booklets and reflection on observations, in the classroom, and discussion with students and teachers. This global sketch can be represented by the schema on the next page. It offers the reader the opportunity to place the next paragraphs in a proper perspective (fig. II.11).

We intend firstly to show clearly the ideas behind mathematization and how mathematization functions in Mathematics A. For this purpose we will not restrict ourselves to an analysis of the booklets, but also consider student's and teacher's work and their reactions.

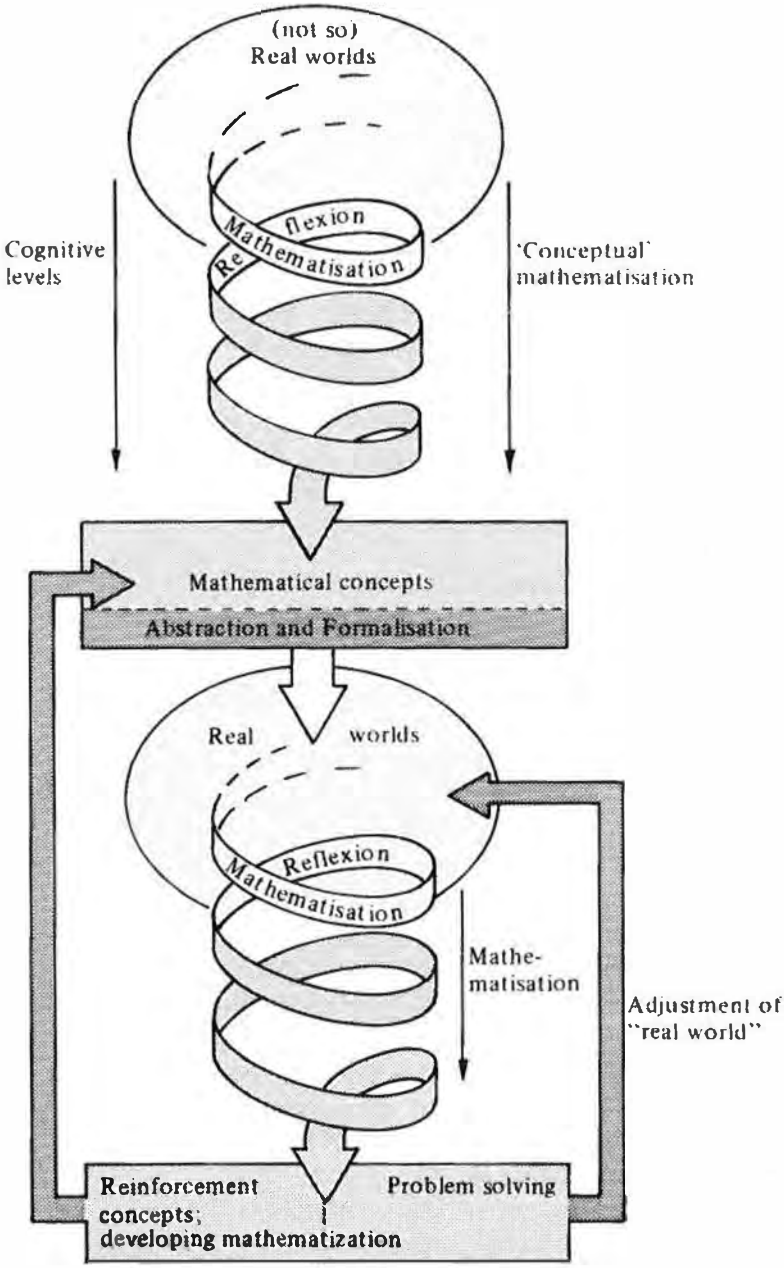
Secondly the role of the context and its relation to mathematization will be discussed.
And, finally, attention will be paid to an aspect of Mathematics A that is not visible at first sight: How to develop a critical attitude?

2.2 Mathematization

2.2.1 Mathematization

The mathematization of science may have its starting point in the 17^{th} century. [5] Newton gives an implicit definition of mathematization in his major work, '*Mathematical Principles of Natural Philosophy*' (1687) as follows:

> "But our purpose is only to trace out the quantity and properties of this force from the phenomena, and to apply what we discover in some simple cases as principles, by which, in a mathematical way, we may estimate the effects thereof in more involved cases.
> We said in a mathematical way, to avoid all questions about the nature or quality of this force, which we would not be understood to determine by any hypothesis."



Global schema of the activities of the experimental math a curriculum

fig. II.11

The sciences involved at that time were physics and astronomy. Only much later did mathematization spread to disciplines like economics, medicine, and social sciences.
The concept of a mathematical model as the result of a process of mathematization was Newton's theory of planetary motion.
In the process of mathematization a known model may be used, or a new one developed, or models may be integrated. It is not easy to describe what a mathematical model is.
Aris gives the following definition:

> "A mathematical model is any complete and consistent set of mathematical equations or structure which are (is) designed to correspond to some other entity, its prototype. The prototype may be a physical, biological, social psychological or conceptual entity." [7]

Certainly the models of Newton comply with this definition. In his case the main purposes in constructing a model were:
(1) to obtain answers about what will happen in the (physical) world;
(2) to influence further experimentation or observation;
(3) to assist the axiomatization of the (physical) situation.

But there are two more reasons that play an especially important role in the *conceptual* mathematization:
(4) to foster conceptual progress and understanding;
(5) to foster mathematics and the art of making mathematical models.

Not until the 19th century was there a sudden growth in applied mathematics and mathematization. But the stimuli that mathematics received came again from the field that today is known as mathematical physics.
It should be noted that throughout history mathematics always has had a strong connection with trading, coinage, borrowing and lending. These activities have been a source of concept formation in mathematics.
In more recent times the ideas of interest, compound interest and discount are strongly related to calculus and, more precisely, to theories of growth.
The theory of probability entered mathematics through gambling, including the notions of expectation and risk.
Today economics is one of the sciences where mathematization occurs regularly. Not always successfully, which causes mathematizing in economics to be a controversial subject. Unlike mathematical physics, mathematizing in economics is almost a one-way activity, although there are a few contributions from economics to the mathematical literature; one of the first examples of these is the motions at the Stock Exchange in the early work of L.

Bachelier.
Freudenthal also speaks of mathematizing mathematics itself. [8] Primordially, mathematizing is organizing the reality with mathematical means. Today, Freudenthal states, mathematizing mathematics is one of the main concerns of mathematicians. In no other science has the habit of recasting become a second nature as it has in mathematics. He also objects to restricting mathematizing as a privilege to mathematicians and excluding it from instruction because it is considered a scribblers' activity. Mathematizing should not be the business of the learner they claim, but of the adult mathematician. However, Freudenthal argues, there is no doubt that pupils should learn mathematizing, too, and certainly on the lowest level, where it applies to unmathematical matter, to guarantee the applicability of mathematics, but no less on the next level, where mathematical matter is organized, at least locally.

It cannot easily be traced when and where in Mathematics Education the idea of *conceptual* mathematization arose and was recognized, in particular if the term 'mathematizing' is not explicitly used by the author or if it is used in various ways, for instance for axiomatizing.

In March 1962, some 75 well-known American mathematicians produced a memorandum, entitled *'On the mathematics curriculum of the High School'* that was published in the *American Mathematical Monthly* of March 1962. [9]
We quote:

> "To know mathematics means to be able to do mathematics: to use mathematical language with some fluency, to do problems, to criticize arguments, to find proofs, and what may the most important activity, to recognize a mathematical concept in, or to extract it from, a given concrete situation"

From this quote it may not be completely clear if this 'recognizing activity' coincides with our ideas of 'conceptual mathematization'. But if we look further, the following remarks give some basis to the opinion that the ideas in this memorandum come close to 'conceptual mathematization':

> "Premature formalization may lead to sterility; premature abstractions meets resistance because critical minds wish to know why it is relevant and how it could be used."

According to Choquet mathematical thinking proceeds in cycles of four phases: observation, mathematization, deduction and application. [10]

Krygowska adds that in *education* we should not forget any one of these four stages. She stresses in particular the 'mathematization' phase:

> "L'enseignement qui ne met pas en evidence cette particularité, presente aux élèves une image tronquée et faussée des mathématiques." [10]
> ("Instruction which does not show this aspect presents to the students a truncated and false image of mathematics.")

Krygowska mentions the well-known "la description schématisante d'une situation réelle" but also, and this is important especially from an educational point of view, "la concrétisation d'une notion mathématique (formellement) élaborée". These activities are to be found in the lower section of our schema of Mathematics A.

More interesting are her remarks on what is called 'conceptual mathematization' in our schema, or at least a similar notion.
We mention:

> "La formation génétique d'une notion mathématique".

Although this is not completely clear from her article one is tempted to draw the conclusion that she means that "mathématiser une situation se relèvent en dehors de mathématique".
This idea is reinforced by Freudenthal's remarks in a panel discussion in which he states that the goals of teaching mathematics as to be useful can only be reached by starting with students from situations that have to be mathematized.[11] Without mentioning the word mathematizing Hilton states that a genuinly relevant course would have to account for how one chooses a mathematical model to tackle a problem from the real world, how one reasons within the mathematical model, how one checks the results of one's reasoning against the original problem in order to verify the appropriateness of the model, and how one modifies the model in the light of an unsatisfactory fit between theory and practice. [12]
The above description fits well with the general idea on mathematization, but fails to notice the conceptual mathematization that is essential for mathematics A.

Lesh, Landau and Hamilton [13] make a strong plea, not only for mathematization, but more specifically for conceptual mathematization. In their opinion, based on research, applications (and problem solving) are unlikely to be fully accepted in the school mathematics curriculum unless teachers and other practitioners are convinced that they play an important role in the acquisition of the basic mathematical ideas. The authors believe that applications should not be reserved for consideration only *after* learning has

occurred (mathematization in our schema); they can and should be used as a context *within* which the learning of mathematical ideas takes place. This approach will contribute significantly to both the meaningfulness and the usability of mathematical ideas.
Treffers [14] and Treffers and Goffree [15] describe mathematizing as an organizing activity according to which acquired knowledge and skills are used to discover unknown regularities, relations and structures. We will start from this 'definition' in describing the process of mathematization in Mathematics A in the next chapter.

We will do this in the following way:
- first a general discussion about mathematization in Mathematics A; (2.2.2)
- followed by a discussion about conceptual mathematization seen in a wider in instruction theoretical context. (2.2.3)

2.2.2 Mathematization in Mathematics A

Mathematizing is an organizing and structuring *activity* according to which acquired knowledge and skills are used to discover unknown regularities, relations and structures.
We may distinguish two components in mathematization, according to Treffers and Goffree: the horizontal and vertical components. [15] First we can identify that part of mathematization aimed at transferring the problem to a mathematically stated problem. Via schematizing and visualizing we try to discover regularities and relations, for which it is necessary to identify the specific mathematics in a general context.
Activities containing a strong horizontal component are:
- identifying the specific mathematics in a general context;
- schematizing;
- formulating and visualizing a problem in different ways;
- discovering relations;
- discovering regularities;
- recognizing isomorphic aspects in different problems;
- transferring a real world problem to a mathematical problem;
- transferring a real world problem to a known mathematical model.

As soon as the problem has been transferred to a more or less mathematical problem this problem can be attacked and treated with mathematical tools: the mathematical processing and refurbishing of the real world problem transformed into mathematics.

Some activities containing a strong vertical component are:
- representing a relation in a formula;
- proving regularities;
- refining and adjusting models;
- using different models;
- combining and integrating models;
- formulating a new mathematical concept;
- generalizing.

Generalizing may be seen as the top level of vertical mathematization. We mean, with Hilton, that when we are reasoning within the mathematical model we may feel compelled to construct a new mathematical model which embeds our original model in a more abstract conceptual way. [16]

A division of the clusters of activities of mathematization into two distinct components is rather arbitrary, to say the least. As we will show in our examples, the two components are always intertwined. But a bipartition in a descriptive sense can be useful, not only to describe mathematization more clearly in concrete examples, but also to discriminate between different methodologies, as will be shown later.

As we indicated before in our schema of the activities in Mathematics A, mathematization always goes together with reflection, as indicated by the spirals in the schema. This reflection must take place in all phases of mathematization. The student must reflect on his personal process of mathematization, discuss his activities with other students, must evaluate the products of his mathematization, and interpret the result.
Horizontal and vertical mathematizing comes about through students *actions* and their *reflections* on their actions.
Thus mathematization does not go without reflection, as indicated in our schema. But mathematization can only be efficient if it is realized in interactive instruction; that is, instruction where there is the opportunity to discuss, to consult, to cooperate.
The interactive aspect will be discussed in our examples.

Before giving an example of the process of mathematization in Mathematics A we wish to make a remark about the distinction *horizontal* vs. *vertical* mathematization.
In practice we will seldom meet a situation where first the process of horizontal mathematization is carried out, followed by exclusively vertical mathematization.

So we will never use a model picture like (fig. II.12):

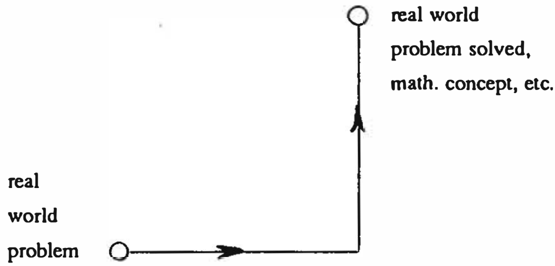


fig. II.12

We may expect quite different pictures in classroom situations, depending on the real world situation perceived by the students and their skills, interaction, problem solving abilities (fig. II.13).

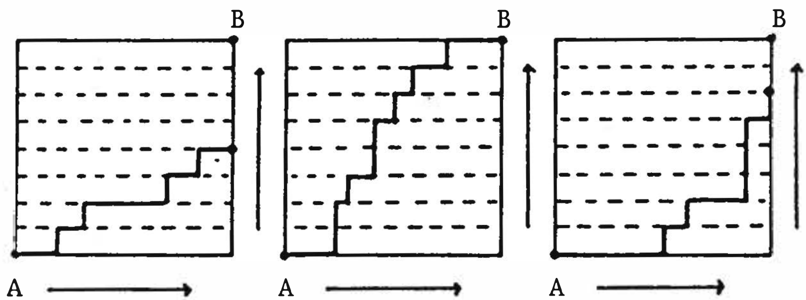


fig. II.13

One cannot expect students to travel the whole route from A to B. In the vast majority of exercises the students will be satisfied with a short part only. And the route may include primarily horizontal steps and few vertical ones, or vice versa.
In fact, the following example is no representative of exercises in Mathematics A, but one that highlights the power of mathematization as carried out by both teachers and students. And it gives us the possibility of observing clearly horizontal and vertical mathematization phases. Later on, when we discuss tests for Mathematics A, we will encounter similar exercises.

The example displays another specific characteristic feature of Mathematics A. In fact it is the process (of mathematization) we are interested in, rather than the product. Learning is best conceived as a process, not in terms of outcomes, as Kolb puts it. [17]
In the example that follows the answer the product is given beforehand. We

agree with Lesh et al, that many of the problems should be designed in a way that the critical solution stages would be 'non-answer-giving' stages. For example, in many realistic problem situations tryout and refinement are crucial to the process of problem formulation, therefore, in many cases, the goal is to make non-mathematical decisions, comparisons or evaluations using mathematics as a tool rather than producing a numerical 'answer'.

Let us now turn to our example. It starts with a page from the book 'Rats' by Maarten 't Hart, a well known Dutch author/biologist.

GROWTH OF RAT POPULATIONS

As regards the progeny of one pair of rats during one year the numbers given vary considerably. In the next chapter I shall discuss the scanty information supplied by research into the fertility of rats in nature, but at this point it might be interesting to estimate the number of offspring produced by one pair under ideal conditions. My estimate will be based on the following data. The average number of young produced at a birth is six; three out of those six are females. The period of gestation is twenty-one days; lactation also lasts twenty-one days. However, a female may already conceive again during lactation, she may even conceive again on the very day she has dropped her young. To simplify matters, let the number of days between one litter and the next be forty. If then a female drops six young on the first of january, she will be able to produce another six forty days later. The females from the first litter of six will be able to produce offspring themselves after a hundred and twenty days. Assuming there will always be three females in every litter of six, the total number of rats will be 1808 by the next first of january, the original pair included.
This number is of course entire ficticious. There will be deaths; mothers may reject their young; sometimes females are not in heat for a long time. Nevertheless, this number gives us some idea of the host of rats that may come into being in one single year.

The question asked to the teachers as well as students was:

"Is the conclusion that there will be 1808 rats at the end of the year correct?"

The teachers concerned were teachers who were taking part in the teacher-training courses of the Hewet project. These teachers had no previous experience in teaching mathematics A, other than from the course. The students usually encounter this problem in their last year.
The general trend was that only a very few teachers were able to prove it.

By a very few we mean less than 20% at best, in a time span of 20 or 30 minutes.
With regard to the students, no overall impression is available of their ability to solve the problem. But considering information from some schools one is tempted to say that certain students did very well on this problem.
Results also depend, of course, on the conditions: in the classroom, with a limited amount of time, students find it very difficult to solve, let alone to schematize the problem. But with no time limit, for instance by giving the problem as homework, some fine results were obtained. This gives once more an indication that such process-oriented activities are not well suited for testing by means of restricted-time written tests.
The first impression, when observing teachers as well as students, is that they are overwhelmed by the amount of information, all relevant. So the main difficulty was to select and organize information that is 'useful' in order to find a schema, as the next phase of mathematization.
Although for some teachers the biological context was familiar - the rabbit problem and Fibonacci numbers - they were unable to use this knowledge effectively. So most of the vertical of horizontal mathematization aspects the transformation of a problem to a known model did not emerge initially. But the author 't Hart has already structured his text:

> "For that I use the following facts."

and

> "The average number of young produced at a birth is six; three out of those six are females."

This text is experienced as relevant from a mathematical point of view. So in a number of cases we see schemas appear such as (fig. II.14a,b):

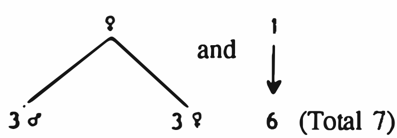


fig. II.14a

Or, more complete:

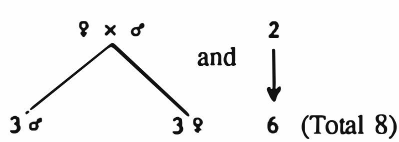


fig. II.14b

This, then, is their first activity.
The activities involved were horizontal: extracting the relevant mathematics, and schematizing and visualizing the problem, or at least part of it.

The next part of the text was more difficult to mathematize: what is relevant for solving our problem?

"The period of gestation is twenty one days; lactation also lasts twenty one days.
However, a female may already conceive again during lactation, she may even conceive again on the very day she has dropped her young."

A more concrete start for mathematizing is the following sentence:

"To simplify matters, let the number of days between one litter and the next be forty."
"If then a female drops six young on the first of January, she will be able to produce another six forty days later."

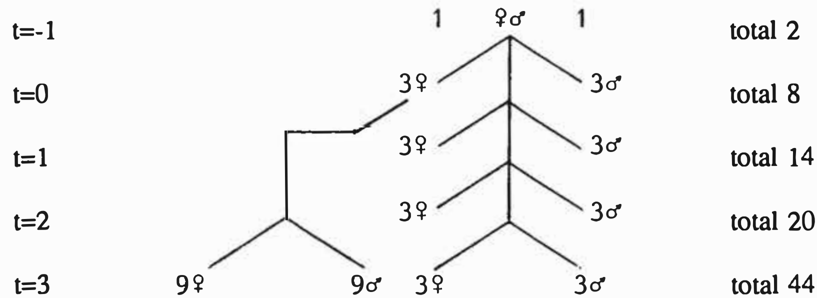
This leads to schemas like:

t=0	2 parents + 6 young ones:	8 total
	↓ ↓	
t=1	2 parents + 6 + 6 young ones:	14 total
	↓ ↓ ↓	
t=2	2 parcnts + 6 + 6 + 6 young ones:	20 total

It is at this stage that many students and teachers run into problems.
At t=3 the six young ones from t=0 start to reproduce as well, while the original female continues reproducing.
The need for schematizing, for discovering regularities and relations becomes more urgent.
One possibility:

t=-1	2 p	total 2
t=0	2 p + 6 y	total 8
t=1	2 p + 2. 6 y	total 14
t=2	2 p + 3.6 y	total 20
t=3	2 p + 4. 6y + 3. 6y	total 44
t=4	2 p + 5. 6y + 3. 6y + 2. 3. 6	total 86

Or, a more visual one:



In both cases the chosen road leads to a very complex structure in which the learner gets lost or makes mistakes.

Two students offered original ways of solving the problem.
The first one is a schema that surprises by its simplicity (fig. II.15):

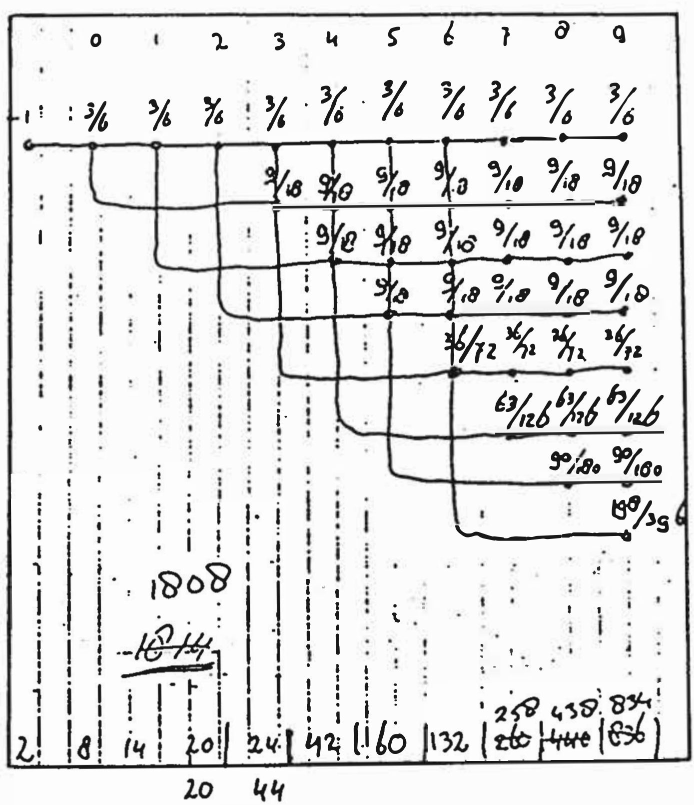


fig. II.15

The entries in the schema are the new borns only. $\frac{36}{72}$ means: 72 newborns of which 36 are women. At the same time it is clear where those 72 young ones are coming from. The schema is so simple, indeed, that one wonders why this solution had not been found earlier.

Another solution - not a teacher's as is easily seen but a student's. She wanted at any moment to be able to trace back where the young came from. This way of notation - her own invention - was perfectly clear to her, but she was initially unable to explain her notation convincingly to her mates.
Let us first present her solution (fig. II.16):

1 worp : 6 jongen : 3 vrouw + 3 man
draagtijd 21 dagen + 21 dagen zogen
periode tussen 2 bevallingen = 40 dagen
jong vrouwtje na 120 dagen (3 x 40 dagen) jongen voortbrengen.

t = -1   man + vrouw (0) = 2
3 meisjes x 6 = 18 jongen
t = 0   2 + 6 jongen (0) (1) = 8   (0) = van 0   (1) = nr. ugen
40 t = 1   8 + 6 jongen (0) (2) = 14
80 t = 2   14 + 6 jongen (0) (3) = 20   18 → 9 meisjes x 6 = 54 jongen
120 t = 3   20 + 6 jongen (0) (4) + (3 x 6 jongen (1) (5)) = 44
160 t = 4   44 + 6(0)(6) + 18(1)(7) + 18(2)(8) = 86
200 t = 5   86 + 6(0)(9) + 18(1)(10) + 18(2)(11) + 18(3)(12) = 146
240 t = 6   146 + 6(0)(13) + 18(1)(14) + 18(2)(15) + 18(3)(16) + 18(4)(17) + 54(5)(18) = 278
27 meisjes x 6 = 162 jongen
132
280 t = 7   278 + 132 + 18(6)(19) + 54(7)(20) + 54(8)(21) = 536
258
320 t = 8   536 + 258 + 18(9)(22) + 54(10)(23) + 54(11)(24) + 54(12)(25) = 974
430
360 t = 9   974 + 430 + 18(13)(26) + 54(14)(27) + 54(15)(28)
+ ...)(29) + 54(17)(30) + 162(18)(31) = 1808

fig. II.16

She explained her notation as follows:
Firstly man and wife are labelled (0).
The first young from these parents are labelled (0)(1).
The second litter from these parents is labelled (0)(2).
So at t=4 we see:

44 + 6(0)(6) + 18(1)(7) + 18(2)(8) = 86

This means:

44 from t=3

plus six newborns from the original parents which are labelled (0); these six come from the 6th birth this year;

plus eighteen newborns from parents (1); being the 7th birth this year;

plus eighteen newborns from parents (2); being the 8th birth.

Actual her mental construction and schematization is the same as the one from the previous solution:
Her statement:

t=4 : (44 +) 6(0)(6) + 18(1)(7) + 18(2)(8)

is similar to (fig. II.17):

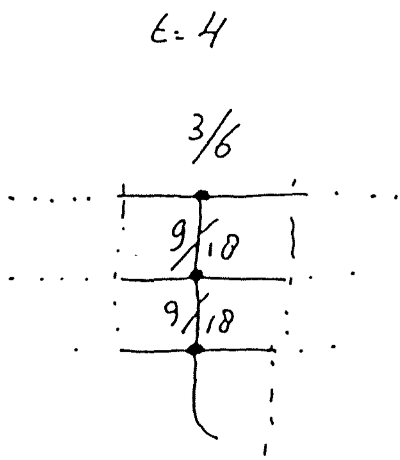


fig. II.17

Both solutions show a high ability to mathematize the problem, the first more visually than the second. Both mathematizings have strong horizontal components, but the vertical components are not negligable: both solutions show a high degree of organization and structure.

Let us now look at one of the teacher's solutions which already shows some shortcuts in the schemas that were presented as their solution.

The first one (fig. II.18):

t:	totaal aantal ratten:
0	2 + 6 = 8
1	2 + 2·6 = 14
2	2 + 3·6 = 20
3	2 + 4·6 + 1·18 = 44
4	2 + 5·6 + (1+2)·18 = 86
5	2 + 6·6 + (1+2+3)·18 = 146
6	2 + 7·6 + (1+2+3+4)·18 + 54 = 278
7	2 + 8·6 + (1+ ... +5)·18 + (1+3)·54 = 536
8	2 + 9·6 + (1+ ... +6)·18 + (1+3+6)·54 = 974
9	2 + 10·6 + (1+ ... +7)·18 + (1+3+6+10)·54 + 162 = 1808

fig. II.18

Progressive schematizing leads another teacher to the following solution (fig. II.19):

t	-1	0	1	2	3	4	5	6	7	8	9
N	2	6	6	6	½·8·6= 24	½·14·6= 44	½·20·6= 60	½·44·6= 132	½·86·6= 258	½·146·6= 438	½·278·6= 834
T	2	8	14	20	44	86	146	278	536	974	1808

fig. II.19

These two solutions are already so strictly organized and schematized that the stage of presenting the relation in a formula is almost reached.
All four solutions of the problem "Is 1808 correct?" have strong horizontal components of mathematizing: although the problem was already pre-organized by the author 't Hart a good deal of "identifying the specific mathematics" remained necessary.
The problem was reformulated and visualized in different ways. Relations and regularities were discovered. But no doubt vertical components are

visible as well: regularities are proven, first steps are taken representing a relation in a formula.
The schema in these cases would look like (fig. II.20):

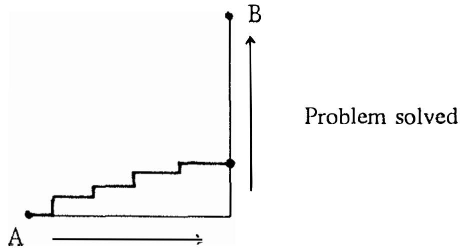


fig. II.20

The problem, as stated, was solved. But teachers especially were not satisfied until they had found a formula, or some kind of generalization. And the booklet requested students to do the same.
The following solution (a teacher's) leads almost directly to the formula (this teacher had already read an article [18] on the problem:

$$
\begin{aligned}
A(-1) &= 2\\
A(0) &= A(-1) + 6\\
A(1) &= A(0) + 6\\
A(2) &= A(1) + 6\\
A(3) &= A(2) + 4 \cdot 6\\
A(4) &= A(3) + 7 \cdot 6\\
&\vdots
\end{aligned}
$$

The only missing link is now to replace:

$$A(3) = A(2) + 4 \cdot 6$$

by

$$A(3) = A(2) + \frac{1}{2} \cdot 6 \cdot A(0)$$

which gives:

$$
\begin{aligned}
A_{n+3} &= A_{n+2} + 3 \cdot A_n = 2;\\
A_0 &= 8\\
A_1 &= 14
\end{aligned}
$$

Now we have found a formula representing the growth - a typically vertical component of the mathematization process.

So a picture of the process so far may look like (fig. II.21):

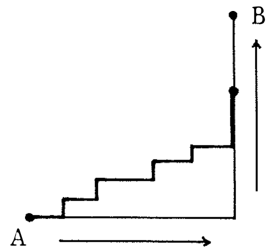


fig. II.21

Some teachers - albeit few - had the feeling that another way of mathematizing and modelling was possible if not preferable. This feeling is prompted by the important part played by the fact of the period of forty days as well as by the two periods, that of 40 and that of 120 days. It suggests a 'natural' partitioning of the population of rats into three age groups. Wouldn't it be advisable to avail oneself with population projection matrices (Lesliematrices)? Was the problem not isomorphic to other ones encountered in the booklet with beetles and hooded seals?
The start was a division into newborns (0-40 days), middle ones (40-80 days) and old ones (> 80 days).
Starting from the schema on page 52 we get:

time	-1	0	1	2	3
newborns	0	6	6	6	24
middle	0	0	6	6	6
old	2	2	2	8	14

with the structure:

newborns	0	6	6	6	24
middle	0	0	6	6	6
old	2	2	2	8	14

which, in turn, can be represented in a graph (fig. II.22):

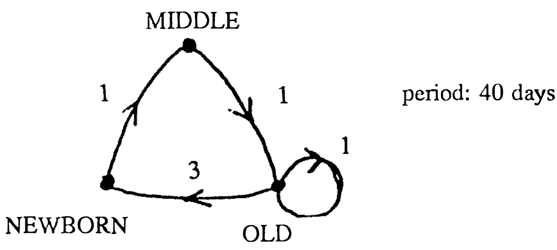


fig. II.22

The meaning of the graph:

a. Newborns are certain to promote to the middle group (100% = 1).

b. The middle aged rats are certain to promote to the old group.

c. The old rats are certain not to die (!)

d. The old rats reproduce three females per period.

So the "1"'s represent survival-probabilities, and the "3" the reproduction rate.

This visual representation can easily be translated to a matrix:

$$\begin{array}{ccc} & \text{from} & \\ \text{to} & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} & \begin{matrix} n \\ m \\ 0 \end{matrix} \\ & \begin{matrix} n & \quad m & \quad \text{old} \end{matrix} & \end{array}$$

Our starting population of $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ will give us 1808 as the total population after one year.

The calculation can, of course, be done with the aid of a computer. The advantage of this solution is that the structure of the population is visible at all times. So one may ask whether the population pyramid is stable.

Another question that might arise:

"Is the population growth exponential?"

The simplest solution - certainly for the students - is to check whether the growth factor $\frac{A_n}{A_{n-1}}$ is constant.

Or, equally simple, one might notate the growth on logarithmic graph paper. The graph for the first year (see fig. II.23).

Obviously, the first three periods hardly suggest exponential growth. But later on in the year the points fit pretty well into a straight line. So we do find exponential growth after all.

One might even try to guess the growth factor from the slope of this line. Most students, however, prefer to calculate it.

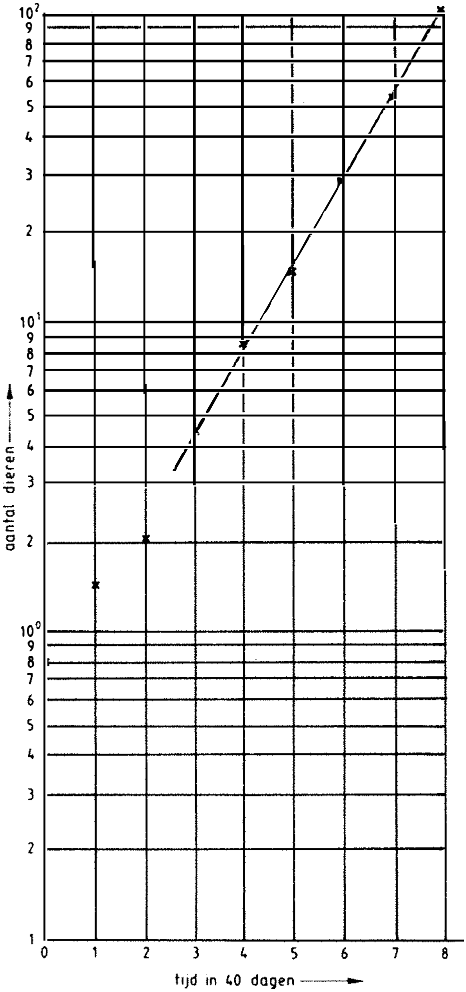


fig. II.23

One student, who wanted a good approximation, felt he had to continue a little farther than one year.

He used the computer to find:

t = 18	494342		
		}	g = 1,863
t = 19	921362		
		}	g = 1,864
t = 20	1717598		
		}	g = 1,863
t = 21	3200624		
		}	g = 1,863
t = 22	5964710		
		}	g = 1,863
t = 23	11117504		
		}	g = 1,863
t = 24	20719376		

So, clearly, the growth-factor is 1,863.
The majority of students and teachers, restricting themselves to the first year only, suggested a growth-factor: g = 1,89.
Proposing a formula for this exponential growth is no problem:
One suggestion:

$A_n = 44.1.89^{n-3} \qquad n = 3, 4, ...$

In classroom practice, this is mostly the point to terminate, apart from necessary reflection and discussion. But some teachers push the process of mathematization as far as conceptualizing eigenvalues and eigenvector.
Others continue with the difference equation:

$A_{n+3} - A_{n+2} - 3A_n = 0$

and try to solve it explicitly. Using the computer again one can find, using the Newton-Raphson method, that

$A_n = \lambda^n$ with $\lambda \cong 1.8637$

presents a solution of the difference equation, as well as the dominant eigenvalue of the Lesliematrix; the characteristic equation being

$\lambda^3 - \lambda^2 - 3 = 0.$

Finally, we show the solution of a teacher who used Pascal's Triangle to find a formula for the growth of the rat population. (See next page).
This solution is an example of strong vertical mathematization.

From our examples it will be clear that, although we 'solved' our initial problem already at an early stage, there was still a lot of vertical mathematization ahead. We may recall that:
- representing a relation in a formula;
- using different models;
- combining and integrating models;

$$A = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

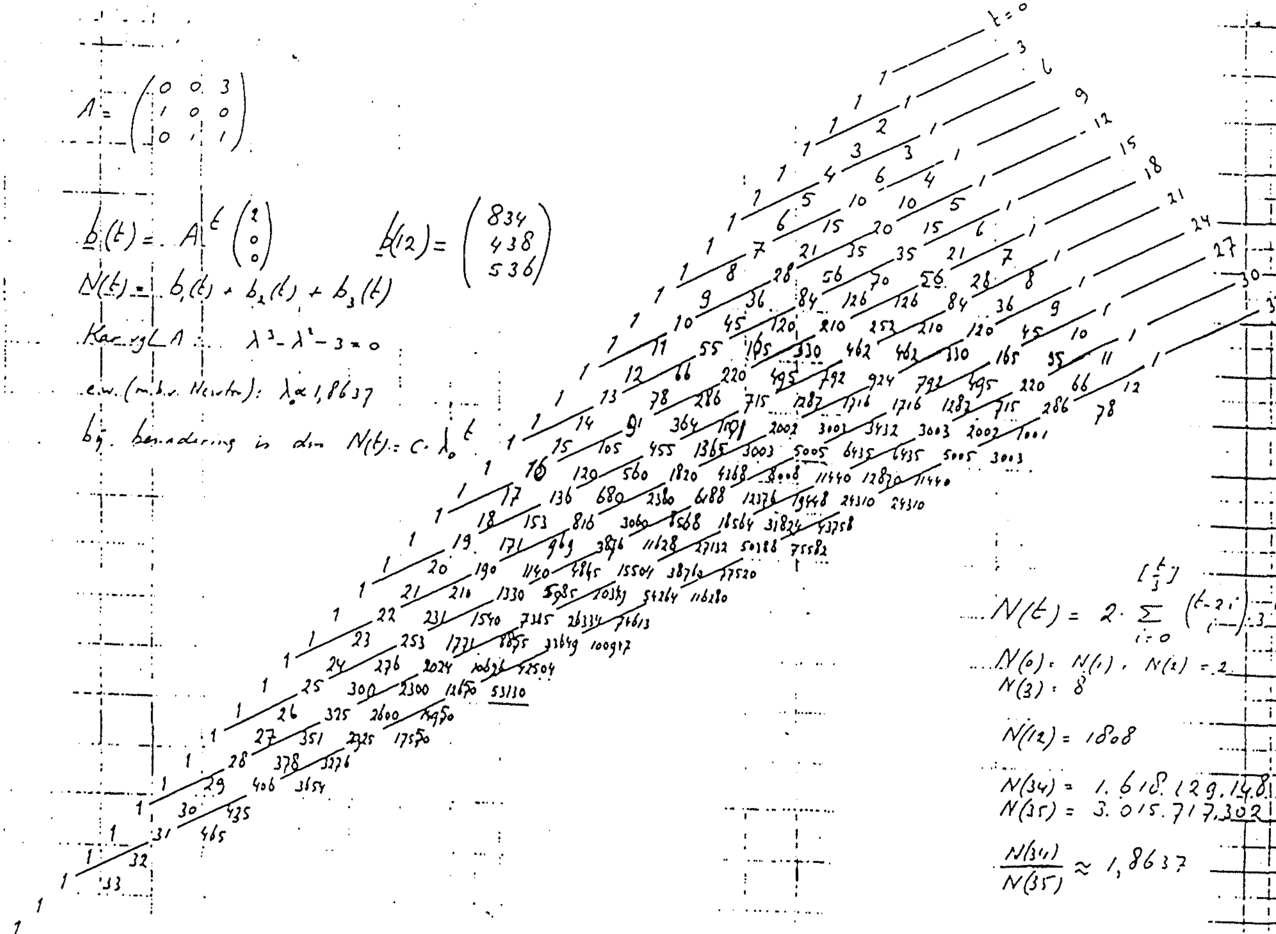
$$\underline{b}(t) = A^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \underline{b}(12) = \begin{pmatrix} 834 \\ 438 \\ 536 \end{pmatrix}$$

$N(t) = b_1(t) + b_2(t) + b_3(t)$

Kar. vgl A: $\lambda^3 - \lambda^2 - 3 = 0$

e.w. (m.b.v. Newton): $\lambda_0 \approx 1,8637$

b.g. benadering is dan $N(t) \approx c \cdot \lambda_0^{\,t}$



$$N(t) = 2 \cdot \sum_{i=0}^{[\frac{t}{3}]} \binom{t-2i}{i} \cdot 3^i \quad (t \in \mathbb{N})$$

$N(0) = N(1) = N(2) = 2$
$N(3) = 8$

$N(12) = 1808$

$N(34) = 1.618.129.148$
$N(35) = 3.015.717.302$

$$\frac{N(34)}{N(35)} \approx 1,8637$$

- generalization.

are all vertical mathematizing.

It came as a surprise to some teachers as well as students that the original Dutch text by Maarten 't Hart left more interpretations open than the one resulting in the 'correct' answer 1808. So, even in this already pre-mathematized problem, organizing the situation gave more than one correct structure. This makes clear that restricted-time written tests are not suited to test mathematizing.

The value of the Leslie matrix as a population projection model has been discussed in the classroom at several opportunities. In general, the entries in the matrix will not remain constant during longer time-intervals. Survival-probabilities will change, and also the reproduction rate will not be constant: as soon as there are too many rats, reproduction will drop sharply, resulting in logistic growth.

In fact, 't Hart had already admitted that he was dealing with an ideal situation: no rat died in the whole year, no young were rejected, mothers gave birth regularly. Nevertheless, he concluded with the remark that idealization somehow reflected the growth of a rat colony.

The question arises whether this is true.

To check this the students were asked to study a more realistic situation that doesn't seem too far away from the ideal one:

- An average of one in each litter of six dies at an early stage.
- Among the 'newborns' some 80% reach 'middle age'; the remaining die.
- Among the 'middle age' some 75% reaches the 'old age' class.
- The 'old-age' rats have an 80% chance of staying alive in that class after forty days.

This leads to a new matrix:

$$M_1 = \begin{pmatrix} 0 & 0 & 2{,}5 \\ 0{,}8 & 0 & 0 \\ 0 & 0{,}75 & 0{,}8 \end{pmatrix}$$

During discussions with teachers at the teacher training course a different model - more realistic, they felt - was presented:

$$M_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0{,}6 & 0 & 0 \\ 0 & 0{,}9 & 0{,}9 \end{pmatrix}$$

During numerous courses teachers were asked to guess to which degree the result from this matrix would differ from the 1808 rats, resulting from:

$$M = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

The guesses varied from 850 to 1800, with most values around 1400-1700. The actual result was calculated by the computer (fig. II.24):

```
                         R E S U L T A A T

                                     * inlezen gegevens *
orde? 3
rij 1    0 0 2
kental 2 0,6
rij 3    0 0.9 0.9
aanvangspopulatie 0 0 2

bevolgingsopbouw herleiden tot percentages? (ja/nee) nee


                                     * afdrukken lesliematrix *
 0.000 0.000 2.000
 0.600 0.000 0.000
 0.000 0.900 0.900
                                     * afdrukken kolommatrix *
    0     0     2
       totaal:      2
druk return en door te gaan, anders stop gevolgd door return
                                     * volgende kolommatrix berekenen *
                                     * afdrukken kolommatrix *
    4     0     2
       totaal:      6

                                     * volgende kolommatrix berekenen *
                                     * afdrukken kolommatrix *
    4     2     2
       totaal:      8

                                     * volgende kol
                                     * afdrukken

                                                        erekenen *
                                                        trix *
   29    12    20
       totaal:     60

                                     * volgende kolommatrix berekenen *
                                     * afdrukken kolommatrix *
   40    17    28
       totaal:     85

                                     * volgende kolommatrix berekenen *
                                     * afdrukken kolommatrix *
   57    24    41
       totaal:    121
```

fig. II.24

The result in this more realistic situation:

only 121 rats.

And even this result looks controversial because it involves fractions of rats rather than whole ones:

$$\begin{pmatrix} 0 & 0 & 2 \\ 0{,}6 & 0 & 0 \\ 0 & 0{,}9 & 0{,}9 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1{,}8 \end{pmatrix}$$

whereas the computer output is $\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

The example of the rats gives students as well as teachers ample opportunities for reflection. Not only during each step of the mathematizing process but certainly also when producing the result. The (own) productions may differ considerably as we have shown. But they all have in common that reflection was necessary before the 'final' version concretized. But there is more: students were asked to explain their solution to the other students and to the teacher. This forced them into another round of reflection in order to be able to act as a teacher. This became especially clear when the girl that had produced the solution presented on page 50 was asked to explain her solution. Initially she was unable to make her point clear to her fellow students. The interaction with and between the students forced her to reflect once again until finally a new way of notation was agreed upon among the students.

These situations of social interaction also play a major role when two or more learners are confronted with the joint task of finding a solution to a problem. [19] The rat problem is sometimes used this way.

As Balacheff states:

> "Pairwork is not only a source of explanation but also a source of confrontation with others. This adds greatly to the dynamics of the activity. Contradictions coming from the partner, due to the fact that they are explained, are more likely to be perceived than contradictions confronting the solitary learner, derived only from the facts. They are also harder to refute than in a conflict resulting from individual and temporary hesitations between two opposing points of view that the solitary learner experiences when confronted with a problem."

Doise and Mugny speak in this situation of a sociocognitive conflict. [20]

Their research shows clearly that inter-individual encounters lead to cognitive progress when sociocognitive conflict occurs during the interaction. The social and cognitive poles are inseparable here, because it must clearly be a matter of conflict between social partners about the ways to resolve the task. In Mathematics A students are often at different cognitive levels - our example gave an indication. This means that they have a greater chance of disagreeing over specific responses that are derived from different schemata.

Another important fact is the major conceptual and perceptual conflict: how can such minor changes in the model of the rats yield such dramatic results. This will lead to heuristic behaviour, and to more motivation. And, returning to our schema of Math A on page 39, the reflection on the student's (or teacher's) real world will lead to an adjustment; the discussion on population control is reopened. Teachers in third world countries feel that mathematics functioning in this way could play a major role in making people aware of the effects and impact of birth control. [21]

Among teachers as well as students there was little doubt that understanding of the concepts of Leslie matrix and exponential growth were reinforced by working through this problem. This is in agreement with ideas of Lesh and Vergnaud and others. Vergnaud's standpoint is that concepts develop gradually through problem solving processes. [22]

Another point in the discussion about the rat problem was the fresh light that was shed on the author's claim of how big a rat population can grow in one year: 1808 rats cause more problems than 121.
This helped to create a more critical attitude towards mathematically oriented presentations. This aspect will be discussed in 2.4.

Finally, students and teachers were surprised to see how much mathematics was involved; the initial problem was soon out of sight and one was occupied with the integration, adjustment and generalization of models: vertical mathematizing.

This section showed how mathematizing can take place in Mathematics A. It is certainly not a perfect example: the problem was already premathematized, as is often the case in the booklets. But it shows how individual the process of mathematizing can be, using all kinds of combinations of horizontal and vertical mathematizing. Attention was paid to the fact that mathematizing without reflection is impossible, that social interaction is essential, that conceptual conflicts can play a motivating role, that the problem solving process may improve the understanding and development of concepts and that this can lead to adjustment of one's perceived real world. And the analysis indicates that achievement testing is a difficult task: restricted-time written tests are not able to operationalize these activities. But before diverting our attention to the problems of testing we will first describe some other aspects of Mathematics A, starting with an important aspect: Conceptual mathematization.

2.2.3 Conceptual Mathematization

> *Thus all human cognition begins with intuitions, proceeds from thence to conceptions, and ends with ideas.*
>
> *Kant.*

We have already indicated earlier what we mean by *conceptual* mathematization. In short, one may describe it as 'mathematizing aimed at developing mathematical concepts'. This activity was already mentioned when we discussed the rat problem. A wider exploration of the rat problem may end up either in the introduction of exponential growth, population projection matrices or eigenvalues.

The concrete real world as a start toward mathematics is by no means a new idea. In 1600 in the Netherlands a book was published on 'Arithmetica oft Rekenconst'. Its contents - which we now should place at primary school and junior-high school level - consisted of a lot of practical arithmetic although its goal was stated as "enseignant l'art à chiffrer". In an article about this book that appeared in 1925, Hallema notices that "practical problems are treated before displaying a general theory. From a methodological and scientific point of view disgusting." [23]
W. Lietzmann, a well-known German didactician of mathematics presented some ideas about mathematics education that seem to bear resemblance to our conceptual mathematization. These ideas, which were not widely accepted at that time, follow from an affirmative answer to the following question:

> "Soll man an den Beginn der Beschäftigung mit dem Funktionsbegriff eine gewissermassen propädeutische Behandlung stellen, wobei man an der Hand empirischer Funktionen mit einer Reihe grundlegender Begriffe vertraut macht?"
>
> (Should we introduce the basic concepts of functions by means of empirical functions?)

One of his arguments is that in this way mathematics has something to offer to everybody and becomes part of general education.
In his booklet *Funktion und Graphische Darstellung* [24] he proposes the following exploration of empirical functions in order to develop *Begriffe* like continuity and increasing functions and rate of change.

During the First World War people who earned less than 9000 Mark received support according to the following graph (fig. II.25):

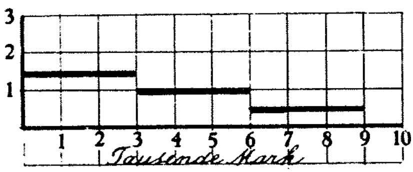


fig. II.25

The amount was M 1500 if one earned less than M 3000, M 1000 for those earning between M 3000 and M 6000 and M 500 for those earning between M 6000 and M 9000.

For the 'man in the street' it is important to know what happens exactly at the discontinuities. To make this clear one may sketch a graph with the *income* plotted along the horizontal axis, and the *income* + *bonus* along the vertical one.
One gets (fig. II.26):

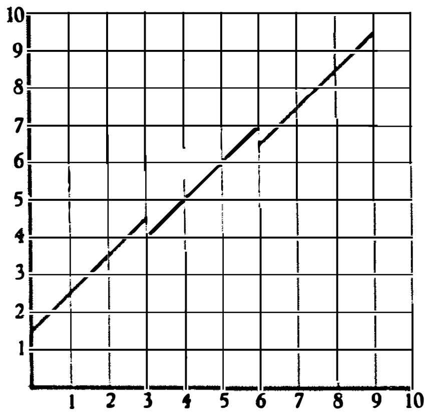


fig. II.26

Obviously the situation represented by this graph is far from ideal.
A possible, simple solution would be (fig. II.27):

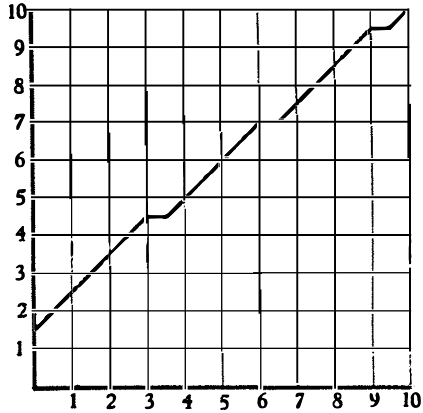


fig. II.27

But even this solution can be improved.
Lietzmann suggests a solution similar to (fig. II.28):

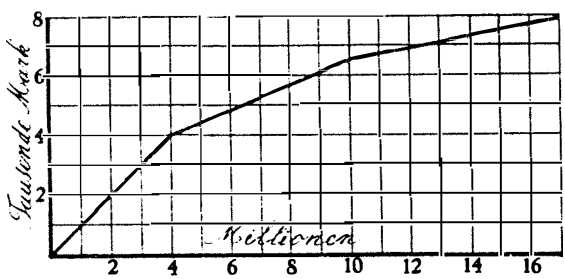


fig. II.28

This solution seems 'fairer'. The *why's* and *how's* are excellently suited for class discussion.

Let us now switch to the Math A program in which the previous example from 1925 could fit immediately. As an example of conceptual mathematization we will describe - in short - how the rules of differentiation are treated.
Hilton presents the way differential calculus is taught to undergraduates as an example of poor mathematics education.
He said:

"Calculus is made to appear as a set of skills - one might almost say tricks - and the student learns those skills and, once he is trained in them, his study of calculus is deemed to have reached the requisite level of maturity.

To make calculus relevant the student is offered examples which are tailor-made; that is to say the examples are designed simply to illustrate the particular skill which the student is currently endeavoring to acquire."[25]

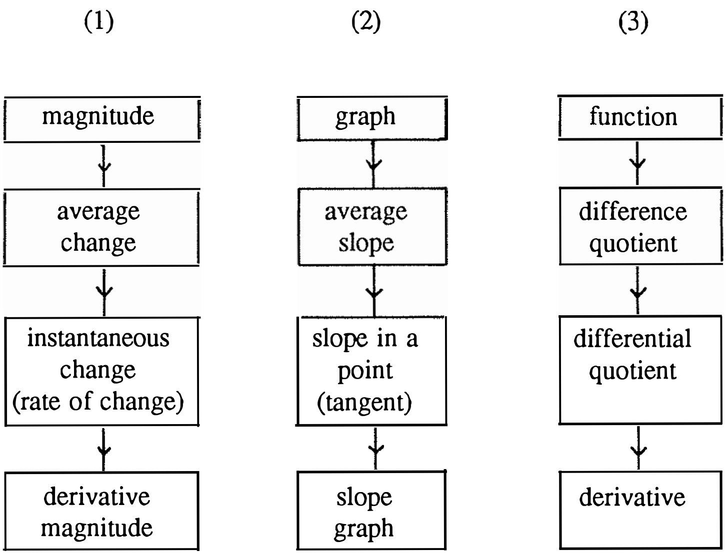
Let us now turn to the way differential calculus is treated in Math A.

From the teacher's guide [26] we quote:

"In the numerous mathematical and didactical tracks leading to differential calculus one may find three distinct main tracks:

- the physics line;
- the geometry line;
- the formal-analytical line.

These tracks have the following characteristics:



In our treatment of differential calculus we have chosen a three-track approach, in which the formal-analytical track will be picked up only at a later moment."

The background philosophy is - completely parallel to Hilton's view - that an early analytical treatment, followed by a rapid algorithmization (rules or 'tricks' for differentiation) will lead to problems when interpretation questions are asked, and the students have no previous insight into nor understanding of the concept of derivative.

So the booklet starts with real world situations which are explored intuitively.

The phenomena investigated are:

- growth of man (height);
- growth and decline of unemployment;
- the circadian temperature cycle;
- acceleration and speed of animals.
- speed and speeding in traffic;

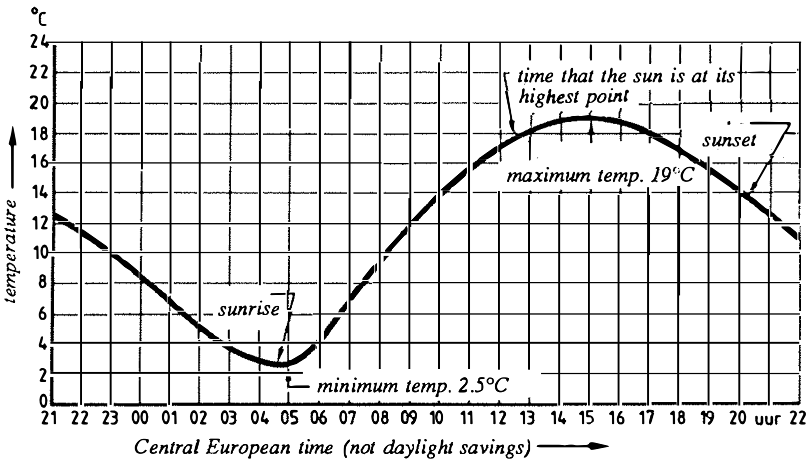
Four of the problems are situated in more or less real situations. The fifth - speed and speeding in traffic - plays in a not so real world. A cartoon - a Dutch classic - is used to show that driving at an average speed of 40 km/h does not necessarily mean that the driver at no time speeded, that is, surpassed the 50 km/h limit. The story is easily transferred to the real world. The cartoon is expected to have a motivating undertone.

The goals of this intuitive exploration of these phenomena are:

- how can a graph show whether a magnitude is increasing or decreasing;
- how can a graph show whether the rate of change of a magnitude at one time is larger or smaller than that at another time;
- how can the rate of change be quantified.

A representative problem from this first explorative phase is the following:

We experience our daily weather through a combination of determining elements, such as the sun's rays, the temperature, wind, humidity and precipitation. Of these we will take a look at the temperature.
Everyone knows from experience that temperature depends strongly on the time of day. Meteorologists speak of the 'daily course': the temperature rises throughout the morning and falls during the afternoon, evening and night. As a rule the temperature is lowest just after sunrise, this is the minimum temperature. The highest temperature (the maximum) is usually reached around 14.00 (2:00 PM).
Figure II.29 shows the temperature during any odd day in May.



Daily course of the air temperature on a clear day in May in the Netherlands.

fig. II.29

Exercises

\> *8 a.* *At what time on that day was it the warmest?*
And the coldest?

b. *How do you explain the fact that the warmest moment of the day takes place after the sun has reached its highest point?*

\> *9 a.* *The temperature T (in °C) is a function of the time t (in hours).*
Calculate $\frac{\Delta T}{\Delta t}$ *for the period (interval) from 5:00 AM until 3:00 PM.*

b. *Calculate this quotient as well for the time between midnight and 5:00 AM.*

c. *At what time of day did the temperature rise the fastest?*
And at approximately what speed (in degrees per hour) did the temperature rise?

This exercise is rather straightforward. The continuity of the graph is self-evident, negative differences are natural.
Even question c. about the point of inflexion offers no problems for the student at this early moment: the context gives the students the support needed.
A quite different case and an example of mathematization at micro-level is the problem of the horse and the cheetah (fig. II.30).

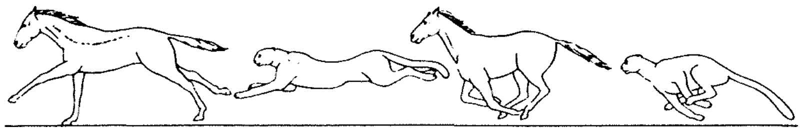


fig. II.30

Two runners. An animal's speed depends partly on the length of its stride which is why runners such as horses have long legs. A cheetah's legs and body are both shorter than a horse's, but the more flexible joints of its fore and hindlegs enable it to take longer strides. The cheetah can quickly reach speeds of more than 110 kilometers per hour while a horse's top speed lies around 70 kilometers per hour.

FAST-FOOTED HUNTERS AND HERBIVORES

Some animals which dwell on grassy plains are safeguarded against attacks by their large size (such as elephants) and others are so small that they protect themselves by burying into the ground. Many animals, however, must count on their speed to escape their enemies.

An animal's speed depends on its size and the frequency of its strides. The tarsal (foot) bone of animals of the horse family is lengthened and the number of toes reduced to one because of the fact that one thick bone is stronger than a number of thin ones. This single toe is surrounded by a solid hoof which protects the bone against jolts while the animal is galloping over hard ground.

The powerful leg muscles are joined together at the top of the leg so that just a slight muscle movement at that point can freely move the slim lower leg.

The fastest sprinter in the world is the cheetah. Its legs are shorter than those of a horse but it can reach a speed of more than 110 kilometers per hour in 17 seconds and maintain that speed for more than 450 meters.

The cheetah tires easily, however, while a horse, whose top speed is 70 km/hour can maintain a speed of 50 km/hour for more than 6 km.

From: The Reader's Digest Animal Atlas.

Exercise

> 15 *A cheetah is awakened from its afternoon nap by horse's hooves. At the moment that the cheetah decides to give chase, the horse has the lead by 200 meters.*
The horse, travelling at its top speed, still has plenty of energy. Taking the above data on the running powers of cheetah and horse into consideration, will the cheetah catch the horse?

You can assume that the cheetah will need around 300 meters to reach its top speed. Solve this problem by using graphs. Let the vertical axis represent distance and the horizontal axis time. Use, for instance, millimeter paper: 1 cm ≅ 100 cm and 1 cm ≅ 5s.

As Freudenthal [27] lamented: "This story of the cheetah seems rather complex. There is an abundance of numbers ... and nowhere an indication of which operation to perform on which numbers.
Indeed, is there anything like a solution?
The only question to be answered is: "Does the cheetah catch (up with) the horse? It is 'yes' or 'no' - no numbers, no kilometers, no seconds. Is that a solution in the usual sense?"

Those who know Freudenthal's ideas on mathematics education know the answer: "This *is* mathematics: look for the relevant information, and for a solving strategy."
We leave it to the reader to 'solve' the problem, either in the indicated graphic way, or in the '*distance = speed • time*' way, or however is desired.

The next stage of conceptual mathematization is the study, qualitatively and quantitatively, of slopes. Obviously this happens on the geometrical track.
The notion of 'slope-function' is introduced, and 'slope-functions' are drawn by measuring the slope of different points in a geometrical way.
The next example illustrates quite well the activities that take place in this chapter.

Figure 8 shows the profile of a mountain slope, accessible to skiers.

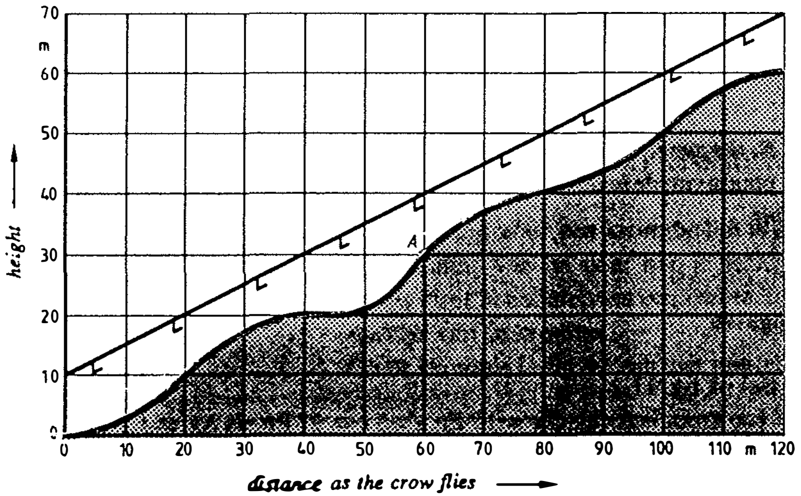


Fig. 10

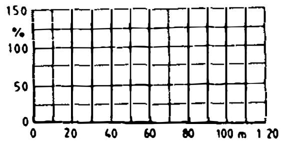
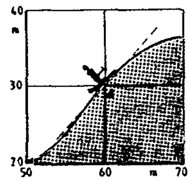


Fig. 9



Exercises

> 20 a. *The steepness of the slope varies, but the chair-lift maintains a constant gradient. How steep?*

b. *What is the 'average gradient' of the profile?*

c. *Where is the slope the steepest?*

> 21 *At point A - 60 meters in a straight line from the lowest point - a skier is heading downhill (fig. 9).*
How can you measure in the illustration how steep the slope is at that spot?

> 22 a. *Measure in fig. 8 the gradient at the points 0, 20, 40, 60, 80, 100 and 120 meters in a straight line.*

b. *Draw the graph of the gradient percentage as a function of the distance in a straight line (in meters), calculated from the lowest point (see also fig. 10).*

> 23 *The skier wishes to descend the slope without ever having to surmount a gradient of more than 50%.*
How can he do this?

After the circumstantial way of drawing the graph of the 'slope-function' the need arises for a more economical description of the function. So the next - natural - step in the process is:

"How can we quantify the slope at any point?"

This is done in the classical context of the free fall. The rules for differentiation of elementary functions are 'discovered' or 're-invented' before the formal rules are finally given and proved.

Looking back one notices how, after the intuitive exploration of the phenomena of increase and decrease, the need for quantifying the rate of change arose. Average and instantaneous speed were discussed: a context familiar to the students.

The geometrical approach led to global slope functions, circumstantially obtained and lacking precision. So a more quantative way of describing the slope function was 'invented': the derivative.

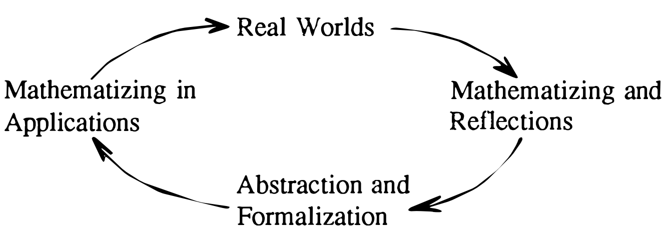
Finally, the students embarked on the formal-analytical approach. The process of conceptual mathematization ends up with the differentiation rules, and their formal proofs.

As indicated before, the calculus stream starts with the booklet *Differentiation 1* for the 10th grade, followed by *Differentiation 2* for the 11th grade. Not all schools, however, used *Differentiation 1* in the 10th grade, but, rather, the 'usual' books. This leads to some problems in the 11th grade. For the students there seemed to be no connection between the 'usual' books and the booklet *Differentiation 2*.
They had missed the conceptual mathematizing phase. At the teacher's request, a special very compact booklet *'Slope-functions'* was developed in order to offer these students a crash-course before starting the booklet *Differentiation 2*. This incident indicates clearly how essential conceptual mathematizing is in Mathematics A.

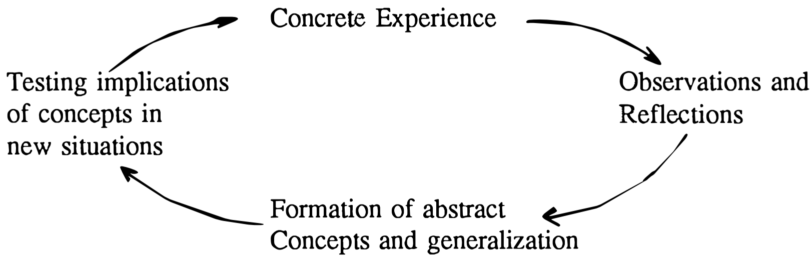
2.2.4 Conceptual Mathematization; Some theoretical remarks

At the time when the experimental material was developed, instruction theory concepts existed only as implicit in concrete student material. Only as the project advanced, and afterwards was a theoretical framework constructed. Admittedly it is a hard thing to put the philosophy behind mathematics A into a theoretical framework, partly because of the availability of a number seemingly valuable theoretical frameworks.

Firstly, we would like to mention the iterative character of mathematics A. If we adjust our basic schema of page 39 slightly we may describe the learning cycle for Mathematics A in the following way:



In this way the learning cycle shows a remarkable similarity to the Experimental Learning Model of Lewin [28]:



Two aspects of this learning model are particularly noteworthy:
Firstly its emphasis on concrete experience to validate and test abstract concepts: in Mathematics A this is the phase of applied mathematization in the problem solving process.
Secondly, the feedback principle in the process. Lewin used the concept of feedback to describe a social learning and problem-solving process that generates valid information to assess deviations from desired goals.
In a more recent study (1984) Kolb adapts this Lewinian model and compares it with Dewey's Model of Experimental Learning and Piaget's Model of Learning and Cognitive Development. [29]
In his opinion all the models suggest the idea that learning is by its very nature a tension- and conflict-filled process. New knowledge, skills, or attitudes are achieved through confrontation among four modes of experimental learning. Learners need four different kinds of abilities - *concrete experience, reflective observation, abstract conceptualization* and *active experimentation.* This means that they must be able to involve themselves in new experiences and to reflect on and observe their experiences from many perspectives.

It is clear that we can find similar abilities in Mathematics A - our previous section made this clear. The weakest link in the cycle seems to be the active experimentation. Students do work with real-world problems like the rat problem. But it seems worthwhile to consider measures to improve this link. One way to achieve this would be to have students make more productions - not only mental contributions. As we will point out later, when describing the results and products of alternative tasks, this production seems to have a very beneficial aspect on the learning process. This point is also emphasized by Treffers. [30] He stresses the fact that by producing simple, moderate and complex problems, the student reflects on the path he himself has taken in his learning process and, at the same time, anticipates its continuation.
We conclude this part of our framework with Kolb's definition of learning:

Learning is the process whereby knowledge is created through the transformation of experience.

We note the emphasis on the *process* as opposed to *content* or *outcomes.* Moreover, that knowledge is a transformation process, being *continuously created* and recreated, not an independent entity to be acquired or transmitted.

Let us turn now to a description of a theoretical framework at micro-level. Here we will lean heavily on the framework for instruction theory as developed by Treffers. [30] This should come as no surprise as both the Wiskobas program (described by Treffers) and the Hewet project find their roots in I.O.W.O. This Wiskobas project ran from 1971-1980 and was aiming at improving arithmetic/mathematics education at primary school. In 1975 the group published an experimental curriculum model for mathematics instruction at primary school. Up until 1985 the curriculum was elaborated upon and concretized in a number of commercially available textbooks: more than 50% of these use Wiskobas ideas. [31] In 1986 the framework for instruction theory was published. For both Wiskobas and the Hewet project the framework is constructed *a posteriori.*

The theoretical concepts were only visible during the projects by paradigms of teaching/learning units. Gradually the concepts became clear by observing the materials in action, by discussion and on reflection. In fact, we went through Lewin's learning cycle. Although the level theory of Van Hiele had hardly played any role in the Hewet or Wiskobas group we would still like to start by mentioning his theory.

According to Van Hiele, the process of learning proceeds through three levels: [32]

1. A pupil reaches the first level of thinking as soon as he can manipulate the known characteristics of a pattern that is familiar to him.
2. As soon as he learns to manipulate the interrelatedness of the characteristics he will have reached the second level.
3. He will reach the third level of thinking when he starts manipulating the intrinsic characteristics of relations.

Traditional instruction is inclined to start at the second or third level. This is not the case in the Mathematics A curriculum. Its aim is to have the final level emerge *during* the process of learning and from concrete situations. So, in our terminology, we should start with a phenomenological exploration of real appearances of mathematical concepts and structures on the first level and proceed slowly from there to the formal operations on the second level and then to the third.

Treffers correctly points out that Van Hiele in his theoretical work, and in the resulting textbooks, did not sufficiently answer the following two questions:
- How should we concretize the phenomenological exploration?
- What didactical actions are needed to help students to proceed as efficiently as possible from one level to the next?

Freudenthal's didactical phenomenology [33] helps to answer the first question:

"What a didactical phenomenology can do is prepare the following approach: starting from those phenomena that beg to be organized and from that starting point teaching the learner to manipulate these means of organizing.

In order to teach groups, rather than starting from the group concept and looking around for material that concretizes this concept, one shall look first for phenomena that might compel the learner to constitute the mental object that is being mathematized by the group concept."

By this phenomenological orientation one may answer the first question: Freudenthal's prescription to use reality as a source for mathematization, together with the macro-structure according to the three Van Hiele levels, give a first idea for framework of instruction theory.

The answer to the second question:

"What didactical actions are needed to help students to proceed as effectively as possible from one level to the next?"

must be looked upon in the light of the preceding sections on mathematizing. But first we must mention that Freudenthal does not distinguish levels in the learning process which are attained by reflection and recursion in the same way as does Van Hiele. Also, in Mathematics A the Van Hiele levels are hardly visible or not at all. There is no rigid tripartition. There is, however, an unrestricted progression according to micro-levels which are only relatively delimited against each other. Treffers subscribes to this view, too, as 'progressive mathematizing'.

Progressive mathematizing as understood by Wiskobas, is inspired by five educational tenets which are logically related to the realistic starting point of the level theory and the didactical phenomenology.

1. The phenomenological exploration or intuitive orientation in a real context.
2. 'Vertical' instruments such as models, schemas, diagrams and symbols are developed, used, offered and explored; in order to bridge the gap between the first level (intuitive, informal, too narrowly context-bound and the

second (reflective, formal) and to make the 'jump' to the third level.
As we have seen in our examples, the word 'jump' is inappropriate: the process is a chain of horizontal and vertical activities; one may work at different levels at the same time.
3. The constructive element is very important during the process of mathematization: the students follow different routes, find solutions at different levels, construct their own concepts. This becomes evident as soon as we have the students compose essays.
4. There is strong interaction at all times; students are confronted with different constructions and productions, which may influence their own ones in a positive way.
5. A considerable amount of intertwining of learning strands.

All of these principles become more concrete when looking back at our examples. More examples will follow in the next chapters.

2.3 The role of the context

The Mathematics A curriculum is distinguished from many others by the dominating place occupied by context problems, serving both as a source for conceptual mathematization and as a field for application of mathematical concepts.
According to Treffers and Goffree [15] context problems in 'realistic' Wiskobas-like instruction fulfill a number of functions, to wit:
- Concept forming: in the first phase of a course they allow the students a natural and motivating access to mathematics.
- Model forming: they supply a firm hold for learning the formal operations, procedures, notations, rules and they do so together with other models, which have an important function as supports for thinking.
- Applicability: they uncover reality as source and domain of application.
- Exercise of specific abilities in applied situations.

In an earlier article [34] we tried to discriminate between the uses of context in a way that seems to be in accordance with the four functions mentioned above.
Three different uses of context were distinguished.
The most significant - and characteristic for the conceptual mathematization process - is the use of the context to introduce and develop a mathematical model or concept. This we call *third order* context use.
Of course the context may play a less important role. The role of the context in mathematization is less essential, but still very important: a real world problem is presented to the student, and the student is expected to find the

relevant mathematics, to organize and structure and solve the problem. In this kind of context, the real world is essential and mathematics is the tool to organize reality. In this case we speak of *second order* context use.
Finally we speak of *first order* context use if the mathematical operations are embedded in contexts. A simple transition from the problem to a mathematical problem is sufficient. These kind of problems are often found in traditional schoolbooks. Before continuing our discussion we will illustrate this classification of context uses by examples.

Third order context use
Our example: the context of 'growth' in *Exponential and logarithmic functions.*
As a starting point for introducing the logarithms we take the following graph (fig. II.31):

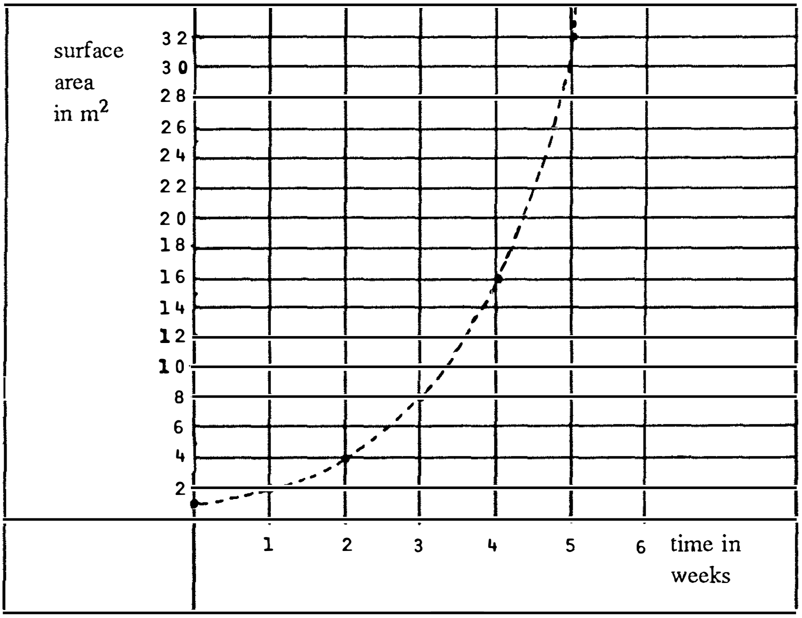


fig. II.31

This graph shows the growth of aquatic plants, starting from 1 m².

> 52 *Estimate by means of the graph after how many days there were about 20* m² *plants.*
The student's answer is: "after about 4.3 weeks."

The next question is:

> 53 *After how many days is there 40* m² *plants? (To be solved without the graph).*
And 80 m²*? And 10* m²*?*
Check the last answer graphically.

In answering this question it is essential to understand that one can (and is requested) to find the answer without using the graph:
the fact that in one week the total area is doubled, related to the answer to question > 52, yields the answer:
"It takes one more week to grow to 20 m². Thus 5.3 weeks."
Analog reasoning gives 3,3 weeks as the time needed to grow to 10 m².

It was striking to observe that many teachers were hindered by their 'context isolated' readily available knowledge of logarithms, which they were inclined to use here. To many students this 'doubling in one week' was obvious.
The following context bound 'definition' is presented a little farther along.

$^2\log 10$ is defined as the moment where 10 m² *plants are formed,*
2 being the growth factor (and starting at 1 m²*).*

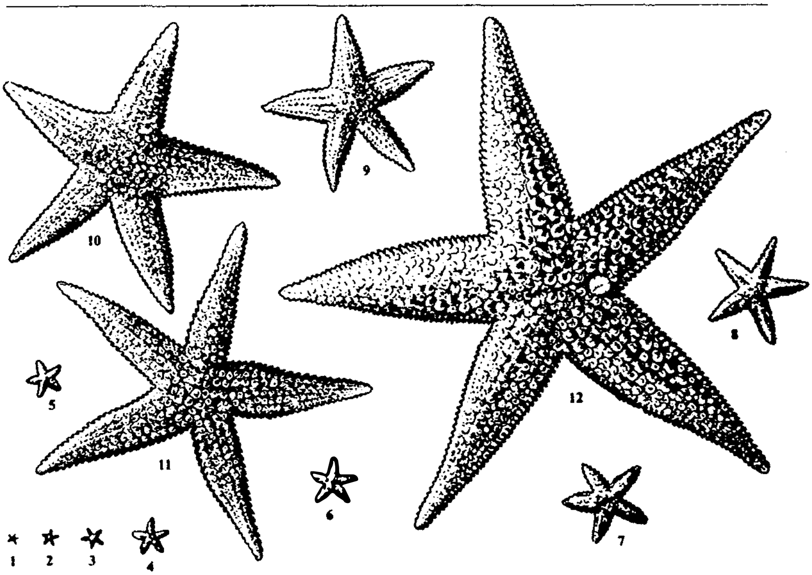
The following context-free exercises are solved by most students within the context:

> 56 *Explain:* $^2\log 16 = 4$
Also: $^3\log 27 = 3$
And: $^5\log 25 = 2$

A typical solution was argued as follows:
"$^2\log 16 = 4$ because it takes 4 (days) to grow to 16 m², with growth factor 2 (starting with 1 m² at t = 0)."
Some, more analytical students, already made the abstraction to $2^4 = 16$.
We conclude this third order context example by describing how the students were anticipating the main property of logarithmic functions.
The first indication follows from:
Explain why:
$^2\log 3 + 1 = {}^2\log 6$
$^2\log 7 + 1 = {}^2\log 14$
$^2\log 6 + {}^2\log 2 = {}^2\log 12$
The explanation:
$^2\log 3 + 1 = {}^2\log 6$
follows directly from the fact that it takes one day to double the 3 m² of

plants to 6 m^2.
A different way to express this :
$^2\log 3 + {}^2\log 2 = {}^2\log 6$.
From this moment on the students start to rely on their intuition that:
$^2\log a + {}^2\log b = {}^2\log ab$.
Before formally proving that property a final discussion takes place to 'prove' that:
"time for fourfolding" + "time for fivefolding" =
"time for twentyfolding".
Starting from a real situation we end up via the process of conceptual mathematization with the main property of logarithmic functions.

Second order context use



1:july 3, 2:july 5, 3:july 7, 4:july 15, 5:july 16, 6:july 18,
7:july 26, 8:aug. 2, 9:aug. 18, 10:sep. 12, 11:sep. 26, 12: oct. 19

fig. II.32

Questions:

You see one starfish during twelve phases of its growth. Is there any time interval where growth is exponential?

This is a straightforward question that leaves a lot open to the students. But there is no conceptual mathematization. Nevertheless the process of mathematization in this case will reinforce the concept of exponential

growth.
As a second example we refer to the 'rat problem', which was discussed intensively in the chapter '*Mathematization*'.

First order context use
As stated before, the context in this situation is only used to 'camouflage' the mathematical problem. Nevertheless, such use is important as a pre-stage in order to familiarize students with applications in the sense of second order context use and it may show students different problems that are mathematically isomorphic:

The growth factor of a bacterium type is equivalent to 6 (per time-unit).
At moment 0 there are 4 bacteria.
Calculate the point in time when there will be 100 bacteria.

The interest percentage for a year is equivalent to 8%.
f 4000 is deposited at moment 0.
At what point in time will this amount have increased to f 5000?

That the second and first order context uses are not always distinguishable will be clear. We conclude our examples with an exercise from the booklet '*Differentiation 2*', that lies somewhere in between:

A supermarket sells cartons of fruit yoghurts (contents $1\frac{1}{2}$ liters) for f 1.90.
Around 500 of these cartons are sold each week. This amounts to f 950.
The manager estimates that a price decrease of 10 cents will result in an extra turnover of 100 cartons. The supermarket's purchase price is f 1.00 per carton.
Assume while doing the following exercises that the manager's estimation is correct.

a. What will the turnover be if the cartons of yoghurt are sold at purchase price? What will the proceeds amount to?
b. Make a computer program for calculating the weekly proceeds of cartons of yoghurt priced at f 1.90, f 1.80 .... f 1.00.
c. The same exercise for the total profit per week.
d. Draw graphs of respectively the proceeds (per week) and the profit (per week) as a funtion of the price.
e. Which price would you advise the manager?

Thus far regarding the functional classification of context-use. But there are more remarks to be made.

In the first place, we would like to note that context use of third order, or in conceptual mathematization, requires very careful consideration.
A quite trivial aspect is that the context should be motivating. However, students have a personal idea about what is motivating. So in general it is quite difficult to say what makes a motivating context. In 1981, after some three years of observations, we made the following two remarks concerning the motivating aspects of contexts: [35]

- For younger students artificial contexts are acceptable and - under certain circumstances - motivating.
  For older students contexts must be more realistic to be acceptable.
- To make sure that the context is motivating to all, or to as many students as possible, one should offer a whole range of different contexts.

We would like to add that observations during the Hewet experiment confirm these conclusions. In conceptual mathematization especially the context is of utmost importance. Sometimes one has to start with 'not-so-real-world' problems in order to avoid complexity. This fact increases the requirements for contexts.

Empirical research by Kaiser [36] shows that context use in general improves motivation and interest in mathematics. But she also underscores the dangers:

> "Negative Auswirkungen auf die Motivation der Schüler(innen) durch Anwendungen sind möglich, wenn die Art der Beispiele und/oder die Methode der Behandlung im Unterricht dem Mathematikbild der Schüler(innen) bzw. ihren Erwartungen an Mathematikunterricht stark zuwiderläuft. Diskussionen und entsprechende Unterrichtsführung können diese negativen Auswirkungen recht weitgehend verhindern."

A good example of a difficult context in the phase of conceptual mathematization is a discussion on the defense budget in our next section. No doubt some students - as well as teachers - learn a lot because of the context. It leads in many cases to *conceptual* conflicts - that means a conflict within the individual about possible different solutions to a problem.
But there are also possibilities for sociocognitive conflicts: conflicts as a result of interindividual encounters, between students or between students and teachers. In general these conflicts will lead to cognitive processes, but on the other hand, many teachers, and students as well, are turned off by this context, and a complete blockade may occur. This places the teacher in a difficult situation: he is not in the first place a teacher but a conflict regulator.
But there are more examples of use of contexts that may demotivate students

though they are still worth being discussed in the classroom. Although a discussion about these exercises may prove to be fruitful in the learning process in general, this may not be the case for mathematical learning.

A well known controversial context is an exercise in which the rates of abortion in different countries were compared. Although the question was of a strictly mathematical nature, discussion rose high, especiallly at calvinistic protestant schools: one school used the book only after removing the page with the 'abortion-exercise'.

Another controversial use is the photo used to visually illustrate the acceleration in free fall (fig. II.33):

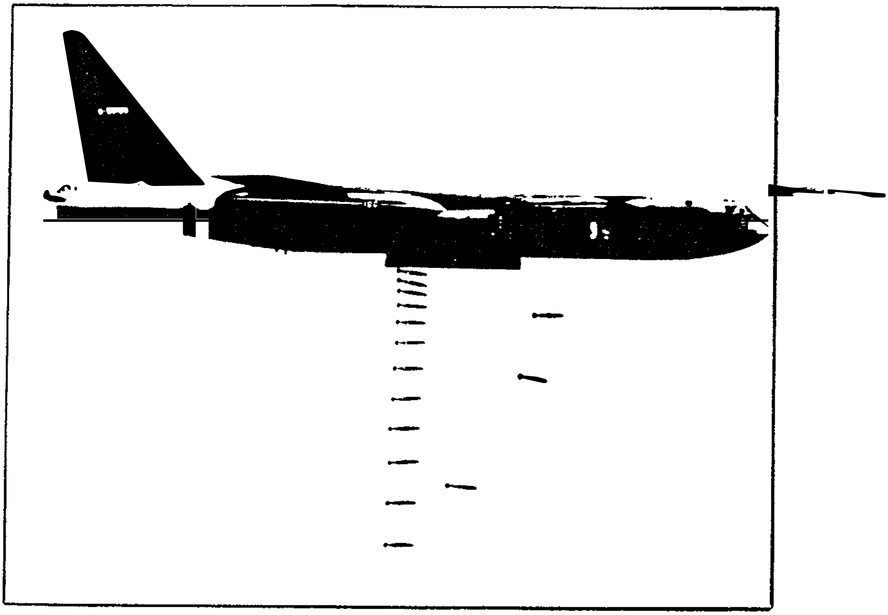


fig. II.33

Although the photo is an 'excellent' illustration of the mathematics involved, the context may give some problems. In relation to this exercise we may mention the fact that in the German translation of *'Differentiation 1'* the editors omitted the rich and realistic problem of the hunting and sinking of the German battleship *'Bismarck'*.

As we have indicated before, a realistic context does not always mean that there has to be a direct connection to the physical world. The Hewet team also accepts mathematical activities derived from topics that owe their existence to a story, cartoon or any creation of an 'imagined' reality. Wiskobas often made use of 'imagined' reality. [37] But our observations over the past ten years make clear that one must be careful in upper secondary education when using cartoons, stories etc. [35]
A 'somewhat' realistic context seems a minimal condition for students of this age. One may get away with using a comic that supposedly is very popular in the Netherlands but even in this case students sometimes complain. But - maybe surprisingly - in the intuitive explorative phase students are forgiving when the real context is not so real. This is exemplified by the introduction to the booklet *'Matrices'*.

The first problem in that booklet is situated on the non-existant but realistic looking island Hau in the Pacific. All following islands are real. The background to this was even clear to some students:

> "If you want to change the infra-structure on an island, or build a school, it is easier and *more realistic* to do this on a fantasy island instead of doing irrealistic things on real islands."

The Netherlands has islands too. Why not use the Dutch islands instead of islands in the Pacific?
One reason is to have all students start from the 'same' reality knowledge. They have knowledge of the Dutch islands different levels of which can cause undesired problems at this stage. This was discovered when we studied - similar to the Pacific islands - a road network in an island-like polder (reclaimed land). The students, or at least some of them, knew the local situation too well, so that they could not imagine that there could be reasons to:

- consider one-way traffic;
- consider building a railway;
- build a new school.

If one chooses a context in the students' world, not only the context has to be real, but so have the problems to be solved. For most conceptual mathematization one is better advised to avoid this kind of real-world situations. However, in general, for mathematization one can and should use these problems.

The real world of one child may differ considerably from the real world of another. Or, as Thomas called it in Symbolic Interactionism: "the definitions of the situation" may differ. [38] Although Thomas never formulated precisely the meaning of this key term, the conception of definition of the situation provided a simple and powerful rationale for the significance of the subjective aspect in social life and thus provided symbolic interaction with the prime methodological rule which seems important in the philosophy behind Mathematics A:

"If men define situations as real, they are real in their consequences" [39]

One of the consequences for context-use is to change the context often. Apart from having a motivating aspect it offers a way to minimize the differences in the students' real-world perception.

Theory and practice sometimes give opposing results. In theory we should use all kinds of different contexts - this will lead to sociocognitive conflicts, strong interaction, group discussions and will definitely have strong learning effects. However, it places the teacher in a very difficult position: his main activity will be conflict regulation, and the question remains whether the learning effect will be strong enough in the discipline of mathematics.
Thus, our observations lead to the following recommendations - given the present state of teacher training and traditions in teaching:

- One should avoid - especially in the phase of conceptual mathematization - contexts that are emotionally disturbing (defense, war, illness, ethic affairs).
- One should avoid artificial contexts.
- One should avoid too neutral contexts, which might fail to motivate unless the mathematical content is stimulating enough.
- One should not expect from the students too much background information; most, if not all, information should be contained in the text.
- One should choose the context and edit the exercise in a way that stimulates interindividual actions (in order to promote sociocognitive conflicts).

One final remark must be made: there is a danger of a mathematical concept being too strongly, or even forever, attached. Take, for instance, our introduction of logarithms; the danger that the student sticks to logarithm as "the time necessary ..." is no danger at all since students who do not perform the required abstraction can work well with this context-bound idea.

2.4 Critical attitude

According to the Hewet report the students should be capable of judging a mathematically oriented presentation.
This in turn means that they
- should get used to common mathematical language;
- should get used to speak in formulas;
- should get accustomed to all kinds of graphic representation;
- should learn to use mathematical models and to judge their relevance.

Our discussion about mathematization, and the examples given, have made clear that most of these goals are recognizable in the material.
Furthermore, we hope to give more examples, when discussing different methodologies in mathematics education. Before doing that, however, we would like to discuss an explicitly mentioned goal that has a special place in the whole of the program of Mathematics A. The importance of this goal is illustrated by the fact that the well-known Cockcroft Report mentioned it also in the following way:

> "Pupils should be encouraged to discuss critically information presented (in diagrammatic form)." [40]

The Hewet Report states it as follows:

> "Critical judgement of statistical data."

However, it is no easy task to operationalize this goal. The booklet on graphic representation starts in the following way:
Two (seemingly) opposing statements are presented and the students are requested to try to find out which one is correct.

> "Defense spending has *decreased* to the pre-Vietnam-War level." (Brown).
> "I'm delighted to be able to inform you that under president Carter the real defense spending has continuously *increased*." (Jayne).

The students are asked to find out who speaks the truth. Of course, a lot of statistical material is offered to the students. In the first place, the material on which the statements were based.
Brown based his statement on the following:
- For the year 1980 the amount for defense shows a real increase of 3,2% compared to 1979, but it is only 4,6% of B.N.P., and that is the lowest since 1940.
- The amount for defense is 23% of the total of Federal expenditures, and that is, apart for 1978, the lowest since 1940.

Jayne used the following graph as a base for his statement (fig. II.34):

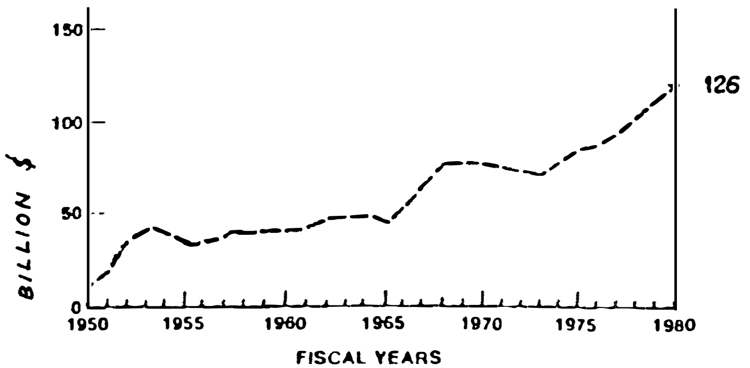


fig. II.34

Similar statements are often made in the media without further clarification. The students, however, are 'forced' to study the statements more carefully. This of course, is not very easy. The students, knowing that defense spending rose from 115 billion US dollars in 1979 to 126 billion US dollars in 1980, have the feeling that Brown is 'less' speaking the truth than Jayne, but some are uncertain:

"Brown's statement seems not to be in agreement with the graph, but he may not be lying if he looked at it in a relative way. You need more money now to buy the same." [41]

The discussion in the student material now focuses on the 'absolute'-'relative' concept. This again produces a lot of problems. One of the bottlenecks is that students in that age bracket (16,17) have no idea at all of inflation. Although they know the expression from the media, they are unable to work with it or even explain verbally what it is. In the first rounds of the experiments the best answer was the following:

"Everything becomes more expensive."

The fact that inflation plays such a vital role in many public discussions, especially among politicians and economists, made the authors decide to pay extra attention, in a very basic way, to inflation. This was done in two ways.

The first way was showing the decreasing purchasing power of the dollar (fig. II.35):

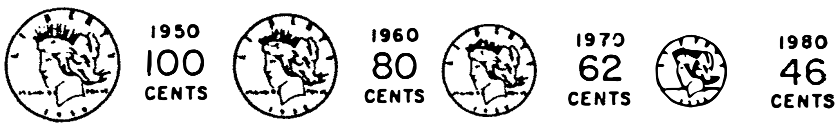


fig. II.35

and discussing this picture.
More stimulating was the next clipping from a well-known Dutch paper, and the discussion that followed.

ARGENTINIAN INFLATION 540,115 PERCENT

From our correspondent

MEXICO CITY
Inflation in Argentina during the past 40 years amounts to 540,115.064 percent. This was calculated by the Buenos Aires newspaper "Tiempo Argentino" after having analyzed price developments during the first Peron government in 1944. In order to portray this one would need a graph 80 centimeters wide and 54 kilometers high.

The greatest contribution to this development was provided during the government of General Videla (1976-1981) with a total of more than 9000 percent and a year average of 208 percent. Second place goes to Peron's widow who, regarded absolutely, scores even higher: 893,5 in one year and eight months. During the last seven months, since General Bignone has become president, inflation has amounted to 8,3 percent, while the total for 1982 came to somewhat more than 200 percent.

At the end of this chapter of the booklet most students agreed that neither Jayne nor Brown had really lied but that they had spoken the truth in their own way: Jayne was talking in absolute terms, Brown in a relative way.
The problem with a goal such as "discussing critically statistical material" is to establish the result, if any, with the students.

The authors of the experimental material chose to solve this problem by having the students misuse statistical material themselves, without lying.

The exercise from the booklet:
In a certain country the defense budget is 30 million dollars for 1980. The total budget for that year is 500 million dollars. The following year

the defense budget is 35 million, while the total budget is 605 million dollars. Inflation during this period between the two budgets was 10%.

- *You are invited to hold a lecture for a pacifist society. You want to explain that the defense budget has been decreasing this year.*
  Explain how to do this.

- *You are invited to lecture to a military academy. You want to explain that the defense budget has been increasing this year.*
  Explain how to do this.

This exercise stirred up a number of conflicts in the classroom originating from one point:

- Is it ethical to teach students how they can manipulate rather than to show them solely 'manipulation' by others?
- Should we discuss such a controversial matter like defense spending in the math lesson? (Context problem).
- Should we spend our mathematics lesson on such (marginal) activities? (Content problem).

The following group discussion makes some of the problems and positive effects clear. [42]

After all of them have read the problem individually, Marijn says: "I think you have to see it as a percentage - 30 of the 500 and 35 of the 605."
Marc: "500 of the 30, that's $\frac{100}{6}$."
Marijn: "The other way around - that's 0.06."
Marc calculates $\frac{35}{605}$ on his calculator: 5.78.
Marijn says to Susan: "Write that down."
Susan asks: "What is that the answer to?"
Marijn tells her and then dictates the answer.
Servaas: "This one is really too simple."
Marijn: "Aren't we supposed to do something with the inflation?"
Marc: "Oh shit."
Servaas: "If you ask me, that has nothing to do with it."
Marc: "The inflation applies to both amounts so they cancel each other out."
Servaas: "In the second one, it just increased from 30 to 35."
Susan doesn't agree: "The inflation lies in between the two estimates, so you do have to figure it for the second one."
Marc: "And you have to add 10% extra to that 605."
Servaas: "It doesn't say that."

Marijn: "But in the next one you do have to do it."
Marc: "You add on the inflation, but you don't mention that there was any inflation, so the difference is even greater."

The arguments held by the pacifist group and by the military academy overlap each other somewhat, making the role to be played by inflation rather unclear. The following four minutes hardly contribute to the solution. Marijn calculates 605 and 35 backwards on basic-year level ("605, that's 100%, so you divide that by 11 and then subtract it, so that used to be 550") and then establishes that $\frac{31,8}{550}$ is *5.78*. But Marc had already pointed this out in the beginning with his "cancelling out". Marc does, however, get an idea from Marijn's calculations, which appears in the second section of the tape recording of this discussion:

Marc says to Susan: "If you subtract the inflation from 35 you get 31.8. This 31.8 is much less in relation to 605 than 35. So you do have to subtract inflation from the 35 but not from the 605."
Susan: "That sure is stupid."
Marc: "Yeah, but you have to do your best to sell it, so it should bo o.k. to fiddle it a bit.
Susan: "That's ridiculous."
Servaas: "You can't do that."
Marc leafs through the book: "They're doing that the whole time."
Marc: "O.k., it's not right, but if you're on the side of the pacifists ......"
Servaas, referring to the book: "But then calculating the inflation for both."
Marc asks Marijn what she thinks, but she had lost the thread of the conversation. He explains it once more but Marijn, too, rejects his solution:
"That's no longer being objective."
Marc: "But they don't have the data."
Servaas: "And at a certain point you just say whatever you like."
Susan: "I'll write it down."
Marijn: "Marc sure knows how to deal with pacifists."
Susan: "Double points. Go ahead, Marc."
Marc dictates his solution.

Of course it depends - most of the time - on the teacher to make the decisions in this case. Those who consider this as one of the highlights of their Mathematics A course defend their opinion with arguments that mathematics education has also a general educational component: this part of mathematics has value for everyone, as is proven in the classroom, where vivid

discussions and sociocognitive conflicts lead to fruitful lessons, sometimes resulting in essays or classroom presentations.
On the other hand, a number of teachers just skips this chapter. Their reasons of doing so can be read from the three questions that were just posed: The context is considered controversial and lacks mathematics that can be formalized. This becomes even clearer when another example of misusing statistical material is presented.

A survey on the population of the Orkney islands shows population pyramids for each of the islands.
One of those is that of the isle of Eday (fig. II.36):

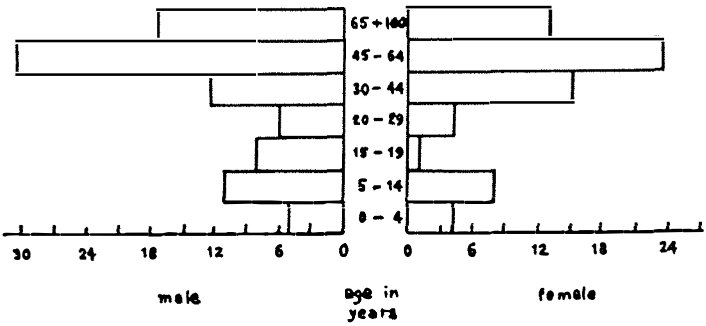


fig. II.36

This pyramid was used to show the 'ageing' problem of the island: according to it there are hardly any children and young adults, but numerous old and very old people.
The students are asked whether these conclusions are correct.

The students, in turn, construct their own more justifiable pyramid, working with frequence-densities. The resulting pyramid shows quite a different picture (fig. II.37).

There seems to be no problem in treating *this* kind of 'lying with statistics'. The context is neutral and there is some concrete mathematics involved.

There is no doubt that this example and similar ones are 'safer' from a teacher's point of view. But experiences from certain schools that take up the challenge to cover the 'defense' chapter show that it may prove to offer excellent lessons that students may recall years later. The most often mentioned effect is: the readjustment of the real world. As a student states:

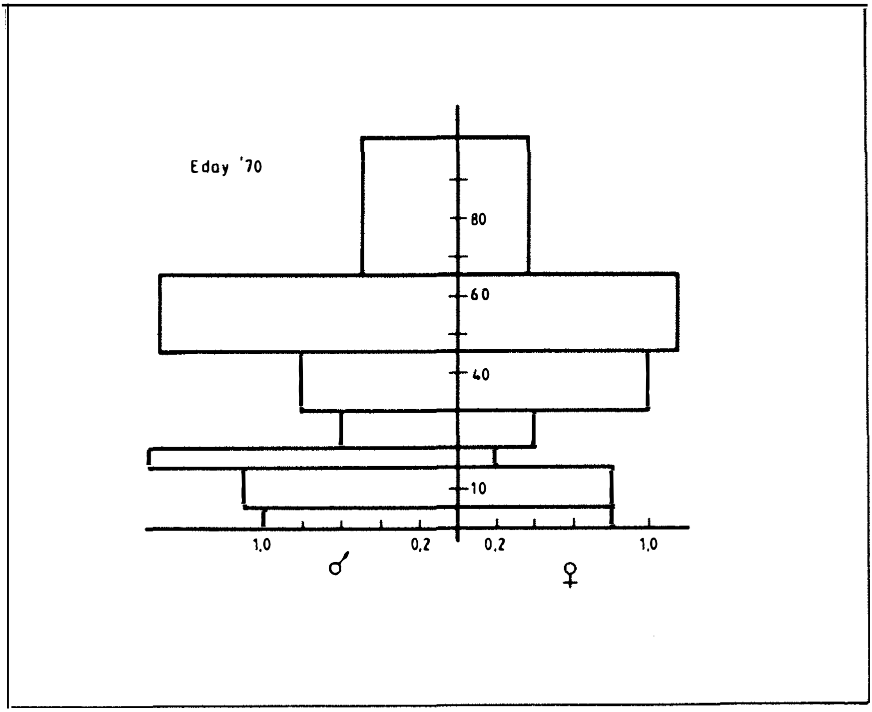


fig. II.37

"The world is different from what it seems. One should not believe all statements from newspapers and T.V. One should look critically at statements based on statistical material."

It will be clear from our discussion that, with these examples, the main focus is not on conceptual mathematization. Here we are concerned mainly with an attitude aspect. Students should be aware of the fact that mathematics can be used in an improper way as the two examples clearly show.
That this can prove to be a serious problem, especially as mathematics is used in more and more disciplines and in the media, can be illustrated by the following example.

Professor Samuel Huntington lectures on the problems of developing countries. In his book *Political Order in Changing Societies* (1968) he suggests various relationships between certain political and sociological concepts:

(a) "social mobilization"
(b) "economic development"

(c) "social frustration"
(d) "mobility opportunities"
(e) "political participation"
(f) "political institutionalization"
(g) "political stability"

He expresses these relationships in a series of equations:

$\frac{a}{b} = c; \frac{c}{d} = e; \frac{e}{f} = g$

Does Huntington mean that:

> "Social mobilization is equal to economic development times mobility opportunities times political institutionalization times political instability!?"

Mathematics is used as a means for mystification, intimidation, an impression of precision and profundity, according to Koblitz. [43]
He, Koblitz, comes to the following implications for teaching:

> "Whether mathematical devices in arguments are used for fair ends or foul, a well-educated person should be able to identify misuses of mathematics as well is to know about the correct uses."

This critical attitude is one of the goals of Mathematics A. And, once again, it seems difficult to operationalize this goal by restricted-time written tests.

3 MATH A IS REALISTIC MATH

In this section we will show that Math A fits the description of the methodology referred to as 'Realistic Mathematics' education.
From the Royaumont seminar on there was a world wide movement towards a more Bourbakist (structuralist) approach to math education. But right from the beginning of this movement there was sharp criticism. This criticism and developments in research in math education led to the development of - among other things - realistic math education.

The first part of this paragraph outlines the discussion leading up to and the description of the characteristics of realistic mathematics education.
The second part gives an example of how the structuralist/mechanistic approach was materialized in school textbooks in the Netherlands. This example is placed against an example of Math A which represents the realistic approach.

3.1 Realistic Math Education

Early in 1959 two activities - a survey and a seminar - were initiated by the O.E.E.C. for the purpose of improving mathematical education. (See Chapter I.1). This seminar - known as the Royaumont seminar - had great impact on mathematics education during the following decades. Looking back on this seminar and on its implications for math education one cannot but conclude that there the foundations were laid for structuralist mathematics education, based on set-theory.

According to the structuralist conception, mathematics as a cognitive attainment is an organized, closed deductive system. For mathematics education this means that the structure of the system is the guide-line of the learning process. Insight into the structure of mathematics is of fundamental importance for this systematically directed education. Not surprisingly, one of the most outspoken advocates of this approach was Dieudonné. In his address to the conference, titled: "*New Thinking in Social Mathematics*" he proposed to offer the students a completely deductive theory, starting right from basic axioms. [44]

On the same occasion he rejected the 'patchwork' of traditional school mathematics with the - now famous - slogan "A bas Euclide!"

His goal for mathematics education was to train the students in logical deduction and instill some ideas on the axiomatic method. Choquet, although less extreme than Dieudonné (according to the editor of the Royaumont report), also advocated a structuralist point of view. He claimed that the set $\mathbb{Z}$ constitutes an excellent basis for young students:

> "The set of positive integers $\mathbb{N}$ or, better still, the set of integers $\mathbb{Z}$ is endowed with numerous structures, such as order, group or ring and each of them has special characteristics which make $\mathbb{Z}$ particularly clear to specialized arithmeticians. We have available an excellent example of a structure in which the main concepts of algebra can be studied." [45]

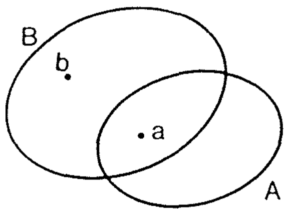
Servais' ideas again were less extreme, and were approved by many among the participants. Mathematics is seen as a structure, a complete building.

And in education, Servais argues:

> "The whole edifice must be rebuilt from the foundations and erected in accordance with modern ideas. The modern ideas are mainly set theory." [46]

Servais' ideas became operationalized in his textbooks (fig. II.38) [47]: Also Felix, very influential in France and closely related to the Bourbaki-group, expresses a structuralist view for mathematics education:

> "The laws of mathematical structure are the laws of logical thought. It is



Soit A une droite. D'après l'axiome G_3, elle n'est pas vide. Désignons par a un point de A.

En vertu de l'axiome G_2, A est une partie propre de Π.

Soit b un point qui n'appartient pas à A.

Comme $a \in A$ et $b \notin A$ on a $a \neq b$.

En vertu de G_1, la paire $\{ a ; b \}$ est contenue dans une seule droite, soit B.

Comme $a \in A$ et $a \in B$ on a $a \in A \cap B$.

L'intersection $A \cap B$ peut-elle comprendre un point c distinct de a ?

On aurait dans ce cas

$$a \neq c \qquad \{ a , c \} \subset A \quad \text{et} \quad \{ a , c \} \subset B.$$

D'où en vertu de l'axiome G_1

$$A = B.$$

Mais cette égalité est impossible puisque

$$b \notin A \quad \text{et} \quad b \in B.$$

En conclusion, $A \cap B = \{ a \}$.

DEFINITIONS.

Deux droites qui ont pour intersection un singleton $\{ a \}$ sont dites **sécantes au point a** qui est leur **point d'intersection**.

Elles se **coupent** en a.

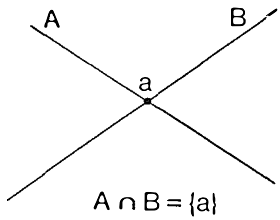


fig. I.38

for this reason that the early chapters of advanced mathematics describe, in abstract form, what the infant school teacher draws to her pupils attention to help them learn to think." [48]

One of the three Dutch participants at the Royaumont conference was Vredenduin. Following the concept of structure, at least in his geometry books, he carried out experiments at secondary schools.

His conclusions, many years later:

"A beautiful edifice, but I do not think there was one student who shared that opinion." [49]

Critical remarks on the ideas of these 'structuralists' were made by Thom in 1972. [50] Reacting to the "Euclid must go" slogan he defended the *Euclidean geometry:*

"They - the Bourbakists - abandon that terrain which is an ideal apprenticeship for investigation: Euclidean geometry, that inexhaustible mine of exercises, and to substitute for it the generalities of sets and logics, that is to say, material which is as poor, empty and discouraging to tuition as can be. The emphasis placed by modernists (structuralists) on axiomatics is not only a pedagogical abbreviation - which is obvious enough - but also a truly mathematical one."

The fact that *pedagogical principles* had been ignored was recognized by Beberman, who admitted that the new curriculum (developed by himself in the U.S.A.) had failed to relate mathematics to the real world. [51]
On another occasion he cast further doubts on the wisdom of his program:

"I think in some cases we have tried to answer questions that students never raise and to resolve doubts they never had, but in effect we have answered our own questions and resolved our own doubts as adults and teachers, but these were not the doubts and questions of the students." [52]

In the same year that Beberman expressed these feelings a memorandum was published in the '*Mathematics Teacher*' and the '*American Mathematical Monthly*'. [9]

"Mathematicians may unconsciously assume that all young people should like what present day mathematicians like or that the only students worth cultivating are those who might become professional mathematicians."

"The mathematics curriculum of the high school should provide for the need of all students: it should contribute to the cultural background of the general student and offer professional preparation to the future users of mathematics."

"To know mathematics means to be able to do mathematics: to use mathematical language with some fluency, to do problems, to criticize arguments, to find proofs and, what may be the most important activity, to recognize a mathematical concept in, or to extract it from, a given concrete situation. Therefore, to introduce new concepts without a sufficient background of concrete facts, to introduce unifying concepts where there is no experience to unify, or to harp on the introduced concepts without concrete applications which would challenge the students, is worse than useless: premature formalization may lead to sterility; premature introduction of abstractions meets resistance especially from critical minds who, before accepting an abstraction, wish to know why it is relevant and how it could be used."

"Mathematics separated from the other sciences loses one of its most important sources of interest and motivation."

"Extracting the appropriate concept from a concrete situation, generalization from observed cases, inductive arguments, arguments by analogy, and intuitive grounds for an emerging conjecture are mathematical modes of thinking."

"The best way to guide the mental development of the individual is to let him retrace the mental development of the race - retrace its great lines, of course, and not the thousand errors of detail. (The genetic principle)."

"Consistent with our principles, we wish that the introduction of new terms and concepts should be preceded by sufficient concrete preparation and followed by genuine, challenging applications and not by thin and pointless material: one must motivate and apply a new concept if one wishes to convince an intelligent youngster that the concept warrants attention."

Among the undersigned, we see: Bers, Birkhoff, Courant, Coxeter, Kline, Morse, Pollak and Polya.

One might call this memorandum a call for 'realistic' mathematics - realistic in the sense that one starts in the real world.
Likewise, many of the ideas in the memorandum can be encountered again in Mathematics A.
We wish to remind more specifically that the math curriculum should provide for the needs of all students:
Firstly, it should contribute to the cultural background of the general student; it should offer professional preparation to the future users.
Secondly, that Mathematics A tries to bridge the gap between knowing mathematics and using mathematics. [53]
Thirdly, and very important, conceptual mathematization is stressed:

extracting the appropriate concept from a concrete situation.
And finally: mathematics should not be separated from other sciences.

This call for 'realistic mathematics education' was in vain however, at that time.
The following year Papy's (in)famous books appeared: the structuralist approach, based on set theory was gaining momentum.

As we mentioned before, Thom was very critical about the Bourbakist approach. [50] In his words "material which is as poor, empty and discouraging to tuition as can be". And, in his opinion, the emphasis placed on axiomatics was not only a pedagogical aberriation but also a truly mathematical one.

According to formalism, mathematics consists solely of axioms, definitions and theorems. Thom declared it "empty", Davis and Hersh seem to agree that "formal mathematics is authoritarian by nature, is not about anything but just *is*; it grows only by the deductive pattern of formalized mathematics." [55]
They (Davis and Hersh) consider the writing of the group known collectively as Nicolas Bourbaki as the most influential example of formalism as a style in mathematical exposition. Under the pseudonym Bourbaki a series of basic graduate texts in mathematics starting from set theory was produced which had tremendous influence throughout the world in the 1950's and 1960's.

This formalist style - often presented by Dieudonné - gradually penetrated downward into undergraduate mathematics teaching and, finally, in the name of "the new math", even invaded kindergarten - all because of the Royaumont Seminar.
Dieudonné, some ten years after Royaumont:

> "On foundation we believe in the reality of mathematics, but of course when philosophers attack us with their paradoxes we rush to hide behind formalism and say, "Mathematics is just a combination of meaningless symbols", and then we bring out chapters 1 and 2 on set theory. Finally we are left in peace to go back to our mathematics and do it as we have always done, with the feeling each mathematician has that he is working with something real. This sensation is probably an illusion, but it is very convenient. That is Bourbaki's attitude toward foundations." [54]

Lakatos launched an attack against formalized mathematics and Dieudonné, in particular in his book *Proof and Refutations.* [56]

He points out that informal mathematics is *also* mathematics. Informal mathematics is a science in the sense of Popper; it grows by a process of successive criticism and refinement of theories and the advancement of new and competing theories.
Lakatos argues that formalism (Dieudonné) disconnects the history of mathematics from the philosphy of mathematics. "Formalism denies the status of mathematics in most of what has been commonly understood to be mathematics, and can say nothing about its growth."
Lakatos' arguments, however, influenced some math educators more than mathematicians. As Feferman [57] noted:

> "Many of those who are interested in the practice, teaching and for history of mathematics will respond with lay sympathy of Lakatos' program. It fits well with the increasingly critical and anti-authoritarian temper of these times."

Let us quote in this regard an Argentine mathematician: "Your Math A program is anti-autocratic." [58]

The Nobel-prize-winning physicist Feynman added one more critical characteristic to the formalist mathematician; in his opinion the formalist mathematician starts from a rigid but arbitrarily prescribed set of axioms and cold-bloodedly grinds out all its consequences. He characterized this as a *dehumanized* activity. On the other hand there was the physicist, stepping lightly from context to context with delicate imaginative leaps and every now and again penetrating the unknown with bright exciting shafts of light. [59]
Hilton however, argues that this is the picture of a mathematician as well, that is, of a good mathematician. In his opinion the axiomatic method, as an explicit tool, should only occur relatively late in the educational process. He also rejects the study of mathematical structure for its own sake: "we should reason about the real world." [60]
Freudenthal, being a mathematician influenced by Brouwer's view on mathematics, - the constructive or intuitionistic view - introduces the slogan:

> "Mathematics as a *human* activity." [61]

He argues that mathematics should never be presented to the students as a ready-made product. The opposite of ready-made 'dehumanised' mathematics is human mathematics in statu nascendi. The student should re-invent mathematics. What Freudenthal calls re-invention, is often described as discovery or re-discovery. It will come as no surprise that he discards the structuralist view on the teaching of mathematics. He summarizes the view of the modernist in the following way: if the analysis of mathematics shows

that mathematics has a deductive structure, then mathematics has to be implemented according to that structure and, more precisely, according to that special deductive system in which the teacher or textbook author believes.
This is - in Freudenthal's words - the antididactic inversion: the only didactical relevant element, the analysis of the subject matter, is dropped; the student is confronted with the *result* of the analysis and may watch the teacher who knows the result, putting the analyzed things together.
Freudenthal's view is that one should recognize that the learner is entitled in a fashion to recapitulate the learning process of mankind. This entails instruction not starting with the formal system, which in fact is a final product, nor with embodiments (materializations of structures) nor with structural games.
The phenomena by which the concepts appear in reality should be the source of concept formation, as we made clear in paragraph 2.2.3
According to Treffers, the global starting points of 'Realistically Oriented Mathematics Education' can be summarized as follows:
- Paying much attention to 're-invention', that is recreating mathematical concepts and structures on the basis of intuitive notions in the making of made ('active').
- Carrying on at various levels of concreteness and abstraction ('differentiated').
- The programming of the instruction is guided by the historical-genetic rather than the subject matter systematic method ('vertically planned').
- Reality bound, meaningful ('and mathematically rich') instruction. [62]

The global starting points lead to the description of 'Realistic Curricula', in which we follow Treffers and Goffree. [63]
In the first place we mention the dominating place occupied by context problems, as was shown and discussed in earlier sections.
Secondly, broad attention is paid to the development of situation models, schemas and symbolizing.
Thirdly, there is a large contribution from the students themselves to the course by their (mental) productions and constructions, which lead them from their own informal to the more standard formal methods.
Constructing, reflecting, anticipating and integrating are fundamental functions of the students own production. In structuralist didactics this production is no essential element: on the contrary, after a preparatory phase there is little room left for individual informal methods and their gradual transformation into formal ones: it is instruction in a straight jacket.
We should, however, mention a problem that may occur when one offers the

students the opportunity to construct: the constructions many not agree with the teacher's own development. This requires a great measure of flexibility on the part of the teacher.
As a matter of fact, this may hold for realistic and empiricist instruction as well. But, in the case of realistic instruction, the students' informal methods are used as a lever to attain the formal ones. This method asks for explicit negotiation, intervention, discussion, cooperation and evaluation among pupils and teachers. [64]
This then can lead to a diversity of solutions for a given problem, depending on the reality or basic assumptions as experienced by the student. This, in turn, leads not seldom to feelings of uncertainty in the teacher's mind.

The final point we mention as characteristic for realistic curricula is the intertwining of learning strands. Learning matrices is intertwined with graphs, graphical representation, exponential growth, periodic functions, and probabilistic notions. In general one may state that all subjects of Mathematics A are interrelated, as we showed at the beginning of this chapter.

In the next paragraph we will compare student material stemming from a structuralist and a realistic view of mathematics education.
But first we would like to pay attention to two more methodologies in math education: The *empiricist* and the *mechanistic* view.

In the *mechanistic* approach mathematics is a system of rules. The rules are given to the students, they verify and apply them to problems similar to previous examples. No real-world phenomena as source, little attention paid to applications, much attention to memorizing and automating the 'tricks' and certainly not a methodology, the goal of which is insight into the structure of mathematics. Qualities such as structure, interrelatedness and insight are ignored, according to Whitney [65] in favour of a direct step-by-step construction, characterized by progressive complexity. He strongly denounces this approach.

Finally we mention the *empiricist* view on the teaching of mathematics education. This approach is characterized by much attention for 'environmental' activities, more than for 'mental' operations. Students are offered a stimulating environment in the hope that by maturing they will get the opportunity to develop cognition. Education is rather weakly aiming at formal-mathematical results.

Examples of student material with a strong empiricist component are the

materials of the Foundation for Curriculum Development dealing with global graphing designed for students 12-14 years old.
According to Freudenthal, one stays at the level of intuitive exploration, and hardly makes any effort to formalize the mathematics involved.
As he states it:

> "The design reminds me of teaching the geometry of the cube without using the words 'cube', 'edge' etc. The mathematical terminology is restricted to the word 'graph'." [66]

Another book emphasizing the empirical component, which may be mentioned, is the one by Castelnuovo/Gori Giorgi/Valenti. [67]
Horizontal mathematization is very strong in these books, but less attention is paid to vertical mathematization.

It may be useful to compare the four approaches in terms of horizontal and vertical mathematization.
The following schema can be made:

	hor. math	vert. math.
empiricist	+	-
realistic	+	+
structuralist	-	+
mechanistic	-	-

The '+'-signs stands for much attention paid to that kind of mathematization, and the '-'-sign for little or no attention at all.

As described before, the *empiricist* methods contain a lot of mathematization but most of it is of the horizontal kind.
In the *realistic* vein, considerable attention is paid to both horizontal and vertical mathematization.
In the *structuralist* approach almost all attention is devoted to vertical mathematization.
In the *mechanistic* methods there is hardly any mathematization at all.
The schema suggests some natural kind of seriation, at least between empiricist, realistic and structuralist approach.

In secondary education, especially since about the sixties, much emphasis has been placed on the structuralist approach. We already mentioned Papy's books, which were quite influential. The present trend in mathematics education seems to be moving away from this approach. This movement is, on the one hand, towards a more mechanicist approach - drill and practice - sometimes, for instance in the Netherlands, under the pressure of tests and examinations. (This means that one has to be very careful when developing tests in realistic curricula, as we will discuss later in detail.)

On the other hand, there is also movement towards more realistic education. In the United Kingdom the S.M.P.[7] definitely has certain connections with the realistic philosophy, although the horizontal mathematization seems less outspoken than in the Hewet materials.

The Hewet project itself influences the major book-publishers in the Netherlands to such a degree that in upper secondary education one may speak of a trend towards the realistic approach. But, also at other levels, this trend is clearly tangible. For the age group 12-16 - partly due to the influence of the Wiskivon group of I.O.W.O. - the same trend may be observed, now even stronger than some years ago because of the effect of the Hewet project and the work of the Foundation for Curriculum Development.

For primary education the trend has been uncovered by research carried out by De Jong. [68] He measured the influence of the Wiskobas group on the books used in primary education, and ended up with the following graph (fig. II.39):

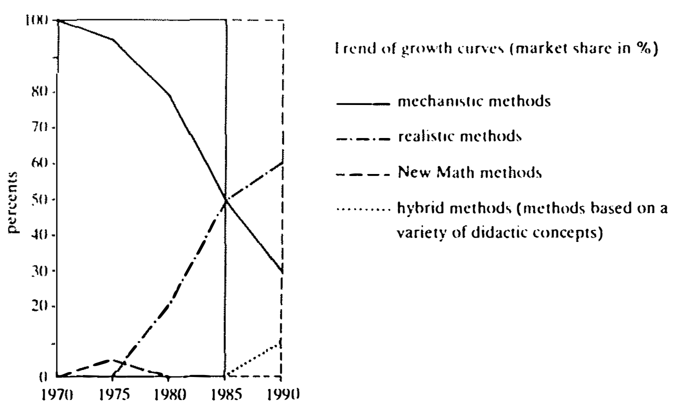


fig. II.39

This graph shows very clearly the increase of realistic methods in primary education, and the decline of the mechanistic approach. One may also note the increase in what De Jong calls 'hybrid' methods: methods mixing the different approaches. The future development seems uncertain. At the upper secondary education level seemingly one 'hybrid' method is on the market: on the one hand using some mathematization, on the other hand emphasizing the structuralist/mechanistic approach.
The other two market leaders seems to have chosen for some kind of a (weakly) realistic approach. At the start of the nationwide effectuation of the new program one had the following approximate distribution. [69]

Title	*Publisher*	*% of Market*	*Classification*
Hewet-Wiskunde	Educabock	20	Realistic
Moderne Wiskunde 4e ed.	Wolters Noordhoff	30	More or less realistic
Getal en Ruimte	Educabock	30	Weakly Realistic
Sigma	Wolters Noordhoff	20	Hybrid

We will now compare the realistic and the structuralist/mechanistic approach with regard to the introduction of matrices.

3.2 Structuralist/Mechanistic vs. Realistic: An Example

As an example we take the multiplication of matrices.
The structuralist/mechanistic approach will be represented by one of the major textbooks [70] for the old curriculum; there matrices were one of the subjects of the vector geometry enrichment program. (See Chapter I).
The multiplication of matrices is introduced in the following steps:

1. Definition of transformation.
   Two-by-two matrices.
2. The product of two two-by-two matrices.
   Examples.
3. Problems/Exercises.

The first chapter (three pages) starts with the definition:

A transformation from $\mathbb{R}_2$ to $\mathbb{R}_2$ is a prescription that assigns to every point of $\mathbb{R}_2$ a(nother) point of $\mathbb{R}_2$

This definition is followed by seven examples.
The first one being:

a Reflection at the x_1-axis.
Since the equation of the x_1-axis is $x_2 = 0$, this line reflection is indicated by $S_{x_2=0}$.
For any point $P(x_1, x_2)$ as original the image is indicated by $P'(x_1', x_2')$.
Under $S_{x_2=0}$ it holds that: $x_1' = x_1$ and $x_2' = -x_2$.
P' being the image of P we also agree that the vector OP' is the image of OP. Consequentially we speak of a mapping of the vector space $\mathbb{R}_2$ on the vector space $\mathbb{R}_2$.
Since $OP = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $OP' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$, we write:

$$S_{x_2=0}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}.$$

Already in this first example we notice generality right from the start. Rather than an example like A(3,4) one immediately takes P(x_1 , x_2) as the original point. Rather than a vector OA = $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, directly the vector OP = $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.
Furthermore there is no drawing to explain the geometrical meaning.
The text is compact: a string of mathematical jargon. It is clearly written from an elevated mathematical point of view: its style is that of mathematical science. No effort is made towards a connection or relation with the students' own experiences outside mathematics.

The next six examples are written in telegram style. For instance, the last one:

g The multiplication with center O and factor $k \in \mathbb{R}$

$$V_{O,k}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} kx_1 \\ kx_2 \end{pmatrix}.$$

Reflecting on their examples, the authors state:

The above mappings have in common that the mapping prescription A can be written in the form:

$$A\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}ax_1 + bx_2\\cx_1 + dx_2\end{pmatrix} \text{ with } a, b, c, d \in \mathbb{R}.$$

Since the mapping is actually determined by the numbers a, b, c, and d, it is usually given by a schema of the form:

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}.$$

Here the place of the single letters is essential.

Now the various examples are translated into matrices (without mentioning the term) and this is followed by the definition of a matrix:

Schemas like $\begin{pmatrix}a & b\\c & d\end{pmatrix}$ are called *matrices*. In general a matrix is a schema of numbers arranged in a rectangle. In our case we speak of a 2-2-matrix, because it has two *rows* and two *columns*.
In the above matrix the numbers a and b are the *elements* of the *first row* whereas a and c are the *elements* of the *first column*.

Followed by:

Instead of $A\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2x_1 + x_2\\-x_1 + 3x_2\end{pmatrix}$ we now write:

$$A\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2 & 1\\-1 & 3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

and consider the righthand expression as the product of two matrices.
In general we define:

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix}ap + bq\\cp + dq\end{pmatrix}.$$

The matrix of the mapping A is also indicated by (A).
Thus the notation:

$$(A) = \begin{pmatrix} 5 & 3 \\ -2 & 7 \end{pmatrix}$$

means that the mapping A is determined by:

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -2 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5x_1 + 3x_2 \\ -2x_1 + 7x_2 \end{pmatrix}.$$

Looking back on this first chapter we may summarize its content as follows:

- definition transformation $\mathbb{R}_2 \to \mathbf{R}_2$
  - examples leading to the
- definition of a matrix;
- definition of matrix multiplication (2x2 times 2x1).

The beginning of the second chapter "The product of two 2-2-matrices" runs as follows:

Not unlike functions in Analysis, mappings can be composed.
We start with a vector x to which by the mapping A a vector Ax is assigned. Then the mapping B assigns to the vector Ax the vector $B(Ax)$. We then speak of the composite of B and A, which is expressed
by $B \circ A$, or otherwise, by BA.
If the mapping A is given by:

$$Ax = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$$

and the mapping B by:

$$Bx = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} px_1 + qx_2 \\ rx_1 + sx_2 \end{pmatrix}$$

we can arrive at the mapping prescription for the product mapping $B \circ A$ as follows:
We put:

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \text{ and } B\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix}.$$

Then:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} \text{ and } \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} px_1' + qx_2' \\ rx_1' + sx_2' \end{pmatrix}.$$

Thus:

$$B(Ax) = \begin{pmatrix} p(ax_1 + bx_2) + q(cx_1 + dx_2) \\ r(ax_1 + bx_2) + s(cx_1 + dx_2) \end{pmatrix}$$

and after some rearrangement:

$$B(Ax) = \begin{pmatrix} (pa + qc)x_1 + (pb + qd)x_2 \\ (ra + sc)x_1 + (rb + sd)x_2 \end{pmatrix}$$

From this it follows that the matrix of the mapping BA is:

$$(BA) = \begin{pmatrix} pa + qc & pb + qd \\ ra + sc & rb + sd \end{pmatrix}$$

This then is named the product of matrix B and matrix A.
We consider this multiplication more closely. Then we notice that the element in the *first* row and *second* column of the product matrix is obtained by multiplying each element of the *first* row of B with the corresponding one of the *second* column of A and afterward adding of these products. Then this element is the scalar product of the vectors $\begin{pmatrix} p \\ q \end{pmatrix}$ and $\begin{pmatrix} a \\ b \end{pmatrix}$.
As a formula we can write this:
If

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

then:

$$c_{11} = a_{11}b_{11} + a_{12}b_{21}$$
$$c_{12} = a_{11}b_{12} + a_{12}b_{22}$$
$$c_{21} = a_{21}b_{11} + a_{22}b_{21}$$
$$c_{22} = a_{21}b_{12} + a_{22}b_{22}$$

This can also be written in the form:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} \; (i, j = 1, 2).$$

This is followed by some remarks in which the:

- sum of two matrices;
- the zero-matrix;
- the opposite of a matrix;
- the scalar multiplication;

are defined by means of examples.

Finally, the chapter ends with two examples containing the definition of a fixed point. The geometrical problems are solved algebraically; no drawing illuminates problems and solution. The second chapter covers a little more than four pages. Altogether, the students must read more than seven pages of text before arriving at the third chapter: the exercises.

We give two representative examples:

1. Solve x_1 and x_2 from:

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

2. Give the matrix of the transformation $A^2 = A \cdot A$.
   The transformation A is given by:

$$\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}$$

Summarizing, we come to the following conclusion:

- Firstly: seven pages of definitions, rules and examples in the best tradition of both structuralist and mechanistic approaches.
  Insight and understanding have no high priority; certainly not into insight into the geometrical context.
- Secondly: one page of exercises; here again much attention is paid to the arithmetical and algebraical aspects, neglecting almost entirely the geometrical aspects. No interpretation (in a geometrical way) of the results.

One will agree that this is a strong example of a structuralist/mechanistic nature.

Next we look at the multiplication of matrices as introduced in the experimental Mathematics A material, which we have already termed as realistic. A superficial look at the table of contents reveals quite a different image:

- *Chapter 1,2*
  1. Orientation, exploration of matrices.
  2. Relation Graph-Matrix.
  3. Addition, intuitively.
  4. Scalar Multiplication, intuitively.

- *Chapter 3*
  5. Addition.
  6. Scalar Multiplication.
  7. Opposite of a Matrix.
  8. Multiplication, intuitively.

- *Chapter 4*
  9. Multiplication: matrix with 'vector'.

- *Chapter 5*
  10. Multiplication.

This enumeration already shows that the problem area is first explored intuitively. The students formulate their conjectures about addition, scalar multiplication, multiplication before the authors do so.
This way the students are allowed to develop their mathematical knowledge, with the intention that this deepens insight and understanding. Besides, the students receive the opportunity to formalize the concepts at their own pace.

Chapter 3 starts with:

JEANS
Jeans come in many brands and sizes. A shop has twenty-three pairs of Wrangler jeans in stock.

sizes:	*28" (28 inch waist):*	*3*
	30"	*11*
	32"	*6*
	34"	*3*

Other brands and sizes:

Levis	*: 5, 5, 3 and 4 resp.*
Club de France	*: 1, 7, 0 and 0*
Bobos	*: 3, 0, 0 and 3*
Ball	*: 3, 0, 0 and 3*

All this information can be written down well-ordered in matrix-form.

> 54 Complete the matrix:

$$\begin{array}{c} \\ 28'' \\ 30'' \\ 32'' \\ 34'' \end{array}\begin{array}{c} \begin{array}{ccccc} W & L & CF & Bo & Ba \end{array} \\ \begin{pmatrix} .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \end{pmatrix} \end{array}$$

> 55 *How many jeans in your size does the shop carry?*
> 56 *Clarify.*

The shopkeeper wants to enlarge his supplies, and he has to take into consideration that:

- *he sells about twice as much Wrangler and Levis jeans as other brands;*
- *sizes 30" and 32" sell twice as fast as 28" and 34";*
- *he does not want to keep more than forty items of one sort in stock.*

> 57 *Make up the list of orders with the above information in mind and write this down in matrix-form.*

The discussion starts in particular with reference to > 57.
Referring to the given information the students arrive at two different solutions:

$$A = \begin{pmatrix} 6 & 6 & 3 & 3 & 3 \\ 12 & 12 & 6 & 6 & 6 \\ 12 & 12 & 6 & 6 & 6 \\ 6 & 6 & 3 & 3 & 3 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 6 & 6 & 3 & 6 & 3 \\ 12 & 12 & 7 & 6 & 6 \\ 12 & 12 & 6 & 6 & 6 \\ 6 & 6 & 3 & 3 & 3 \end{pmatrix}$$

The difference between the two solutions is that solution A is not related to the real world *starting* situation:

$$A_0 = \begin{pmatrix} 3 & 5 & 1 & 6 & 3 \\ 11 & 5 & 7 & 2 & 0 \\ 6 & 3 & 0 & 2 & 0 \\ 3 & 4 & 0 & 2 & 3 \end{pmatrix}$$

Students with B as solution, defended their solution as follows:
"Answer A gives the ideal but theoretical situation for the owner. But the starting situation A_0 is such that it 'surpasses' the ideal situation A in two

entries. And, in view of the fact that it seems unlikely that the owner will throw away four pairs of jeans, we consider B the realistic answer."

After an interlude about the (im)possibilities of 'adding' matrices of different dimensions [71] the text returns to the jeans:

Back to the jeans shop. The supplies have been replenished. Now twelve pairs of the best-selling brands are in stock.

> 61 *Write down the supply-matrix (if this is different from the previous described one).*

> 62 *In one week there has been sold:*

$$V = \begin{pmatrix} 1 & 3 & 0 & 1 & 2 \\ 5 & 8 & 6 & 1 & 2 \\ 2 & 3 & 5 & 6 & 0 \\ 0 & 1 & 1 & 0 & 3 \end{pmatrix}$$

Write down the supplies at the end of that week in matrix-form.

> 63 *The (average) profit per pair of Wrangler jeans is f 30,--; Levis f 35,--; Club de France f 40,--; Bobos f 25,-- and Ball f 40,--.*
What is the total profit in those weeks on the smallest size?

> 64 *And on size 30"? And in total?*

Clearly the idea exercized here, albeit implicitly, is that in this context the 'unnatural' way of multiplying matrices bocomes 'natural'. (By 'unnatural' we refer to the fact that, when adding matrices, one adds the corresponding entries, but with multiplication we do *not* follow this obvious rule).[8]

After this first exploration the multiplication is further schematized and formalized at the beginning of chapter 4.

In the previous chapter the number of jeans a shop sold (in one week) was:

$$V = \begin{pmatrix} 1 & 3 & 0 & 1 & 2 \\ 5 & 8 & 6 & 1 & 2 \\ 2 & 3 & 5 & 6 & 0 \\ 0 & 1 & 1 & 0 & 3 \end{pmatrix}$$

The profit on Wranglers was f 30,--; Levis f 35,--; Club de France f 40,--; Bobos f 25,--; Ball f 40,--.
The profit on the smallest size can be computed as follows:
The profit per brand is described in a 5x1 matrix:

Profit 28" $\begin{pmatrix} 1 & 3 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 30 \\ 35 \\ 40 \\ 25 \\ 40 \end{pmatrix}$

$$= 1.30 + 3.35 + 0.40 + 1.25 + 2.40 = 240.$$

> 65 Compute in the same way the profit on size 30", 32" and 34".

> 66 The profit can be described in a 4x1 matrix-form:

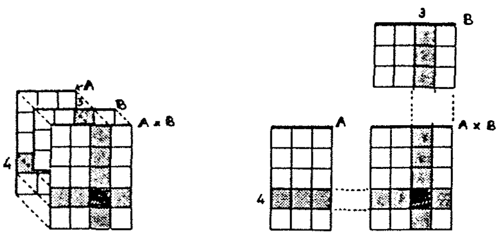
$$\begin{pmatrix} \textit{Profit on size 28} \\ \textit{Profit on size 30} \\ \textit{Profit on size 32} \\ \textit{Profit on size 34} \end{pmatrix}$$

The product of multiplying a 4x5 matrix by a 5x1 matrix is a 4x1 matrix.

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 2 \\ 5 & 8 & 6 & 1 & 2 \\ 2 & 3 & 5 & 6 & 0 \\ 0 & 1 & 1 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 30 \\ 35 \\ 40 \\ 25 \\ 40 \end{pmatrix} = \begin{pmatrix} .... \\ .... \\ .... \\ .... \end{pmatrix}$$

Multiplying matrices seems rather complicated, but in application this has many advantages. You are not allowed to multiply the corresponding entries as is the case in addition.

Compared with the structuralist/mechanistic example, the above realistic one shows a repeated interchange between text and exercises which, in the former example are strictly separated from each other. The first stage of matrix multiplication is column matrices multiplied by general ones, which in a natural way is followed by multiplication of matrices by matrices. The final formalization and abstraction exhibits two different levels.
First we get (fig. II.40):



Here you see a diagram of matrix-multiplication:
the 5 x 3 matrix A is multiplied by the
3 x 4 matrix B. This results in the n the
5 x 4 matrix A x B.
The fourth row of A "by' the third column of B
gives the entry of A x B in place 4,3 (double-
shaded) or $P_{4,3}$.
The condition for multiplication is that the
number of columns of A equals the number of
rows of B.

fig. I.40

Here you see a diagram of matrix-multiplication:
The 5x5 matrix A is multiplied by the 3x4 matrix B.
This results in the 5x4 matrix AxB.
The fourth row of A 'by' the third column of B gives the entry of AxB in place 4,3 (double shaded) or $P_{4,3}$.
The condition for multiplication is that the number of columns of A equal the number of rows of B.

Secondly, in order to finish with the 'usual' formal definition of matrix multiplication this definition was added - on teachers' request - in later versions.

Rather than concluding our comparison of two philosophies we would like to add some remarks. In the summer of 1985 a different approach to matrix multiplication for Mathematics A appeared in the series analyzed above representing the mechanistic/structuralistic approach. Earlier we have classified this new method as hybrid. The theory and exercises are strictly separated. But now the examples are borrowed from the real world. However, rather than being left to the students, the process of mathematization is performed by the authors themselves, as well as the process of concept development.
Of the seventeen exercises analyzed only three are more than bare arithmetic and algebra. Interpretation and reflection play at best a minor role. At this

to the roots of the series, and whether or not books like these will influence the examinations.

4 OTHER ASPECTS OF MATH A

4.1 Pure vs Applied Mathematics

The way applications are integrated into the Math A curriculum leads us to the discussion of the dichotomy 'Pure' vs 'Applied' mathematics.
In our opinion such a discussion should not be necessary because both aspects should be present in a well-balanced, successful mathematics education. Nevertheless it is interesting to note the sometimes emotional and seemingly never-ending discussion.
The disagreement starts already with Euclid. Although it may look strange, even Euclid's Elements have recently been claimed to be applied mathematics.
Not surprisingly, this was done by the 'Champion of Applications' [72], Kline.

In Kline's own words:

> "The real goal was the study of nature. In so far as the study of the physical world was concerned, even the truths of geometry were highly significant. It was clear to the Greeks that geometric principles were embodied in the entire structure of the universe, of which space was the primary component." [73]

The partisans of the 'pure' point of view find an advocate in the work of Proclus (410-485), who formulated the then prevailing view on the philosophy behind Euclid's Elements. Although Proclus - a Neoplatonist - recognizes the practical side of the elements, the principles are not to be found in the applications but in the *'beauty and ordering of mathematical reasoning.'*
This seems to be exactly the reason why mathematicians love mathematics.

If Kline is the Champion of Applied Mathematics, Hardy will have a good chance of being the Champion of Pure Mathematics. His near famous quote makes clear why:

> "I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly, for good or ill, the least difference to the amenity of the world ... Judged by all practical standards, the value of my mathematical life is nil; and outside mathematics it is trivial anyhow." [74]

The truth in the case of Euclid may be somewhere in the middle. To make our point clear we cite the following imaginary dialogue on the work of

Euclid between Archimedes and his patron Hieron [75]:

> *Hieron:*
> Speaking about gains, I recall a story about Euclid. One of his students studying geometry asked Euclid: "What shall I gain by learning these things?" Whereupon Euclid called his slave and said: "Give him a coin, since he wants to gain from what he learns."
> It seems to me that this story shows that Euclid thought it unnecessary for a mathematician to bother about the practical use of his results.
>
> *Archimedes:*
> I have, of course, heard the anecdote, but you will be surprised when I tell you that I sympathize completely with Euclid.
>
> *Hieron:*
> Now I am confused. Up to now you spoke enthusiastically about the applications of mathematics, and now you agree with the purists who think that the only reward which a scientist should expect is the pleasure of knowledge.
>
> *Archimedes:*
> I think you and most people misunderstand the story about Euclid. It does not mean that he was not interested in the practical consequences of mathematical results and that he considered them unworthy of a philosopher. This is pure nonsense; he has written, as you certainly know, a book called '*Phaenomena*', about astronomy, and a book on optics, and he is probably the author of the book '*Catoptrica*' too, which I used in constructing mirrors; he was also interested in mechanics.
> Mathematics is like your daughter Helena, who suspects every time a suitor appears that he is not really in love with her, but is interested in her only because he wants to be the son-in-law of the King.
> Similarly, mathematics reveals its secrets only to those who approach it with pure love, for its own beauty. Those who do this are, of course, rewarded with results of practical importance."

Society tolerates mathematics because of this practical importance. Greco-Roman technology certainly required more mathematics than did that of Babylon and Egypt previously, but it was still a poorly applied mathematics and this hampered the development of pure mathematics.
As Archimedes stated in the dialogue with Hieron:

> "You remember I told you that the Romans would never be really successful in applying mathematics. Well, now you know why; they are too practical-minded."

Mathematics in the Roman time was almost identical to astrology; it continued to play a modest role in society until the 16th century. According to Freudenthal [60] this growth was prepared by the fact that between 1200 and 1500 more things were invented than ever before in human history - a chain of very fundamental inventions in which the art of printing is the last link.
The same attitude of mind which concocts inventions, plays tricks on nature, and searches for the secrets in numbers and figures caused this sudden growth.

Mathematics was used more and more.
A large number of textbooks with often a practical touch were written. It was around 1713-1715 that the trend towards useful mathematics was concretized in new 'Elements' : Christian Wolf's *'Elementa Matheseos Universae'*.
According to Bos, [76] the main difference between this and the 'Elements' of Euclid lies in the fact that Wolf accepts very clearly applications as part of mathematics; this was the generally accepted idea about mathematics in the 17th and 18th century. But Euclid and Wolf agree on one point:
Primarily not the contents of the theorems is important, but the method and style of reasoning. Although Bos argues that mathematics is at its 'widest' in the 17th and 18th century, the really great influence of applications on the problems of mathematics dates from the beginning of the 19th century.
Freudenthal [60] sees a key role for Fourier, Poisson and Cauchy in the shift towards applications that turned out to be enormously beneficial for the development of mathematics in the 18th century.

But, too, in this century pure mathematics was created. The vast expansion of mathematics and science made it more and more difficult to be at home in both fields. The idea that mathematics is not a body of truths about nature was another source that altered radically the mathematicians' attitude towards their own work. The idea arose that it was not necessary to undertake problems of the real world. Man-made mathematics would surely prove useful.

Abstraction, generalization, and above all, specialization are three types of activity undertaken by pure mathematicians. The divergence from 'reality', the study of mathematics for its own sake provoked much discussion almost directly from the beginning. Many physicists and mathematicians - like Lord Kelvin [1867], Klein [1895], Poincaré [1905], Courant [1924] - warned against the danger of mathematics becoming more and more isolated.
As Von Neumann stated in his essay *'The Mathematician'* [1947]:

> "As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired from 'reality' it is beset with very grave dangers. It becomes more and more

pure aestheticizing, more and more purely l'art pour l'art." [77]
Although completely inspired by Von Neumann, and admitting that virtually all of mathematics is rooted in the physical world, Halmos not only sees a strong dichotomy between mathematics and applied mathematics, but also regards mathematics as 'good' and applied mathematics as 'bad'. [78]
Or, more precisely:

"Mathematics is security. Certainty, Truth, Beauty, Insight, Structure, Architecture."

"Applied Mathematics is much too often. Bad, Ugly, Badly Arranged, Sloppy, Untrue, Undigested, Unorganized, and Unarchitected mathematics."

The theory of structures, as we have noted before, received a great impetus from the work of N. Bourbaki. This group of French mathematicians published a series of books, again with the title: *'Eléments de Mathématique'* [1939].
This structuralist approach, based on set theory, was of great influence after the Second World War. But when the influence spread to primary and secondary education, via the 'New Math' and the introduction of set theory, the amount of criticism increased as we have described in 3.1.
Eventually this criticism has had considerable effect. We already quoted Dieudonné advocating at the Royaumont Conference in 1959, the introduction of some structuralist aspects into secondary education; some twenty years later he seems to make a correction:

"Never has Bourbaki given any opinions about the possibilities or the desirability of implementing the concepts written in this book at a lower level, and certainly not at primary or secondary schools.
People, proposing such aberrations have often, to say it politely, a poor knowledge of modern mathematics." [79]

In general, one can state that in the first half of this century pure mathematics stood higher than applied mathematics. Or, as Davis and Hersh put it: "the mental universe stands higher than the physical universe." [54] This leads to more and more pure mathematics, which in turn leads to critical remarks from people outside the mathematical scene. In their isolated ivory tower they do not seem responsible to society and avoid answering questions about the meaning of their work.
Or, as E. Bishop once said in a discussion to Kahane:

"Most mathematicians feel that mathematics has meaning, but it bores them to try to find out what it is." [80]

But, in recent decades, there seems to be a change in attitude. Not only is there a trend towards unification in mathematics, but there is also the feeling that true applied mathematics may be an art as well.
Or, as Hilton says:

There are many examples of the art of applied mathematics, and we see that it is correct to speak of the science of mathematics, and of the art of applied mathematics. For, since mathematics incorporates a systematic body of knowledge and involves cumulative reasoning and understanding, it is to that extent a science. And since applied mathematics involves choices which must be made on the basis of experience, intuition, and even inspiration, it partakes of the quality of art. Thus in mathematics there are certainly to be found both art and science, and there is science in both pure and applied mathematics, as there is art." [25]

As indicated before we consider the dichotomy 'Pure' vs 'Applied' as a false one, and much of the discussion not very fruitful, not excluding the present one.
One problem lies in defining just what we mean by 'applied' mathematics.
Pollak's contribution to *New Trends in Mathematics Teaching* clarifies this in the following way [81]:

1. *Applied Mathematics means classical applied mathematics,* by which he means those classical branches of mathematics such as analysis (including calculus), ordinary and partial differential equations, integral equations, theory of functions, in fact those branches of mathematics most applicable to classical physics, although he stresses that no actual connection with physics problems is implied.

2. *Applied Mathematics means all mathematics that has significant application* which entails (1) more especially by including such topics as set theory and logic, linear algebra, probability, statistics and computing.

3. *Applied Mathematics means beginning with a situation in some other field or in real life,* making a mathematical interpretation, or model, doing some mathematics on the model and applying the result to the original solution.
   The 'other field' is by no means restricted to physical science and includes areas of application in biology, social studies, management and business studies, and so on.

4. *Applied Mathematics means what people who apply mathematics in their livelihood actually do.* (Like (3), but more so!).

This means that (1) and (2) are defined *within* mathematics; it is only in definitions (3) and (4) that we see interaction between mathematics and the rest of the world. [82]
The division between (3) and (4) comes close to the distinction made by Kemme between Applied Mathematics (3) and Users Mathematics (4). [71]
It will be clear from our previous description that Mathematics A fits well in Pollak's category (4).
Next we will turn to Pure vs Applied in Mathematics *Education.*

4.2 Pure vs. Applied in Education

During the last 80 years there has always been discussion about the desirability of including applications in mathematics education. And certainly during recent decades there is an obvious trend in literature towards more applications, although Blum doubted whether one could speak of a similar trend in the actual classroom situation. [83]
Pollak also notices a worldwide drive to make mathematics in the school more applied. [81] He adds that in some cases, the applied point of view is a more accurate reflection of the country's social system. "What we demand", wrote Chairman Mao-Tse-Tung, "is the unity of politics and art." [55]

In some cases it is part of a reaction against the 'New Math', as we can see in Kline's *'Why Johnny can't Add'*. [84]
In others it is a recognition of the increased mathematization of many other fields than physics. Many countries have found that the problem of motivating students becomes easier when applications are used as one of the possible motivations.
Finally more and more real problems become available for use in education.
Bell [85] believes that progress is now possible where very little progress has been made in eighty years or more because of the powerful factors mentioned above. But his worldwide survey shows that Blum may be correct when stating that in classroom practice this progress has yet to be made. Characteristic comments in Bell's survey are these:

> "There is no program I know of that succeeds in including 'true' applications of mathematics in school textbooks."
>
> "It is only a slight exaggeration to say that mathematics education for most students beyond age 11-15 is completely devoid of applications except as some elementary school teachers try to show how arithmetic is used."

Very important seems the fact that experimental versions of textbooks sometimes contained strong applications, but that many such textbooks did not survive; for those that did survive, "dilution and corruption" of material difficult to teach had in revised editions eroded the emphasis on applications.
The teachers are often educated with pure mathematics and seem reluctant to teach the more difficult applied mathematics. For this reason, a series of textbooks with applied mathematics for secondary school level in Brasil is not a success. Or as others state it - and this is also our experience during the Hewet project - teachers are afraid of admitting other disciplines into Mathematics.

The problem is certainly not new. We showed already the applied mathematics examples from Lietzmann in Germany (1925), but in the fifties he cancels most of them. About his ideas in the twenties he made the remark that

not all teachers were ready to teach his way.
In the Netherlands a much used book [Van Thijn and Kobus, 1947] contains some interesting problems in the chapter *'Graphical Representation'* which could fit neatly into the Mathematics A curriculum but, one fears, never made it into classroom practice. [86]

One of the few countries where the mathematics curriculum has always been firmly related to classical applied mathematics is Great Britain. But we have to keep in mind that 'applied mathematics' normally meant 'mechanics', later more or less substituted by statistics.
Statistics, of course, has found its way into many curricula. Maybe this is one of the reasons why teachers now are sometimes more open to applied mathematics than they used to be.
The same substitution - 'mechanics' by 'statistics' was made in the Netherlands. Mechanics was represented in one of the streams up to 1963, but when a new curriculum was introduced disappeared in favour of statistics.

Initially - and this will not come as a surprise - the discussion on applications focused on applications from physics. In January, 1967 there was a conference on Reform and Co-ordination of the Teaching of Mathematics and Physics, in June 1973 an ICME Conference on *'Les Applications Nouvelles des Mathêmatiques et l'Enseignement Secondaire'* where computer science was proposed as field of applications.
A more influential conference is: "*How to teach mathematics so as to be useful*" held in 1967 in Utrecht. Here the view was broadened from the world of physics to the real world in general and the usefulness of mathematics is stressed. We find pleas for using this real-world to teach mathematizing, while teaching math concepts along the way.

By the end of the seventies, this kind of conference and discussion has become more and more frequent: we mentioned the role of applications in education at the ICME 3 in 1976, followed in 1978 by a Unesco Conference on the co-operation between science teachers and mathematics teachers.

At ICME 3, in Karlsruhe, Sir James Lighthill presented in his main lecture some examples of the *Interaction between Mathematics and Society* [87]. Furthermore, there was a Section on *The Interaction between Mathematics and Other School Subjects* where Pollak presented his four definitions of applied mathematics as presented in 4.1 [81]. In this Section Ormell presented his 'Mathematics applicable' project.
The conference on *'Co-operation between Science Teachers and Mathematics Teachers'* in Bielefeld in 1978 offered many interesting papers on applied mathematics in education. Jung noticed an increasing gap between mathematics and science curricula [88] but most other presentations were

more positive in respect to a closer relationship. Especially interesting was D'Ambrosio's *'Strategies for a closer relationship of mathematics with the other sciences in education'* because of his aim towards an integrated view of natural, social and cultural phenomena. [89]

In 1980 at ICME-4 the applications were discussed in several groups, and on this occasion the Hewet project was mentioned for the first time. [90]
During this period most people became convinced that including many applications and stressing the usefulness of mathematics is particularly desirable for those students who do not continue to study mathematics. This point was emphasized, among other people, by Howson in 1974 [91] and Van der Blij in 1967 [92].

Another important event took place in 1983, at the first conference on Teaching and Applying Mathematical Modelling in Exeter [82], which was followed by a second one in 1985.

In the U.S.A. attention to applications is increasing. At the University of Chicago Usiskin leads a project on arithmetic and applications. Also at Chicago M. Bell carries out his work by applications that we mentioned earlier.
More fundamental is the work of Lesh in Provo, Utah, who uses the real world in a similar way as we do. [93]
This brings us back to our chapter on conceptual mathematization as Lesh states:

> "We believe that applications and problem solving should not be reserved for consideration only *after* learning has occurred; they can and should be used as a context within which the learning of mathematical ideas takes place."

4.3 Analog vs Analytic

According to Davis and Hersh [54] it is convenient to distinguish two aspects of mathematical experience: 'analog experimental' and 'analytic' mathematics.
Mathematizing by analogy is based on 'intuition', 'understanding', 'feeling' while little if any use is made of the abstract symbolic structures of school mathematics.
In the analytic approach, the symbolic material predominates. It is the great virtue of analytic mathematics that verifying another's intuitions is, if at all possible, harder than unifying his proofs. It is clear that in the Mathematics A much emphasis is laid on mathematization by analogy, especially in its horizontal mathematization stage. Analytic mathematics comes in only at a latter stage.
This is not a surprise because, according to Davis and Hersh and Hilton the

analytic mathematics requires special training and is 'elitist' on the way that only the talented scholars do need and are able to master this kind of mathematics.
In classical textbooks there is not much emphasis on analog mathematization.

In the opinion of some, this is a drawback, if viewed in the light of an intriguing though still much disputed theory about the two distinct modes of thought in relation to the specialization seen in the hemispheres of the brain.[9]
Ornstein [101] has become an advocate of the idea that there are alternate ways of knowing and alternate forms of consciousness. He feels that our intellectual training unduly emphasizes the analytical with the result that we have learned to look at unconnected fragments instead of at an entire solution. Or, as Watzlawick states it:

> "On one side one has the logical-methodological, step-by-step approach (sometimes not seeing the forest because of the trees). At the other side the global, holistic observation (perception) of totalities, of 'gestalts' (sometimes not seeing the trees because of the forest)." [100]

According to Springer and Deutsch [102] the idea that half - more precisely, the right half - of our mental capability is neglected has been appearing with increasing frequency in educational journals, self-help manuels and a variety of other publications. Articles usually include a background summary of some of the data on laterality along with the author's personal interpretation of what the data mean.
They - Springer and Deutsch - question however, the division of styles of thinking along hemispheric lines. It may very well be that in certain stages the formation of new ideas involves intuitive processes independent of analytic reasoning.

> "Preliminary schemes ordering new data or re-ordering preexisting knowledge could possibly arise from even aimless wanderings of the mind during which a connection is seen between a present and a past event or a remote analogy is established. But are these right hemisphere functions?"

Nevertheless they agree on the different functions of the hemispheres. And that there are different modes of thinking: analog vs. analytic. They agree completely with Sagan when he states that the most significant creative activities of a culture - including science - are the result of collaborative work by the left and right hemispheres. And, Sagan states [103]:

> "Mere critical thinking, without creative and intuitive insights is sterile and doomed."

In our opinion the call for collaborative work by the left and right hemispheres - and thus for more creative and intuitive insights - is also valid for mathematics education. In structuralist and mechanistic math education these aspects are almost non-existent. In mathematics and mathematics education the co-operation between the analytic and analog understanding left something to be desired. There seems no doubt that in the Math A program there is more room for intuition - in the first exploratory stage - and creativity. [10]

4.4 Epilogue

We started this chapter with a *description* of the contents of Math A which was followed by a *analysis* of the curriculum based on the experimental booklets and the observations in the classroom. We pointed out that Math A is *process oriented* (mathematization) which will have consequences for its testing as we will describe in later chapters.
Conceptual mathematization was discussed, and a sketch of a theoretical framework was given. Together with the role of the context, the critical attitude of students and the role of social interaction this framework made clear that Math A is an example of realistic mathematics education.
Because of the role of applications we paid some attention to the not very fruitful discussion on the dichotomy between pure and applied mathematics.
We concluded this chapter by paying attention to another debate: on analog versus analytic understanding. Whether or not these modes of understanding are related to left and right hemispheres is not so important in light of the Math A program. Important, however, is the fact that Math A offers ample opportunities for both modes of understanding.

In the next chapter we will see how the experiences with Math A in the classroom were: Math A in action.

III STUDENTS AND TEACHERS ON MATH A

1 EXPERIENCES

1.1 Introduction

The goal of this chapter is twofold.
In the first place it offers an *evaluation* of the two-ten-forty-schools experiences during the experimental period.
In the second place it makes clear that the *problem of finding proper tests* is a real problem.
Or, to present it in a different way: it gives an impression of how the Math A curriculum is experienced by teachers and students - within the experiment. In this perspective it forms a natural sequel to the first two chapters: the experiment - Math A - the experiences.

But the fact that one of the main problems mentioned is the testing of Math A makes this chapter also the starting point of a new sequence: Experiences - Restricted-time Written Tests - Alternative Tasks.

The evaluation is based (roughly) on empirical material, sampled during the experimental stage of the project.
To be more precise, it is based on:

1. Discussions with students and teachers from the two schools and observations at two schools.
2. Articles on classroom activities (mainly in 'Nieuwe Wiskrant').
3. A panel of students from the two schools, with their two teachers.
4. Discussions with teachers from ten schools during their teacher training course.
5. Discussions with teachers from the ten schools during special meetings where progress was to be evaluated.
6. Report on the Hewet Experiment 1981-'82.
7. Report on the Hewet Experiment 1982-'83.
8. Report on experiences in the 10^{th} grade of the forty schools.
9. Conferences for forty schools (two).
10. In-stream-research.
11. Discussions by counsellors from the ten schools.
12. Reflection on all of the above.

These materials were sampled during the period from August 1981 to the end of 1985. All of them are documented by *written* reports. Where suitable we will refer to the sources. In our description we will proceed along the stages of the *two* schools, *ten* schools (twelve schools) and *forty* schools (fifty-two schools).

1.2 The two schools

The first impression dates from the first lessons. [1]
At one of the two schools the students were used to mechanistic - structuralist education and frontal teaching without much interaction. So it came to the students as a surprise that they had to start by reading the booklet.
"Just start at page 1" was the only explanation the teacher gave. No doubt the material looked confusing too. The most frequent question was: "Is this mathematics?"
The following incident demonstrates this confusion: the graph of a road map of the island of Hau is drawn (fig. III.1):

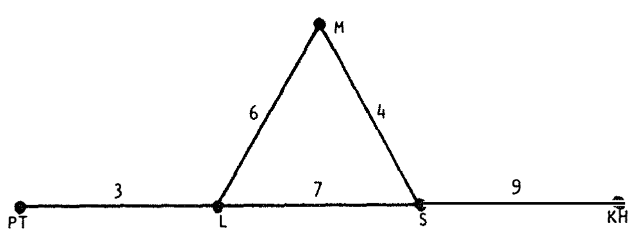


fig. III.1

with the distances in kilometers.
This graph can be used to make a distance-table; it is simpler than the original map. To compute the distance from PT to M one just adds the 3 and 6 to obtain 9 kilometers.
One student, however, who fancied himself in pure-mathematics-land tried to obtain the distance between PT and M by Pythagoras.
Suddenly he was back in the context:

"Oh no, those are roads, you just go from PT to L and then on to M."

Another source of confusion is the abundance of information. The students ask:

"Why don't you just give us exactly the information needed?"

At this school the students are not used to discussing problems with each other, whereas at the second school much emphasis is put on group work. With the new curriculum, however, discussions are going to play a large role at both schools, and interaction is going to contribute to the cognitive development.

Example:
The students are asked to draw the graph of the roads of Malaita (fig. III.2):

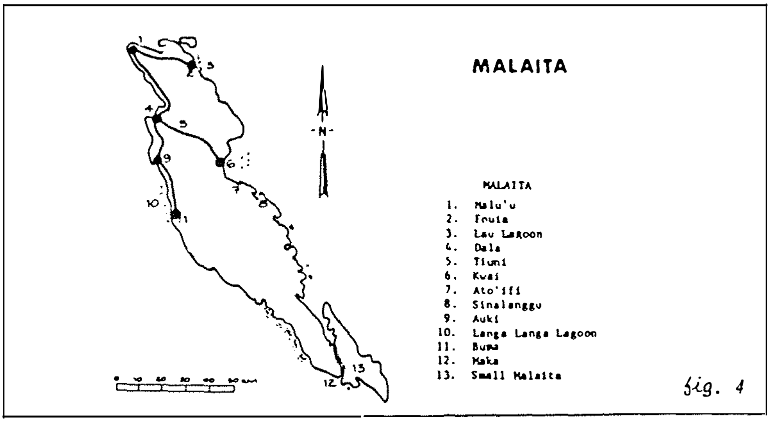


fig. III.2

The teacher collects the results (fig. III.3):

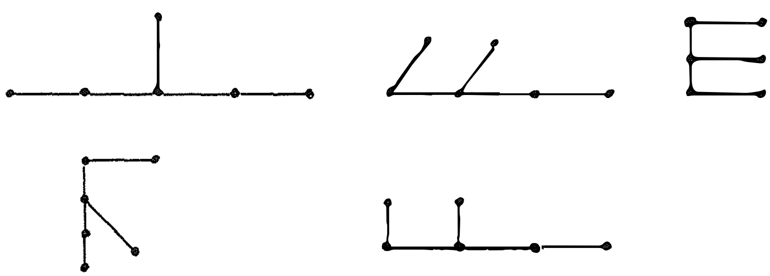


fig. III.3

Rightaway, discussion starts on whether or not the graphs are the same. This helps the students to clear up the graph concept. The next question is closely related to the preceding one:

"What is the maximum number of 'roads' you can draw in this graph?"

Many students interpret geometrically rather than conbinatorically.
In the figure (fig. III.4):

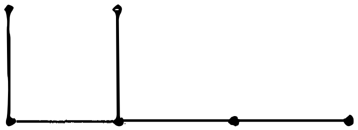


fig. III.4

One can draw lines like (fig. III.5):

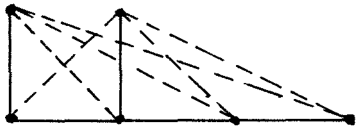


fig. III.5

A few produce even the following 'roads' (fig. III.6):

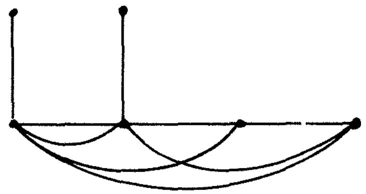


fig. III.6

The discussion boils down to distinguishing roads in 'reality' and roads in graphs. The seeming collinearity of the four points invites real understanding of the difference.

The teacher may try to help the students by drawing the Malaita graph as follows (fig. III.7):

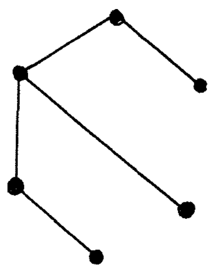


fig. III.7

New light was shed on the problem by another quite convincing solution:

"The distance-table has 36 entries; the diagonal does not count, which leaves us with 36 - 6 = 30. Because of the symmetry we get 15 roads."

This took place during the first two lessons. In spite of some exciting moments, the overall impression was still one of uncertainty. The next weeks, however, showed improvement. The students became familiar with this way of doing mathematics. They became more inventive, exposed their own views less reluctantly, discussed solutions and constructed their own definitions. About the addition of matrices:

"You just lay those matrices on top of each other."

Some seven lessons later the mood has changed:

"It is less boring."

"At least I can understand it."

"You don't have to calculate so much, you have to find out things."

"You see that mathematics is useful."

At the other school, accustomed to group work, the impressions were similar. Comments from students at this school [2]:

"Very nice compared to 'normal' mathematics and very nice on the whole."

"I prefer Math A above ordinary mathematics, though mathematics is still not my favourite."

"Much more discussion. You really have to think about everything, not just formulas and strings of exercises."

There are some critical remarks as well:

"The trouble is you can never know for sure."

"Sometimes questions are not clear enough."

Later on in the same first year the teacher at the Haarlem school, Kees Lagerwaard, told a newspaper about his experiences:

"In general, the students seem to enjoy working with this new mathematics.

To be sure, we have just started. Anyway, the students are attracted: now they can see when and where mathematics may be useful."

One year and a half later it was all over for the first 45 students. In May, 1983 they took their first exam. Directly after the exam four Haarlem students and their teacher joined with four 11th grade students from Zevenaar and their teachers to take part in a panel. The audience consisted of some thirty-five *teachers* from the ten schools and teacher trainers (who would train teachers from the forty schools the following year). Almost three years later one is tempted to conclude that many of the observations would seem to be valid not only for the two schools but also for the ten and for the forty schools.

It is for that reason that we give ample excerpts from this panel.

Participants:

Wim (Kremers), teacher at Zevenaar school, with students Yolanda, Servaas, Monique, Jan.

(Group-work is essential at this school).

Kees (Lagerwaard), teacher at Haarlem school, with students Annemarie, Judith, Laura, Marius.

The Zevenaar students are in the 11th grade; the Haarlem students are in the 12th grade, and have just taken their final exam.

1. Introduction of students and why they chose Math A

Wim: "We didn't prepare this in the sense that each student would tell a long story. I've asked them to introduce themselves and to tell what their background was that led them to Math A. After that the simplest thing is for you to ask questions."

Yolanda: "I first did HAVO and then VWO[11]. At the HAVO I first did 'normal' Math and then Math A. I dreaded it a bit in the beginning but it wasn't so bad at all. It was explained differently and went a lot deeper. That's what I see as the difference."

Servaas: "I did VWO-tenth grade and then Math A. I thought that Math I would be much too abstract for me. In Math A I can at least see what I'm doing."

Monique: "I also did VWO-tenth grade and then Math A. It wasn't so bad at all, lots more examples and stuff."

Jan: "I did Math I the first term and switched after that to Math A. I found Math I too difficult, I had to work too hard. It didn't go so well because I didn't find it so interesting. In Math I you don't really know what you're doing, but Math A has a story so you know what you're supposed to be calculating."

Wim: "I asked the entire class. Everyone in their class without exception finds it a fun course and half of them find it a difficult one."

Annemarie: "I chose Math A because I liked it being more practically-oriented than Math I and it's much clearer. I found Math I a lot more boring."

Judith: "I chose Math A because I didn't think I could really manage Math I. What I liked so much about Math A is that it's practically-oriented. You're really involved in solving a problem and that just interests me a lot more than learning formulas by heart that you forget the week after."

Marius: "I had to do eleventh grade over, I had a B-task but I couldn't handle it. I had to do a lot at home but now I have Math A and I can follow that better in class so I don't have to do so much at home. I'm planning to do a social study in the future and I think Math A is really appropriate for that."

Laura: "I didn't want to do any Math at all because it was too difficult for me. But then Math A was offered and it is much more practically-oriented, like the others said."

2. How difficult is Math A?

Teacher: "I keep hearing that Math A is easy, but I can still imagine that there must be difficult parts too. Can you tell me which parts of the subject matter are more difficult than others?"

Annemarie: " I found probability the most difficult, that was the least clear."

Judith: "Yes, sometimes I see that people think Math A is so easy, but I don't agree at all. I've had to work hard, for probability and that kind of thing. It's not like we can do it one-two-three and that's why we enjoy it."

Marius: "I think that for A-people, who normally wouldn't choose Math, this is a really great subject, because it's much more appealing, it's written more in a story form and I think that you can learn a lot of Math in this way."

Yolanda: "Probability isn't so difficult for me, but maybe that's because I had that at the HAVO. I found computers really difficult, I'd never had that before."

Teacher: "Math A contains many more of those story-sums, compared with how it used to be. With story-sums you often get a long text to read, and then you have to do something with it. We often hear the complaint that, although it's fun, you have to read lots of those stories before you get enough practice. I've noticed that none of you have mentioned that - have you had enough of that to be able to say "I understand that section?"

Marius: "Well, it's not true that there are pages and pages of text, there's just a short introductory text. It's not true that reading takes so much time that we don't get around to the theory."

Judith: "With normal Math you first got the theory and then the sums. Here you usually begin directly with a sum and a story, but the first sum is usually pretty easy. Then they build it up without you noticing so that the difficult things emerge naturally."

Annemarie: "The way to do it, if you know the way - the method - then you can do it."

Jan: "First of all, it's great that you don't have to do one sum after another, it's kind of relaxing if you get a story once in a while, then you're not working so intensively all the time. It's less tiring. When you get those stories all the time you get practice in searching out the essential bits."

Wim: "If you look at this morning's sum, with the proceeds and prices, do you find it difficult because it's a story or do you say that's not so bad?"

Jan: "It's not so bad at all. Maybe it's kind of confusing now and then, but I think that it's a good idea to learn to separate the main issues from the side issues in those stories."

Monique: "The story keeps changing but the sum stays the same."

Wim: "I don't think that this can be separated from what took place in the lower grades. This is, of course, a group that already had worked with this material in tenth grade."

Teacher: "Our experience in tenth grade is that the students have trouble reading those stories. Was that true for you as well?"

Monique: "Well, I don't know if that's because of the stories, I don't think so."

Kees: "At our school they still had Math in the traditional way in tenth grade. So the transition was a pretty big step, the step to Math A."

Laura: "It's because of those stories that it's clear to me what the idea is, what it's about. In my case the lack of anything being told was why I didn't understand it. I didn't know what it was about and had to learn it by heart. Here there's a story around it and it just happens as a matter of course."

Marius: "It's just a lot more appealing. For instance, in tenth grade we had logarithms. You learned it, but I couldn't follow it very well in class because it just didn't interest me, while now it does. Logarithms become much more clear through the examples."

4. Interdisciplinarity

Teacher: "I know of students who didn't like taking Math A because they said "with Math you now already get biology."
Did you find that as well?"

Judith: "Oh, you get those Mendel sums and stuff like that but you can hardly call it biology. If you're not good in biology it doesn't mean that you won't be able to do those sums well."

Laura: "I think that it does help."

Judith: "It's that you can see the connection, that you can apply math to other neat subjects. Just like economics."

Marius: "Yeah, and like physics. I was able to use Math I for physics and now I can use Math A for economics and biology."

5. Conceptual mathematization and Catching up

Teacher: "I want to ask Yolanda something because she also did the senior years of HAVO. With the old math you could let it slide for a week and then catch up. In my experience, that doesn't work nearly as well with Math A. If you let it slide you lose the practice of solving these kinds of problems. Is that so?"

Yolanda: "Yes, that's right, because with normal math it was first explained and after that came the exercises. For instance, you had a row of a's and b's and also h's. You only had to do the h's to see if you'd understood it. You could stop paying attention but if you just understood the last sums then you knew the entire chapter. But now each chapter is a whole sum in itself. So you have to pay attention each time."

Jan: "I think it is also possible to let Math A slide. Usually you'll figure out the formula yourself. In Math I you were given the formula directly -lop - and then you worked with it in problems of increasing difficulty. In Math A you gradually work towards the formula. If you miss that approach work it's not actually such a problem because once you know the formula you can still do the sums. You just don't know why the formula is the way it is. You may find that annoying for yourself, but it won't make that much difference as far as your knowledge of math or of doing sums goes."

6. Cumulative Problems

Teacher: "I'd like to ask the twelfth graders something. I can imagine that it all goes o.k. in class. But when there's a test you're on your own. Don't you have the impression that you collide sooner with chain-questions, questions which are derived from each other? I can imagine that it might all go wrong - I couldn't do a and therefore I can't do b, etc. because you get stuck in that story."

Marius: "On one test we had linear programming. If you weren't able to figure out the restricting conditions in the story then you couldn't get any further."

Judith: "But you always have four or five sums. In Math I you sometimes only had two or three."

Teacher: "There's also 'finding the function'. If you can't do that then you can't get any further either."

Annemarie: "That's just as true for Math I."

Marius: "At one point there was a graph and you had to find the formula for it. But that was the final chapter, it stood apart from the rest."

Teacher: "I wondered if you sometimes developed a method, like, go on like this from here."

Judith: "Of course you go on, you try it out."

Annemarie: "If that's wrong but the rest of the sum is right as far as the method is concerned then that will be counted correct, or at least a part of it."

Wim: "I think that when we put together a test we try to avoid that chain-effect, for instance by saying 'prove that this is it'."

Teacher: "But that's much more difficult in Math A."

Wim: "Yes, at any rate if you want it to be at all true to life."

7. Fun subjects in Math A

Teacher: "Which subject in Math A did you find the most fun?"

Judith: "Spatial drawing, with three axes. Probability and Linear

Programming I didn't like so much."

Servaas: "I liked working with computers the best. Making up a program and typing it in and seeing what comes out."

Monique: "Matrices and computers."

Wim: "We're lucky to have a number of computers at school, and the difference with this class is that the computers are much more integrated into Matrices and Differentiation than last year. That's been a lot of fun and we've let the students work with them a great deal. Then, too, the students developed an awful lot on their own initiative. For instance: how should I alter a program if I only want that printed. Or, how can I make it more extensive. Not every student did that, but a good number of them did."

Teacher: "Do I conclude from this that you've learned a bit of programming?"

Wim: "That's too big a word for translating a flow-chart that's in the book into a program. I think that the students use the word programming, but I don't."

Jan: "We were given a story and we had to make a flow-chart and a program to go along with it. Sure, it's simple, but you yourself are making all of what you're putting into the computer."

Wim: "It was sort of the difference between what was obligatory - which was not programming, and what some students did with it - which was programming. It was nice to see."

8. Group-work

Teacher: "Usually in class I guess you worked mostly in small groups, but the exam you have to do on your own. Just like the tests. How did you feel about that: always in groups and then alone?"

Judith: "Groups, that's a little ... at the most there were two of us. And one works a lot faster than the others, some of them were already ten sums farther. You still have to do it yourself. And you do your homework alone too."

Monique: "Four of us sat together, but I always worked on my own with another girl. And they had a mixed group."

Jan: "I enjoyed it. You were allowed to figure it out by yourself, so you went and sat with people whom you got along with. That worked out just great with us. And it wasn't like one had a big mouth and the other meekly wrote it down. We really worked together so that everyone did part of it and we all understood the whole thing. If someone didn't get it then we just stopped and explained it until he did get it and if that didn't work in the group then we'd call the teacher over. For me it was ideal, working in groups and it wasn't a disadvantage for the test either. But you do have to be lucky enough to sit in a group that works well together."

Servaas: "I think in fact that, when working in a group, if one person doesn't get it you have to explain it, and when he finally does get it then you know you really understand it yourself. 'Cause I've had that a lot: I get it more or less but I don't know why it's like it is. By explaining it to someone else you learn a lot yourself."

9. Homework

Teacher: "Can you give me an indication of how much time you have to spend at home on Math A?"

Marius: "I didn't have to work so long because I understood it in class. In Math I I didn't get it in class at all, so I had to work hard at home.

Annemarie: "You could do a lot in class. If you did a lot there you didn't have to do so much at home."

Marius: "You were forced to do a lot in class because you were told - this has to be finished next time - so it was in your own interest to do a lot."

Laura: "It was your own choice. If, for instance, you were given ten sums and you didn't do them at home and then they went through it quickly in class, it could be rough. Ten sums, that took an hour at home, maybe even longer. It wasn't like the sums were so easy that you could just toss them off in five minutes."

Jan: "You got most of it done during class. What you had to do at home, that depends, four or five sums usually take a half-hour. And if they're more difficult, I didn't find that such a problem, 'cause thinking them over isn't a drag. And if you skipped it once, then it was explained in class so that you could understand it; the only problem is that then you hadn't done it yourself, which is too bad. I don't like that, by the way, I prefer to do it myself, but that doesn't mean that I did my homework every time."

Monique: "I didn't think we had a whole lot of homework. Most of it you managed to get done during class. And the rest, 15 minutes, a half-hour."

Servaas: "I usually finished during class. When there's some left to do at home then it's really tough for me to get it done. But I think that has more to do with me."

Marius: "But that's because we're still in the experimental phase. The tempo is adapted to the students. Maybe you're a little shocked that we only do fifteen minutes to a half-hour of homework, but I think that that'll be increased later."

Yolanda: "Sometimes we'll have done a sum in class and then at home I don't know at all any more what it's about. And then I have to do the whole sum over."

Teacher: "Do you ever doubt if you've done it right?"

Yolanda: "Sure, and then it's a drag that there aren't any solutions at the back. You wonder whether you've got it right and there's no way to check if you're on the right track."

Teacher: "Yes, I miss that, that you doubt 'whether I've got it right'. Because with these kinds of sums in particular it is important what the result is, because it means something. More important than whether the result is a 6 or an 8."

10. Test and exams

Teacher: "You said that Probability is kind of tough. Math A is certainly not an easy subject. My question is, how do you study Math A? Are you able to glean the essential points from the subject matter and to grasp it?"

Annemarie: "I was able to gather the most important things and then maybe do some sums."

Teacher: "Are they shown clearly, is it easy to pick them out?"

Judith: "You gradually get to know what the most important sums are, you know what to watch out for. You knew at any rate that you could expect a sum from Linear Programming, either spatially or with the Simplex method. So you know a little bit how and what. The initial sum is sort of unexpected, those are more introductory sums to help construct something. I heard one student say: "I'm going to learn it really well, I'm going to do all the sums." If you ask me, that's the wrong track. There are certain sums I couldn't do if they were set in front of me. You're working more in a specialized kind of way. Maybe there's a little guesswork."

Teacher: "You have all done your final exams today. Did the questions pretty much meet with your expectations?"

Marius: "Yes, definitely. It linked up completely with the textbooks."

Judith: "Yes, because the last internal exam contained the same material as the final exam."

Teacher: "I have one more question about the exam. Were you particularly nervous for Math A compared to other subjects?"

Students: "No."

11. Teachers

Teacher: "If I've understood correctly, Math A is quite a different subject from the old-fashioned math as you've described it, namely Math I. I don't know if you've had various math teachers during your school years but - and maybe it's a strange question - do you think that all the math teachers you know would be fit to teach Math A?"

Students: "No."

Teacher: "What kind of extra properties should this teacher have compared with the traditional math teacher?"

Judith: "Well, you have to present it in a kind of fun way. The stories are neat but you also have to make something of it."

Marius: "Math A is more connected to reality, so a teacher has to be somewhat interested in society, I would think. And not just live in the world of math."

Judith: "Yes, it's no longer the case that the students do sums, teacher corrects them in class and explains the following theory. It has more to do with guiding the students, and it's also more individual. The teacher no longer stands alone in front of the class, but walks around the room."

Annemarie: "The teacher also has to think up examples of what certain sums are about, I think that's really important."

Teacher: "Do you think from your experience that some teachers might have trouble doing this?"

Judith: "Maybe, yeah. Especially older teachers, I think, who've always ... I mean teachers who've been there for ages. They sometimes get kind of in a rut and, well, they don't seem to enjoy teaching so much any more."

Teacher: "Do others of you agree?"

Yolanda: "I think you really have to be able to put yourself in the students' place. Not so straightlined. Students have to be able to put themselves in the teacher's place, but the other way around too."

Wim: "I think that the way we work in the junior years is pretty comparable to the senior ones. For the rest, it has to do with the teacher's mentality. I've seen indications, based on which I can say that the teacher who used to do it well, still does. And the ones who had trouble, maybe they now have even more trouble. I mean, in a class where there's an authority in front and the students in their places, well, I think that even in more traditional education that's kind of rough nowadays. So that teacher will have it more difficult, but that's all."

12. Students

Teacher: "Are there students who couldn't pass math in the tenth grade who then chose Math A? And how did that work out? How is it compared to the earlier situation: do you discourage students from taking Math A who couldn't handle Math I, or do you say, go ahead and try it. Is anything known yet about this?"

Kees: "There are definitely students in the Math A group who never would have chosen math before."

Teacher: "Maybe they could also say something, not about teachers but about students. Because many of you have admitted to having difficulty with math. What kind of students would you advise *not* to chose Math A?"

Judith: "I think that students who are only in school to get a diploma but otherwise aren't interested, well, I think that'd be rough. I mean, a whole lot of problems are touched on, and you have to show some interest. You really have to exert yourself. Sure, that's true for every subject, but it's not like you can cram at the last moment or crib, that kind of thing, you won't make it that way. You have to be motivated."

Yolanda: "Yeah, but that's true of every subject, that you have to be motivated."

Teacher: "I meant math in particular. There are students who say, I find math tough. To which students should you say, "you can handle Math A." Everybody?"

Marius: "I don't think you can say that. You also can't say "don't do French" if you've never had French. Because Math A is a whole different subject."

Judith: "It's really nothing at all like Math I. For me, it was like I was taking an entirely new subject, especially at the beginning. And later, when we came to logarithms, sine and cosine, then it became math again. But in the beginning there was all this stuff about islands and frequency matrices that we'd never had before. It was just a whole new subject."

Teacher: "Maybe the eleventh graders could say some more about it, because

I assume that they had already worked with those islands in the tenth grade."

Jan: "Yes, I have to admit that the change is therefore no longer so great, you've already become somewhat used to it. But I must say that a disadvantage of already working in the tenth grade with this sort of method is that in that year you still have Math I and if you start weaving stories around that, then it all becomes even more cluttered and complex. If you decide to take Math A the following year, then you'll have a distinct advantage if you've already had this method in tenth grade. But for those who go on with Math I I think that this method in tenth grade is disadvantageous."

Wim: "Could you explain a little more clearly, based on your experience, which students you would advise against choosing Math A?"

Servaas: "I can't really say exactly, but to my mind, if you compare Math A to Math I, for Math I you really have to be able to think logically, at any rate abstractly, and that's much less true for Math A. So there is a much wider range of people who are able to do it. I, myself, can hardly think abstractly at all, that went really badly last year. I got bad grades for anything abstract. If you ask me, most people should be able to do Math A, unless they're completely non-math oriented. But I don't think there are a lot of people who absolutely couldn't do it."

Wim: "In the present group there are about eight people who really have trouble. Last year we said, everyone who wants to do math can choose it. The result was that 92% chose math. The change in character is very important in this context in the sense that Math I is cumulative - you have to know this in order to be able to do the following - while Math A consists of a number of separate things."

Servaas: "Yes, and because it's so complementary I think that after your Math A exam you have a kind of basic knowledge across a wide area. With which you can go on to do a whole lot of things."

It is tempting to compare this panel discussion with the reaction of thirteen students from one of the forty schools - the only reaction of individual students available from this group. We will do this later on in this chapter. First we will discuss the experiences of the ten (twelve) schools.

1.3 The ten schools

At the two schools observations were done at micro-level: almost all the lessons in Mathematics A were observed by members of the Hewet team. The distance between the team and the actual school-practice was the smallest possible. Right from the beginning it was clear that in the next stage this distance was due to be much larger.
The teachers from the ten schools followed a teacher training course given by the members of the Hewet team. This was a facility which had not been available to the teachers from the two schools. During this course the teachers were probed in order to learn about experiences in the pre-experimental tenth grade. Probes were taken in December 1982, March 1983 and May 1983. It was on this last occasion that the discussion with students from the two schools took place. During the following years the contact between the Hewet team and the ten schools consisted of a number of meetings to discuss the experiences. Internal written reports exist of all meetings [3] from which we have extracted a general picture of the trend of experiences at these ten (twelve) schools. These meetings took place in October 1983, January 1984, May 1984, November 1984 and May 1985.
The meeting of November 1984 did not lend itself to having experiences *in general* extracted because of its specific nature: it was solely devoted to the problems of testing and examinations and contributed to this study by signalling the problems.
During the other meetings we had the opportunity to classify the experiences.
The first question at all meetings was the same:
"What are your experiences and your general impression?"
We have divided the answers given into three classes:
a. Teachers' difficulties, troubles etc. were classified as '–'.
b. Teachers' mixed feelings, or neutral feelings, were classified as '0'.
c. In cases where teachers were positive or very positive we classified the experiences with '+'.

In this way we get the following table, resulting in the graph below it (fig. III.8):

Reactions 12 schools (from Dec. 82 till May 85)

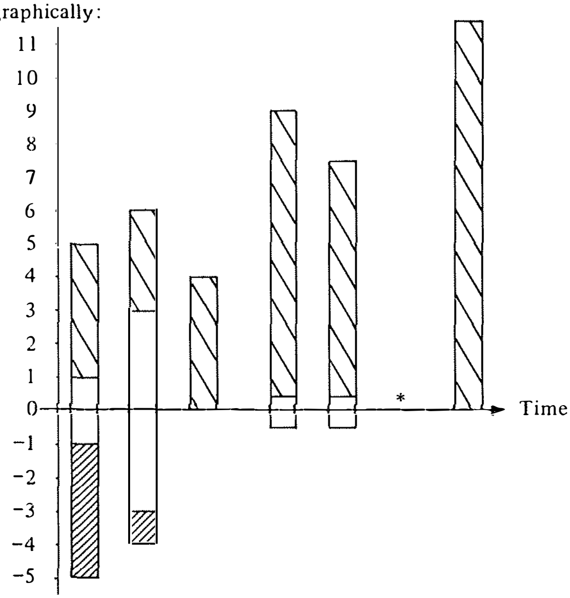
Reaction # / School #	1	2	3	4	5	6	7
1	+	+		+	+		+
2		–		+	+	o	+
3	+	+		+			+
4	–	o			+		+
5	–	o	+				+
6	–	o	+	o	o		+
7	–	o	+	+	+		+
8	+	+		+	+		+
9	o	o		+			+
10	o	o	+	+	+		+
11	+			+	+		+
12				+		+	+
	Dec 82	March 83	May 83	Oct 83	Jan 84	May 84	May 85
	10th Grade			11th Grade			Exam

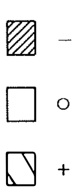
–: Difficult, problems

o: Mixed feelings

+: Positive, very positive

Or graphically:





* Not sufficient data

fig. III.8

We have to be very careful when interpreting this data because of the arbitrariness of the classification. One conclusion, however, is clear: after an introductory period where some doubts were cast on the new program, there is a steady increase in the rating of the experiences. This positive trend becomes even stronger if we look at the ten schools (nr. 1-10) because of the fact the two schools performed the new program for the second or third time.

Some remark should be made about the '0' at (2,6): After two '+' the third mark was a '0'. This was due to the fact that the teacher encountered increasing problems with mixed A/B-stream groups. It was the first time we encountered this problem; in a later stage it became more prominent.
We conclude our report on the experiences of the twelve schools with a quote from a teacher's article about the changes that took place in his classroom practice [4]:

> "The introduction of the Hewet material compells the teacher to compare his own vision on mathematics education with the Hewet view.
> Students, however, have their own view of mathematics. This can lead to confrontations, as we experienced. Learning by 'explanation' and 'reproduction' is quite different from mathematization as a group activity."
>
> "What did we learn from teaching Math A? Quite a bit. Teaching is a dynamic process with unexpected and unforeseen results.
> Mathematics education is less self-evident than one assumes. Teaching 'at-always-the-same-rate' is a fable.
> Obviously, it is possible to lead students to perform actions which make them experience mathematics as an activity."

1.4 The forty schools

At the forty-schools stage the distance between the schools and the Hewet team was considerable. The teacher training course was not given by the members of the Hewet team but by professional teacher trainers. On several occasions, however, the Team was present at these courses, as were the teachers from the ten and two schools to share classroom experiences. This proved to be very useful. At two opportunities, in February and October 1985, the team met with the majority of the teachers from the schools during two twenty-four-hour conferences in order to exchange information and experiences. We received reactions from forty-five schools, out of a maximum of fifty. One of the fifty-two schools that had volunteered for the

experiment withdrew before the experiment actually started[12]. Another school developed their own material and therefore did not use the experimental Hewet student-material which we are discussing.
In a questionaire we asked the teachers the following question:

"What have been your experiences with Mathematics A so far?"

We did not use any 5- or 7-point scale, nor did we point out special points of attention, in order not to influence the teachers.
The answers - of course - were very different in quality and quantity. Long winded ones were compressed, without loss of essentials. We give all the forty-five reactions in order to give a complete picture.

1. Good experiences; positive reports from students continuing at universities.
2. Connection with the real world is very attractive .
   Difficult to judge mathematical contents.
3. Positive experiences; usefulness is stressed.
   Model building is difficult.
   'Weaker' students lack basic skills.
4. Our first impression was negative; at present it is less negative though we are not enthusiastic.
   - Problem: wide spread in levels: easy vs. difficult.
   - Problem: A- and B-students together in one class.
5. Difficult; for A-stream students not always clear what should be learned; continual need for improvisation and imagination.
   Simple for B-stream students.
6. We subscribe to the goals of Mathematics A. Many students have trouble catching the essentials in the often long texts.
   Difficult to summarize the contents of a chapter. Better in the new version of the experimental material.
7. Although time consuming, the students work with pleasure.
8. Even for lower-ability students no problems if they are willing to invest time.
9. Lack of basic skills (from 12-15 year olds) is a problem; some more training necessary.
10. The goals remain vague, mathematizing is difficult; basic skills are lacking.
11. Experiences are positive.
12. Experiences are positive. Beautiful didactical approach. Achievement testing is difficult.
13. Problems with mixed groups of B-stream students and A-stream students. The latter consider tempo too high (this school was the fastest school of

all).
14. Not enough structure, which makes it time consuming.
15. Students have a positive attitude towards Math A.
16. We managed to carry on the whole program without losing one student.
Not only we - the teachers - are enthusiastic about the experimental materials, but the students as well, which may be illustrated by a comment of one of the students:
"The way we were working with Math A the past two years fulfilled me with real interest for mathematics."
17. In certain cases discussions take too much time, which means that the teacher must explain the whole thing. It is an advantage that the students are no longer pledged to x as variable, and clearly understand the usefulness of mathematics.
Students are not as enthusiastic as they seem to be at some other schools.
18. My experiences are positive. Mathematics A gives a new dimension to school mathematics. What struck me is that some students were quite inventive and creative when interpreting results. Both the students and myself are increasingly enthusiastic and we work with pleasure.
19. Our experiences are positive; we hope they will improve even more after choosing another textbook series for the 12-15 years old.
20. The students experience the mathematical methods of solving real world problems as useful.
Sometimes there are great differences in tempi. Not all students cooperate equally well.
Lack of basic skills.
21. The experiences are very positive, also according to the students.
22. The materials are attractive and challenging.
Students work with pleasure.
23. Our experiences are moderately positive.
24. Students are attracted to Math A; it is more relevant than the old program.
Great differences in degrees of difficulty.
25. Pleasant way of working with students. The relation with the real world is a positive aspect.
26. Experiences in general: positive.
27. Our experiences are positive. The initial opposition of the students has disappeared.
28. Positive.
29. We subscribe to the goals of Mathematics A.
Mathematizing is difficult; more theory would be desirable.
30. The program is not too time consuming.

31. Time pressure made it necessary to switch from groupwork to class work. This has improved the results considerably.
32. As a teacher, I have seen some very nice examples of practical applications of mathematics, but I doubt whether the students shared my experience.
33. One group is positive, the other group has motivation problems.
34. Our general impression is very positive; no more students' questions about the usefulness of mathematics.
35. Too many students underestimate the difficulty of Mathematics A; thinking and reflecting are hard things. Math A is less structured than Mathematics I. For some students this is a disadvantage.
    Total impression: neutral.
36. General impression: very good. More interaction between students, more retention because of their own (mental) production. Some problems with mixed A/B-stream groups.
37. We are quite satisfied with the change in the Mathematics Curriculum. We are anxious to get good exercises and tests. There are problems with mixed A/B-stream groups.
38. Our experiences are positive; but the program is overloaded.
    Problem: to produce appropriate tests.
39. Less formal, more practical mathematics: much more attractive for students; we need more good exercises.
40. Students are enthusiastic; some basic skills are lacking.
41. Students appreciate the program; sometimes too difficult for teacher himself; basic skills are lacking; problems with mixed A-B-groups.
42. Mixed feelings; but in general the students appreciate working with real problems.
43. Students are working with much motivation, but it is hard for them to extract the essentials: you need tasks and tests to find out what students know and what they don't know.
44. We are satisfied; but it is a danger that students take Mathematics A too lightly: it is more difficult than it looks.
45. Students appreciate mathematics very much; they enjoy it but find it very difficult. It is hard to produce tests.

Also in this case the reactions to Mathematics A were more positive than half a year before when the first probe was taken. We have tried to tabulate the results of the latest probe available in order to get an overview. For this purpose we initially classified the reactions in *four* classes: *negative, neutral, moderately positive* and *positive.* However, none of the reactions was negative. This leaves us with these classes:

neutral
moderately positive
positive; very positive

At the same time we took into account the most frequent critical remarks. In our analysis we found seven points that were mentioned more than once spontaneously.
As fig. III.9 on the next page shows these points are:

- lack of (basic) skills (6);
- lack of structure (2);
- difficult to extract essentials (3);
- problems with mixed A/B-stream groups (6);
- overloaded (5);
  (or, for one school: ample time) (1);
- difficult (7)/too much variety in difficult and easy parts (3);
- problems how to test (7).

Looking at the totals we notice the following:

Neutral	10	or	22%	
Moder. Positive	13	or	29%	
Positive, Very Pos.	22	or	59 %	based on 90.% of schools

The table leads to the conclusion that experiences are better than moderately positive. At the same time, however, we should bear in mind that all schools volunteered for these experiments[13], and are therefore by no means representative of the total population of 450 schools, which introduced the new program in the summer of 1985. It *only* represents the experiences of the total group of *'experimental'* schools that worked with the *experimental student material* developed by the Hewet team.

This discussion is highlighted by written reactions of the totality of students of one of the forty schools, to wit one whose reaction to the new curriculum is positive. It is interesting to compare these written reactions with the panel discussion with students (and teachers) from the two schools. For completeness' sake, we do not give extracts, but copy the reactions completely.

Reactions of 52 schools

● Neutral ● ● Moderately positive ● ● ● Positive

School #	Impression	Lack of skills	Lack of skills	A/B	Time	Tests	Difficult ●/Easy ○	Difficult to extract essentials
1	● ● ●						●	
2	● ● ●	●					●	
3	● ●						●○	
4	●			●			●○	
5	● ●				●			●
6	● ●							●
7	● ● ●				●			
8	● ● ●							
9	●	●						
10	●	●					●	
11	● ● ●							
12	● ● ●					●		
13	●			●	○			
14	●		●		●			
15	● ● ●				●			
16	● ● ●							
17	●							
18	● ● ●							
19	● ● ●							
20	● ●	●		●				
21	● ● ●							
22	● ● ●							
23	● ●							
24	● ●						●○	
25	● ●							
26	● ● ●							
27	● ● ●							
28	● ● ●							
29	● ●						●	
30	●							
31	● ●							
32	●							
33	● ●							
34	● ● ●							
35	●		●			●	●	
36	● ● ●			●				
37	● ● ●			●		●		
38	● ● ●				●	●		
39	● ●					●		
40	● ● ●	●						
41	● ● ●	●		●				
42	●							
43	● ●					●		●
44	● ●						●	
45	● ● ●					●	●	

fig. III.9

"I find Math A really a pretty difficult subject, although some things are easy.
Compared to the math we used to have I find this more difficult because you have to have more insight for this and that's sometimes difficult for me.
It used to just be explained and then you had to do a lot of problems which was often pretty boring and that's at any rate no longer true so on the one hand I guess I like it better and it's more fun."

"I think Math A is a better system than Math I. In Math A you learn to work from easy to difficult. You have to figure out a lot more for yourself. I don't mean by this that Math A is easier, but that in fact you need to have more insight. Math A is aimed much more towards things that you come across in daily life."

"I like Math A the way it's given now a lot better than the old math; because it's not just a bunch of formulas. You don't need so much foundation for some sections. Matrices, for instance, is a separate section. The new kind of math is a lot more varied."

"I find the new math a lot more pleasant; you see more clearly how mathematic formulas can be used in daily life. In addition I think it's good that you're not just given a formula to use on an x-number of sums but that you have to derive a formula by means of contextual problems. That way you understand a lot better how you've arrived at a formula."

"I think the new system of Math A is pretty good because everything is dealt with step by step and very simply (sometimes too simply). I do think that some things could be summarized more briefly. To do the problems well you really have to spend a lot of time on them."

"Math A is a good kind of math for A-stream students, but for B-stream students who don't take Math B along with it, the absence of integral calculus in Math A can prove to be a problem in Physics. For the rest, I can't find fault with the Math A program; it's very comprehendable."

"I like Math A better than the Math I we had in the junior years. At least I can see what I'm doing now - what I'm calculating and to what purpose. But on the other hand I find it difficult that there are so few formulas. You have to really reason logically, you can't just blindly apply a formula. And I often have trouble with that."

"I like it. The descriptions of problems could sometimes be better (not so many mistakes). For the rest no complaints. Fun to do. Better than Math B. Could have contained more mixed problems, such as matrices mixed with probability. (Only possible, of course if the order of the program is set). I'd already often worked with it before so it was really nothing new."

"Math A shows how you can apply math in real life. That way you understand it much quicker and better. It gives much better insight into math. At any rate it's a lot better than the 'old math' and it also gives A-stream students the chance to take math.
A disadvantage is that you don't get certain subjects which can be useful for Physics and Chemistry (such as integral calculus)."

"What I like about it: it's directed towards real life; when you come across things from daily life you understand it quicker and it's more interesting.
What I don't like about it: the transition from sum to sum is now and then too great, sometimes too much is asked of you if you don't get it immediately. Sometimes I found the phrasing difficult."

"I like the Math A that we have now better than the old method. Now you get a much stronger sense of having figured something out yourself, which is better than being fed 'dry' information that you just have to learn by heart. Also, I have the idea that Math A can be much more useful in daily life. Graphs and that kind of thing now acquire significance. So you become more interested in it - at least I do. And there is a lot more variation in the problems, so that besides the standard sums, you gain more and more insight through your own efforts."

"It often appears complicated, but once you look at the explanation it's not so bad at all. Our old book was more exact, there were more theorems. I think that that would have been easier for me. But I can't say for sure, since I never had matrices, for instance, in that book. I like the work in itself, it's more reality-oriented, you know what you've up to. But sometimes it's also a big mix-up. I think that last year we paid more attention to each sum."

"I like Math A better than what we used to have because you can work more independently. When you work on your own you understand the material quicker and remember it better."

Both the panel students from the 'two' schools and those from one of the 'forty' schools judge the curriculum quite positively. Both groups mention the usefulness of mathematics as a positive feature, and for some students (from both groups) this helps them to understand things better. The process of re-invention is mentioned by both groups, as well as that this improves retention and insight. The need for insight, for thinking, is mentioned rather frequently by the second group; and although students experience this as

difficult, they judge it positively; not just a presentation of a formula and some exercises, but the student has to find the formula by himself or, as one student formulates it: "you are working *towards* the formula."
The didactical method of conceptual mathematization is acknowledged by some students. One of the criticisms brought to the fore by the teachers, the 'lack of structure' is mentioned by one student, who also wants some more formal mathematics in the course. The degree of difficulty of the curriculum gives cause to a seeming paradox: although most students agree that there are some difficult aspects in the course - insight, thinking, long texts - they admit that more (if not all) students can complete this course if compared with the previous Mathematics I course.

Thus far our interpretations of the experiences of the two, the ten, and the forty experimental schools, based on rough empirical material, and collected originally in articles and reports.
We will now discuss the criticisms brought to the fore by the fifty-two schools as displayed in our table on page (147).

1.5 Discussion

1. Lack of basic skills

The skills mentioned by the teachers are quite elementary: fractions, drawing graphs of linear functions, solving quadratic equations.
It is a fact that many teachers (in the Netherlands) complain about the actual level of basic skills supposed to be obtained in the primary grades. In the same way teachers of upper secondary grades complain about the level of skills acquired in the lower grades.

Whatever the cause may be, it is remarkable that this 'lack of basic skills' is mentioned rather frequently by the teachers from the fifty-two schools.
It is our opinion that during the Math A course more attention should be paid to basic skills whenever the occasion arises.

2. Lack of Structure and 3. Difficult to extract essentials

We discuss these two points together because they seem somehow related to each other. As we have pointed out when discussing the methodology of the Mathematics A curriculum, less attention is paid to the structure of mathematics compared with the traditional mathematics curricula. This also

holds for the formal aspects of the curriculum; the Hewet commission explicitly recommended a less formal approach to Mathematics A. [5]
According to some teachers this 'lack' of structure or, properly said, less attention to structure, makes it more difficult for (some) students to find out the essentials. Structure building by the students themselves is considered as a difficult and time consuming task.
Giving definitions, theorems, rules in an earlier stage could help these students.
Another aspect, closely related to this one, is the fact that 'long texts' with much redundant information make it difficult for the students to extract the essentials. Here it is not the 'lack of structure' which is criticized in the first place; rather question-marks are placed behind the process of mathematization or, more precisely, behind the rather prominent place held by mathematization in the Math A curriculum.

It is not completely clear whether or not the teachers mean conceptual mathematization, which is a methodological-didactical choice, or the mathematization in applications, which seems to be a '*conditio sine qua non*' in Mathematics A.

4. Problems with mixed A/B groups

This problem was already mentioned by one of the twelve schools, which experienced increasing problems with groups consisting of students that have chosen Mathematics A ànd B together with students that have chosen only Mathematics A.
The students that have chosen A and B as disciplines have often seven or eight hours of mathematics per week, compared with four hours per week for students with Mathematics A only (and six other subjects).

Not only are the skills of these students better, but some parts of the Mathematics A curriculum are also part of the Math B curriculum. This is especially the case for the calculus part of the A-course.
One can imagine that for most B-stream students, who know all the rules for differentiation, re-invention of these rules is not a very motivating activity. The applications however are also new for Math B-students, as is the process of mathematization.

The teachers try to solve this problem in different ways:
- Avoid mixed groups involving both A- and A/B-students. Stick to 'pure' A-groups and 'pure' A/B groups.

- Treat the differentiation rules as soon as possible during the first weeks of the course, when they are new for the B-students as well.
- Give B-students in mixed A/B-groups extra exercises that stress mathematization and applications; have them make their own productions.

It is too early at this moment to give any conclusions about the success of these approaches.

5. The overloaded program

Five schools find that the program is overloaded. One school, however, claims that there is ample time to complete the program.
The teacher at this school wonders whether Mathematics A is appropriately interpreted. "Maybe we should go deeper", he commented.

The experience of the teacher at the Haarlem school is relevant to this point. When he taught the course for the second time he expected to do it faster. After all, it was the second time and he was familiar with the material and the philosophy. It actually it took him *more* time to complete the course now than in the preceding year. When we discussed 'why' he arrived at the conclusion that the reason was his wish to pay more attention to the goals of Mathematics A. He engaged more intensely in discussion with the students, spent more time on reflection. He considered this to be better teaching, but the exam results were poorer.
The question is, of course, how well the exam operationalizes the goals of Mathematics A.

But let us return to the problem itself.
Certainly the Math A course could be taught faster. But we fear that teachers' feelings about the 'depth' are justified. Students' interaction, student-teacher interaction, discussion, reflection, it all should have its share in classroom practice. Or, even more than this, the students should generate products like essays, articles, booklets etc. As will be discussed later extensively, those activities contributed in a major way to the motivation and understanding of most students and teachers as well.

6. How difficult is the program

Seven schools consider the program to be difficult, especially the process of mathematizing, the breadth of the program and the integration of different subjects. Three schools note (very) easy parts alongside (very) difficult parts. This leads to problems in classroom practice. A probe on the degree of difficulty of the different booklets showed that opinions vary widely, as was the case in the student panel discussion. The only agreement concerned the contents of the booklet 'Matrices', which was experienced as 'rather easy' by most students. But as soon as new exercises which called for mathematization were presented the difficulty increased, even in the case of 'Matrices'.

Remarkably, none of the teachers judged the mathematical contents too easy. Remarkably, considering the fact that some newspapers had characterized Mathematics A as mathematics for girls. The larger participation by girls (see next paragraph) had been explained by its facility. The reactions of the actually participating schools, however, did not point in this direction.

7. Tests

The need for good exercises and tests is mentioned by 7 schools to be a major problem.
We completely agree with the view of these teachers. Taking this into account we devote the remainder of the study mainly to the problem of testing and tasks.

8. General remarks on the role of the teacher

The contradiction between the experiences causes us to guess that the teachers' influence may be significant. As the students in the panel of the two schools already pointed out, high standards are expected from the teacher: he/she should be broadly interested, provoke discussion, reward initiative and creativity, and co-operate with the students.

Most teachers, however, are trained in the traditional way: mathematics is truth, certainty, structure. This is not so in Mathematics A, however.
This mathematics is a dialogue between people tackling mathematical problems. In this view - sometimes named falliblism [6] - mathematics is what mathematicians do and have done, with all the imperfections inherent in any human activity or creation. Mathematicians are fallible and their products, including proofs, can never be considered as final or perfect, but may require re-negotiation as standards of rigour change. As a human activity, mathematics cannot be viewed in isolation from its history, its sociology and its applications in the sciences and elsewhere.

No doubt, teaching Math A is no sinecure and for a number of traditional teachers the teaching is no pleasure. One single in-service teacher training

course will not be of great help to them. Some of them, however, can be convinced by conversations with students from other schools or, even better, with colleagues. Recently, remarkable signals revealed that quite a number of teachers learn to teach Mathematics A *through* the experiences with the students.
The shift that we noticed in the attitude of the teachers from the ten schools seems also to be present in some cases of individual teachers.
One older teacher stated that teaching Mathematics A had revitalized him completely, and he felt gratitude for the fact that he had taken part in the concretezation of such a curriculum.
Another younger colleague felt that by teaching Mathematics A he could easily reach the age of 100.
But let us return to the 52 schools conference.

At the end of the conference of 52 schools many teachers were in higher spirits than when they arrived. The discussion between the various teachers seemed to have been fruitful. The same happened when teachers were invited to speak at teacher training courses. One hour of discussion with an experienced teacher sometimes influences a teacher more than a whole teacher trainer course.
Some future Math A teachers contacted experimental schools to find out how things were going there. This again had positive effects. But the teacher aspect of the project is not restricted to these features. As in most projects, the teachers possess the key to success. Materials are important but teachers even more so. Without their positive attitude one runs the risk of running into trouble.

This is not of course a typically Dutch problem. Griffiths and Howson (1974) already remark a shortage of good teachers for math with applications. They also note that teachers, competent in pure mathematics, are more willing to teach statistics than mechanics, because applications in statistics are more 'real' than in mathematics [7]. Maybe the factor that Math A offers some 'real' applications may have contributed to the moderately positive reception.
M. Bell also notes the world-wide reluctance from the side of teachers to exploit applications-oriented materials, even with examinations which are application-oriented. [8]
They fear, according to Bell, that it is difficult. Moreover, reform in education is never a thing to inspire optimism.
Surprisingly, this general pessimism is given the lie by the general picture in the 52 schools phase. What is the reason for this phenomenon?

There are some indications to answer the question. Firstly, the fact that the teacher training course precedes actual teaching. In general, during the first weeks the teachers, though volunteers, had serious doubts about the materials. But as time went on, and teachers discussed the materials, worked with them, and became accustomed to the background philosophy, the participants became more and more enthusiastic. Reluctance subsided, albeit to various degrees. This was particularly strong in the ten schools training course.
The teachers were given ample opportunity to get used to the material; they may have been influenced to an even higher degree, however, by the presence of teachers that were already using the material. The panel which the students and teachers from the two schools held for the group of teachers from the ten schools certainly motivated the new teachers, as did earlier confrontations with the two teachers.
Discussion between colleagues proved more convincing than reports from the members of the Hewet team.

In later courses both teachers and students, and incidentally also Hewet team members, were invited to take part in discussions at teacher training courses. The strategy of having teachers participate in teacher training must have been of paramount responsibility for the positive change of motivation. In the future, teachers and teachers' experiences should be more structurally integrated into teacher training and innovation.
The same positive shift as caused by the teacher trainers course was clearly visible at the conferences of the ten and the forty schools, for a large part because of teacher interaction. The same development holds for most school experiences: we already mentioned the shift for the forty schools between the two conferences.
An even clearer picture arises when looking at the teachers' reaction at the ten schools, as we have done before.

Teacher training, including presentations by experimenting teachers, thus proved to be a key factor in achieving a moderately positive total picture as a result of the experiments with the new Mathematics A curriculum.

2 STUDENTS' PARTICIPATION IN MATH A

In the second part of this chapter the focus is on students' participation in the new program as compared with the old one. We restrict ourselves again to the experimental 'fifty-two' schools. Data were collected by Verhage, a member of the Hewet team [9]; future data, regarding the nationwide

situation, are to be expected from research carried out by R.I.O.N.[14]
We start by comparing the national situation in 1981-1982 with the situation at the first round for the twelve schools, which is not significantly different from that of the fifty-two schools, one year later.
The national situation is given in percentages, the twelve schools situation in absolute numbers and in percentages (fig. III.10a,b).

MATHEMATICS CHOICE IN 1981/1982

	Math I	Math I & II	No Math	Total in %	Total abs
boys %	84	28	16	100	21900
girls %	63	5	37	100	19281
total %	74	17	26	100	41181

National situation in student participation in math I and math II in 1981/'82 in %.

fig. III.10a

MATHEMATICS CHOICE 12 SCHOOLS

	Math A	Math B	Math A & B	No Math	Total
boys	155	209	226	51	641
in %	24	33	35	8	100
girls	280	75	102	121	578
in %	48	13	18	21	100
Total	435	284	328	172	1219
in %	36	23	27	14	100

Situation at twelve experimental schools in math A and math B in 1983/'84 in absolute numbers and %.

fig. III.10b

In schemas (fig. III.11a,b,c):

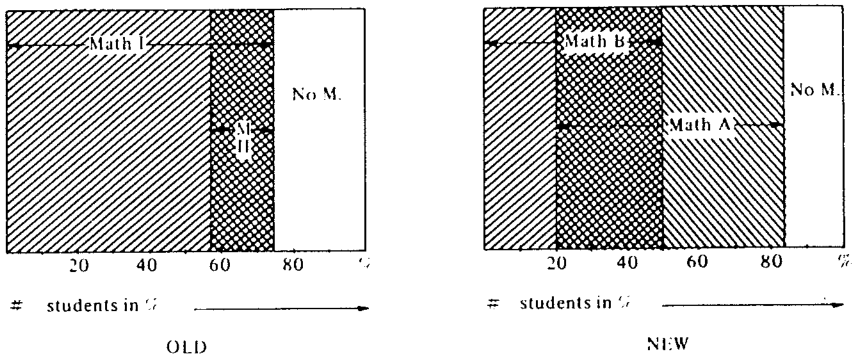


fig. III.11a

It is interesting to look at similar schemas for girls and boys separately; especially because mathematics is traditionally a 'male discipline' in the Netherlands.

For boys:

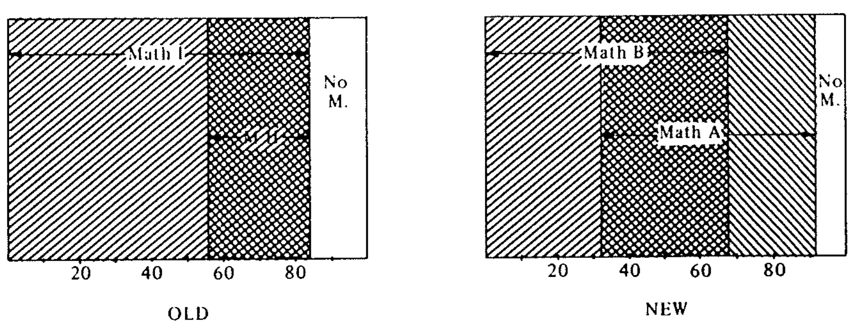


fig. III.11b

For girls:

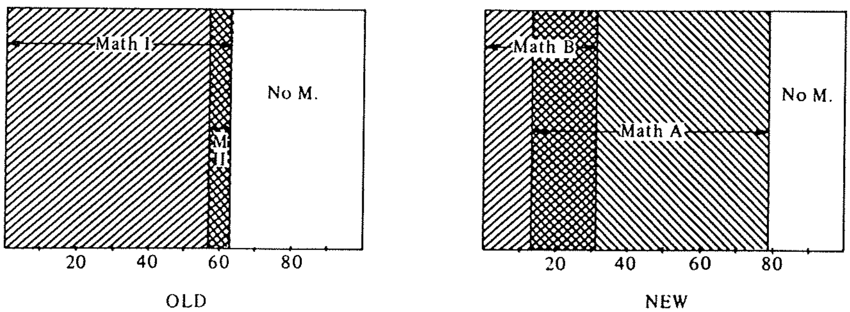


fig. III.11c

A number of conclusions may be drawn:

- Total student participation increased from 74% to 86%.
- Math A and Math B are more or less equally chosen by the students, compared with the uneven distribution of choices in the old situation.
- The distribution of sexes is more or less equal for Math A; Math B is chosen by boys twice as often as by girls.
- In the new situation fewer students chose the kind of mathematics suited for exact sciences than did in the old one:
  - for boys from 84% to 68%;
  - for girls from 63% to 31%.

The last statement confirms the arguments of the Hewet Commission: too many students who needed mathematics only as a tool were forced to study Math I, although this was too formal and abstract and consequently proved problematic in the classroom situation.
This confirmation becomes even stronger if we look at the other choices of subjects made by students. Students, heading for natural sciences, often include physics in their choice. Among the boys 66% chose physics, compared with 68% Math B and from the girls these percentages are resp. 30% and 31%.

From the following tables it becomes clear that the choice of Math B was highly correlated with that of physics; for Math A and physics there is no such correlation (fig. III.12):

		Math A			Math B		
		yes	no		yes	no	
Physics	yes	27	22	49	45	04	49
	no	36	15	51	05	46	51
Columntotal		63	37	100	49	51	100

fig. III.12

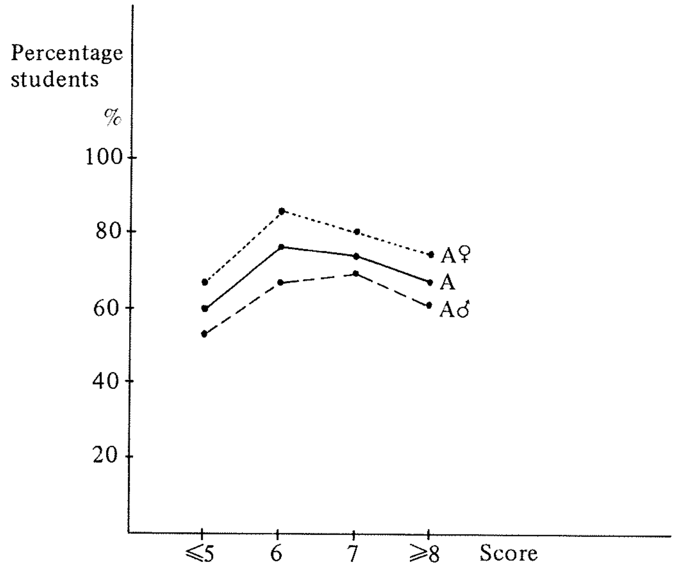
The conclusion seems to be that the fact that girls choose Math B only half as often as do boys is due to the fact that they have a lower interest in natural sciences in general, rather than because of differences between the two mathematics curricula.

Another conclusion to be drawn is that in the past relatively more girls than boys were compelled to choose math I, which was not suited to them.
If any recommendation should be made it is that girls, being under-represented in natural sciences, should be motivated to choose natural sciences, which automatically involves higher participation in Math B.

In the light of these remarks it is interesting to note that even girls who are doing fine in mathematics in the 10^{th} grade are still reluctant to choose Math B. Of the boys with a mark 7 (out of 10) some 80% chose Math B, whereas only 50% of the girls with the same mark did so. Therefore, according to the figures, many more girls should be able to follow Math B successfully.

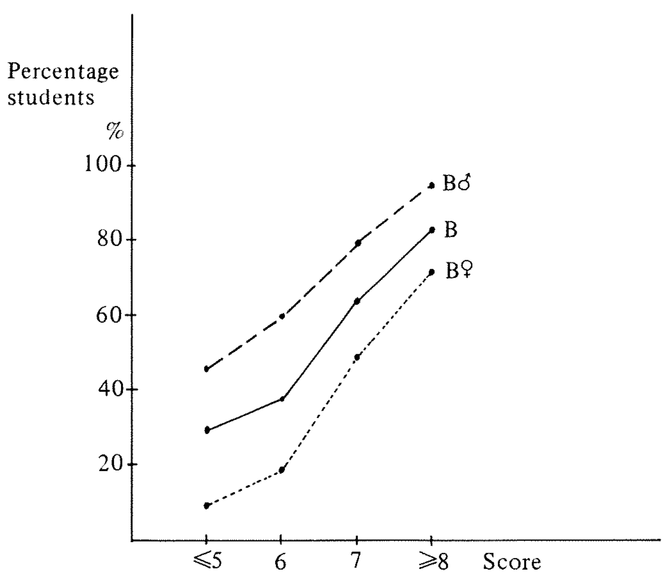
In general, if we look at the relation between the results in the 10^{th} grade and the choice for Mathematics A and B we get the following graphs (fig. III.13a,b).

- The choice for Math A hardly depends on the mark for mathematics in 10^{th} grade. (Between 60% and 75%.)
  The choice of girls is some 5% larger than for the whole group; that of the boys some 5% lower.
- The choice for Math B is strongly related to the mark for mathematics in 10^{th} grade: the lower the mark, the rarer the choice for Math B.
  Among the students with mark 5 or lower, more than 25% still chose Math B, as opposed to more than 80% of those attaining marks of 8 and higher.



Graph, showing the choice for math A (vertical in %) in relation to the mark in 10th grade. ♀ *: girls* ♂ *: boys.*

fig. III.13a



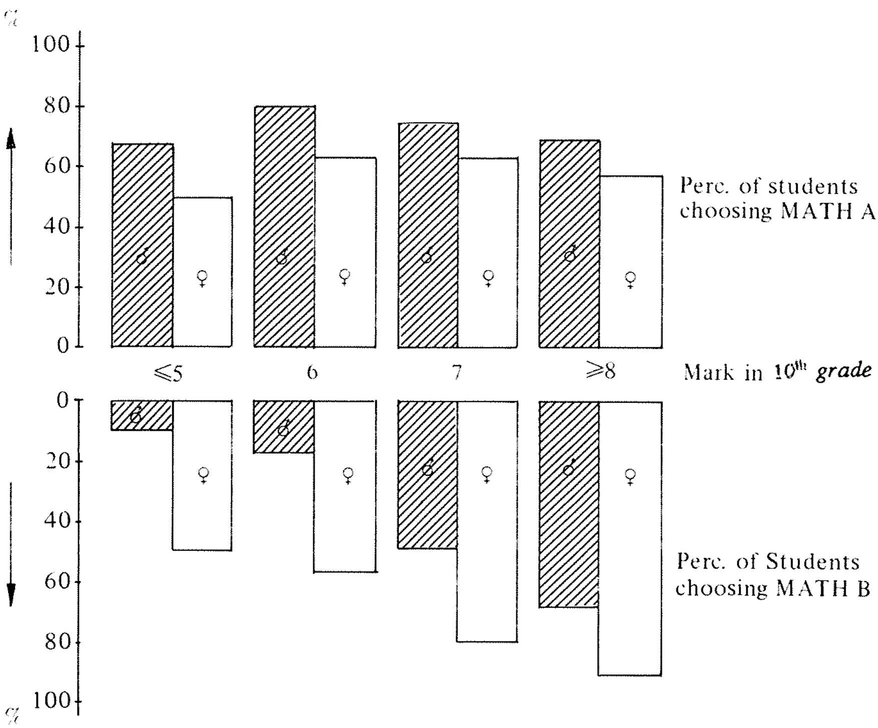
Graph, showing the choice for Math B (vertical in %) in relation to the mark in 10th grade. ♀ *: girls* ♂ *: boys.* fig. III.13b

- As noted before, the differences between boys and girls are highly significant.

  This phenomenon is most striking if students with poor results are considered:

  Among the boys with marks of 5 and lower in 10th grade some 45% still chooses Math B, roughly as many as Math A.

  Among the girls with marks of 5 and lower in 10th grade no more than 10% chose Math B, compared with the more than 60% choosing Math A (fig. III.14).



Graph, showing the relation between the mark for mathematics in 10th grade and the choice for Math A and Math B; boys and girls separated.

fig. III.14

The future development is very uncertain. As mentioned before, the totals for the 52 schools came very close to the figures for the twelve schools, showing that even fewer than 14% chose no mathematics at all. Future research will have to show how representative the experimental schools have been. But there seems little doubt that the new Math A and Math B curricula will have long-lasting effects on student participation.

IV RESTRICTED TIME WRITTEN TESTS

1 INTRODUCTION

In the Netherlands much emphasis is traditionally placed on restricted-time written tests in mathematics; in general these last 50 minutes, but sometimes shorter or longer. In fact, among the fifty-two schools, hardly any other means of testing was used, apart from informal oral testing.
Formative and summative testing is strongly intertwined, and tests are often used both for diagnostic and selective reasons. It becomes even more complex as the teacher prepares the student simultaneously for the nationwide exam.
In fact, the teacher is often mainly concerned with the question:
"How to prepare my students for the exam?"
Our experimental teachers, however, were as much concerned with other aspects of testing Math A.
One question asked:
"How do we operationalize the goals of Math A?"
It was hard to answer this question. Indeed, the Hewet report lacked details, and not until 1985 were the goals as operationalized in the experimental materials made explicit. [1]
Actually, this study contains for the first time a detailed description of the goals and methodology of Math A, as concretized in the experimental material. Without this analysis it is almost impossible to develop appropriate tasks and tests.
But even in the early stage most teachers felt that mathematization was an activity that should be tested and therefore by testing *processes* rather than *products*, at least in certain cases. And how to give students the possibility te reflect, to show their creativity?
In this connection we recall the detailed description of the 'Rat problem'.

Questions like the above ones arose, but the answers were not simple. Modest attention, however, was paid to them at the two schools.
At the Zevenaar school the students were given the task to solve the 'Hooded Seal' problem at home, and to deliver the complete work some weeks later. Excerpts from this can be found in *Hewet & Toets* [2].
At the Haarlem school the students were asked to rewrite an article on migration problems in Indonesia.
We will come back on this later on. (Chapter V,5).

During the teacher training course for the ten schools it became clear that testing Math A involved many problems.
Typical questions from teachers:

- How to prepare students properly for the exam?
- How to know what to test?
  (What goals should I try to operationalize?)
- Should the role of the test differ from the traditional one?

This last question arose when teachers found out that they needed the test as an integral part of the learning process. Rather than 'sampling' marks to give a judgement at year's end, the teachers should see to it that students learned through the tests as did the teacher. Achievement testing should be a learning aid, as formulated by Gronlund. [3]
On the other hand, the Hewet team posed themself the question:

- Are our goals for Math A manifest in the experimental textbooks?
- Knowing the limitations of restricted-time written tests such as the exam, how do we prevent this exam from dictating the program (and the intermediate tests)?

A good answer to this last question is essential for the survival of Math A if the original intentions of the Hewet Commission are to be realized. Surprisingly, the Commission had not mentioned the problems of achievement testing in their report.
The problem, however, is not new at all.
Van Hiele stated in 1957:

> "But soon, much too soon, the influence of the exam becomes noticeable, with the result that the exam makes it necessary to anticipate a long time in advance. If at all, these preparations contribute to concept forming only in a negative sense." [4]

Although the situation was different, the Hewet team had to take into account the influence of the exam on the Math A curriculum in a similar way.
More recently, the National Council of Teachers of Mathematics expressed an opinion similar to Van Hiele's in *'An Agenda for Action'*:

> "It is imperative that the goals of the mathematics program dictate the nature of evaluations needed to assess program effectiveness, student learning, teacher performance, or the equality of materials.
> Too often the reverse is true: the tests dictate the programs or assumptions of the evaluation plan are inconsistent with the programs goals." [5]

The Cockcroft Report *'Mathematics Counts'* put it even more sharply:

> "It therefore seems clear that, if assessment at 16^+ is to reflect as many aspects of mathematical attainment as possible, it needs to take account

not only of those aspects which it is possible
written papers, but also of these aspects whic
some other way." [6]

This "some other way" will be the main point
which will be described in the next chapter. In th
focus on the traditional restricted-time written tests
the teachers from the twelve schools.

2 RESTRICTED-TIME WRITTEN TESTS AT THE

2.1 Two questions

During the first two years of the experiment for
than one hundred tests were collected, including r
remarks. Two questions, related to each other, we
investigation:

- How well are the goals of Math A operationali
  ten tests?
  And, if insufficiently, what is the reason:
  - Were the goals of Math A not obvious to the
  - Are the goals not properly operationaliza
    timed-tests?
- Do the exercises resemble the exercises in these
  texts and new problems offered to find out how
  problems has been mastered?

The last question can be answered more easily t
ourselves to the collected tests. These tests were
from the Hewet team or from the teachers of the
teachers had seen the tests from the two schools
further co-operation should take place in order to
how the different schools constructed their own re

Of the more than one hundred tests that were co
some detail 78 problems from 23 tests on the
present an overall picture of the collected tests.

2.2 Classification of written timed tests

In order to answer our second question we have t

that takes into account the 'distance' of the test exercises from those of the students' material.
A classification as used by Treffers [7] is:
- Identical situation: same context, same contents.
- Related situation: same context, different contents.
- New situation: different context, different or the same content.

This classification does not, however, suit our purpose: in the first place, we need to take into account exercises with no context at all, or hardly any context. It should, however, be acknowledged that a 'related situation' with the same context may be quite different from those in the booklets and thus have a 'large distance', and that a 'new situation' with a different context but with the same contents can show a high degree of similarity and thus a 'small distance'.

We have chosen for the following classification:

Class 1:
Exercises without context; or with hardly any context.

Class 2:
Exercises with substantial use of context:
a. Exercises strongly resembling the exercises from the booklet.
b. Exercises resembling those of the booklet somewhat, but not *strikingly.*
c. Exercises not resembling those of the booklet.
To elucidate our classification let us give examples of every class.

Class 1:

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 4 & 0 \\ 0 & 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

a. Calculate $D = A\text{-}B$
b. Calculate $E = 2B+A$
c. Calculate $F = B{\bullet}A$
d. Calculate if possible a product G of B and C.
e. Determine matrix X such that:

$$X \bullet A = \begin{pmatrix} 0 & 3 & 4 \\ 3 & 1 & 2 \\ 2 & 7 & 4 \end{pmatrix} \begin{array}{l} \leftarrow 3^{rd} \text{ row of A} \\ \leftarrow 1^{st} \text{ row of A} \\ \leftarrow 2^{nd} \text{ and } 3^{rd} \text{ row of A} \end{array}$$

Class 2a

Test exercise:

During the seventies research was carried out on the population development of the animal 'Canis familiaris amathematicus'. Only the females were studied, because they reproduce, and are therefore suitable for the use of Leslie matrices.

Some data:

Age	# In 1970	# Offspring 70-80	# 1980
0-9	3234	643	4758
10-19	6021	4009	2832
20-29	4178	250	4738

a. *Explain why in general there is an entry 0 at the lower right corner of a Leslie-matrix.*

b. *Determine the Leslie-matrix based on the above data.*

c. *Draw the population pyramid of the female population in 1990.*

Booklet exercises:

1. *Explain why in general there is an entry 0 at the lower right corner of a Leslie-matrix.*
2. *Some research results from the female population of the U.S.A. are given. Only females are considered because of the fact that they are responsible for reproduction and therefore suitable for the use of a Leslie-matrix.*

Some data:

Age	# In 1940	# Offspring 40-55	# 1955
0-14	14459	4651	16428
15-29	15264	10403	14258
30-44	11346	1374	14837

a. *Determine the* v_i *of the Leslie-matrix L.*

b. *Determine the* p_i *of the Leslie-matrix L.*

c. *Calculate the population of 1970 and 1985.*

d. *Draw population pyramid of 1970.*

Class 2b

Because of water pollution a certain fish is threatened with extinction. From research it is known that 70% of the newborn fish reached their first birthday, and only 30% of the one-year-olds reached their second birthday. There are no fish older than two years and only this class reproduces; it does so on an average of four newborns per fish.

a. Find out, using a Leslie-matrix, if there is any reason to worry about extinction. Start with 1000 fish in each year group.
b. Determine the Leslie-matrix for a three-year period.
c. Will the starting population be cut in half after nine years?

In an effort to save the fish a big group is brought to a non-polluted lake. Due to this fact the one year olds also reproduce: on the average one young fish.
The survival probabilities, however, do not change. One divides to put 1000 newborn fish in a big fresh water basin, and to count the fish every year; this results in:

Age	Start	5 years	10 years*
0	1000	590	525
1	0	86	180
2	0	176	159

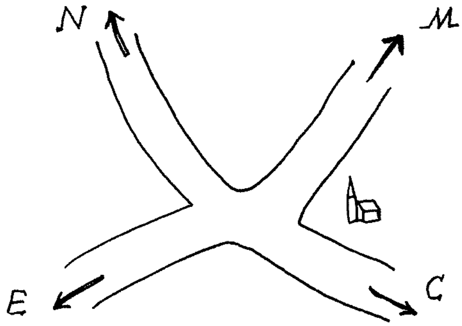
**These numbers have been influenced by a disease of the two-year olds in the 8th year, killing 120 fish.*

d. One wants to conclude whether the experiment is a success or not; in other words: will the fish survive under the new circumstances?
e. Motivate your answer to d. by a calculation for another five years; assume that there will be no new diseases.

Although most of the questions are familiar to the students, the combination of questions is new; as is the context, albeit only slightly.

Finally, we will give an example of a *class 2c* exercise, which is new to the students; there seems to be no resemblance to any of the exercises in the book.

Here you see a crossroads in Geldrop, The Netherlands, nearby the Great Church.



In order to let the traffic flow as smoothly as possible, the traffic lights have been regulated so as to avoid rush hour traffic jams.
A count showed that the following number of vehicles had to pass the crossroads during rush hour (per hour):

$$A: \quad \text{from} \quad \begin{array}{c} M \\ N \\ E \\ C \\ \\ \end{array} \begin{array}{c} \begin{pmatrix} 0 & 40 & 200 & 30 \\ 30 & 0 & 80 & 50 \\ 210 & 60 & 0 & 60 \\ 30 & 40 & 80 & 0 \end{pmatrix} \\ \begin{array}{cccc} M & N & E & C \end{array} \end{array}$$

The matrices G_1, G_2, G_3 *and* G_4 *show which directions have a green light and for how long. In* G_1 *for instance, you can see that the directions* $M \to E$, $E \to M$, $M \to N$ *and* $E \to C$ *can ride simultaneously during a period of* $\frac{2}{3}$ *minute. The other directions are then blocked by a red light.*

$$G_1: \begin{array}{c|c} & \begin{array}{cccc} M & N & E & C \end{array} \\ \begin{array}{c} M \\ N \\ E \\ C \end{array} & \begin{pmatrix} 0 & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad G_3: \begin{array}{c|c} & \begin{array}{cccc} M & N & E & C \end{array} \\ \begin{array}{c} M \\ N \\ E \\ C \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix} \end{array}$$

$$G_2: \begin{array}{c|c} & \begin{array}{cccc} M & N & E & C \end{array} \\ \begin{array}{c} M \\ N \\ E \\ C \end{array} & \begin{pmatrix} 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad G_4: \begin{array}{c|c} & \begin{array}{cccc} M & N & E & C \end{array} \\ \begin{array}{c} M \\ N \\ E \\ C \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

a. How many cars come from the direction of Eindhoven during that one hour? And how many travel towards the City Center?
b. How much time is needed to have all lights turn green exactly once?
c. Determine $G = G_1 + G_2 + G_3 + G_4$ and thereafter $T = 30 \bullet G$.
What do the elements of T signify?
d. Ten cars per minute can pass through the green light.
Show in a matrix the maximum number of cars that can pass in each direction in one hour.
e. Compare this matrix to matrix A (at the top of the page).
Are the traffic lights regulated accurately? If not, can you make another matrix G in which traffic can pass more smoothly?

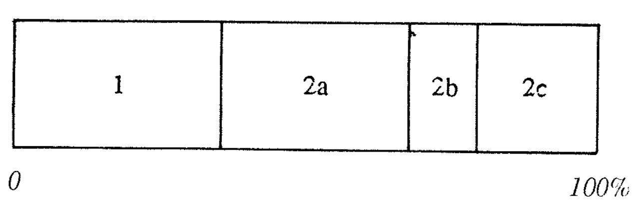
When discussing this exercise in a teacher training course, it was generally agreed that this was a good example of a Math A exercise, if combined, of course, with other exercises more oriented toward basic calculations.
Much emphasis is put on interpreting the results - an activity which many teachers find difficult to test in restricted-time written tests. One also felt that this exercise should only be the starting point of an activity in project form. *The restrictions of this kind of testing were felt, which led to the conclusion that other kinds of tasks should be part of the A curriculum.*

2.3 Results

The distribution of the exercises over the different classes is as follows:

class #	# exercises
1	28
2a	25
2b	9
2c	16

or, graphically:



We notice that almost 80% of the exercises has some or strong resemblance to the exercises in the students' texts, and about 35% of the problems are context-free.

What can we conclude from these figures with respect to our two questions? We will discuss this in our next paragraph.

2.4 Discussion of the results

The *first* group of exercises is the group without any context or with very little context. Those exercises are variations on exercises from the booklet. They serve to test the basic mathematical skills like multiplication of matrices, the rules of differentiation, manipulation with logarithms. The exercises are well suited to their goals; roughly one-third of the exercises belong to this group. With regard to our questions, this group is not the most interesting: skills shall certainly be tested, but this can be done in the traditional way.

The next group: 'Class 2a', consists of exercises closely resembling textbook exercises. Again, this group is fairly large in size: roughly one-third of all exercises fits this pattern. What is the value of these tests? Let us take as an example the above representative of *class 2a.*
The exercise in the *book* is unique in the sense that there is no exercise that resembles it. The main goal of this exercise is the mathematization of the data into a Leslie-matrix in order to reinforce the concept of Leslie-matrices. The students found this problem difficult - as did the teachers on teacher trainers courses. A lot of discussion, thinking, reflection was needed for the final solution.
This experience strongly differs from another one where a strongly similar problem was posed *during a test.* The results were striking: of 28 students one student scored 0 (out of 10), two students scored 2 and the rest 8 or higher (6 perfect scores). What is the value of such exercises and scores?
Obviously, the test goals didn't match those of the booklet. The only activities *tested* were reproducing the method and a little bit of transfer from one context to another.
Well, this is certainly not an unusual feature of tests. Traditionally, in mathematics instruction in the Netherlands, tests do not differ much from the exercises in the book. Or, as one of the students phrased it:
"The tests should represent the exercises in the book."
The problem, however, as we have indicated before, is that exercises from

class 2a allow only a quite limited view of the student capabilities. A well prepared student should have no problem at all with these exercises. The only information the teacher gets is that the student has prepared himself for the test. Again, in itself this is a valid goal, but for our goals more is needed.

Class 2b exercises allow more insight into the student's capabilities: although they still show some resemblance with the exercises in the textbook, more activities are required than mere reproduction of basic skills.

If we look at our above example, we notice that the first three questions are almost *class 2a*. But the remainder of the story is rather complex, and a disease under the two year olds makes the composition of the matrix even more difficult. But many students will feel at ease because of the resemblance: it is clear what kind of matrices are used, the calculations are well-known.
The reactions of the students: "more or less like the exercises in the book."
This was certainly not the reaction of students confronted with the traffic-light problem, representative of *class 2c* exercises. This exercise included in the textbook, would have posed no problem: basically this problem is not too difficult for Math A-students as was proven by the students. The context is completely new, though the real-world situation is not: the crossroads described in the exercise is close to the school buildings. When teachers were given this problem, we noticed how difficult it was to interpret properly the meaning of the matrices. Matrix A, the number of cars per hour, caused no problem. But the four matrices G_1, G_2, G_3 and G_4 posed more serious problems. To understand them one needs some imagination, which is tested by the second question. After these initial questions, which compell the students to reflect and think about the meaning of these matrices, the actual problem is posed:

"Are the traffic lights adjusted in a proper way in order to guarantee maximum traffic flow?"

Here higher goals are tested. Students must mathematize in a new situation, although most of the work has already been taken care of: the matrices are given; the way to solve the problem is indicated. Mathematization skill is required. Can they transfer these skills from the known problems to completely new ones? For the teacher who developed this test it was clear from the start that this was one of the main goals of Math A. Another teacher who tried to incorporate *class 2c* problems experienced that students who had

understood the exercises in the book got into big trouble with exercises of *class 2c*. The test helped these students to grasp the goals of mathematics. A passive understanding of the problems will not suffice. Active understanding is needed, the processes behind the solution have to be understood, while the product may be less important.
One teacher tried the following complicated *class 2c* exercise where most of the mathematization had to be carried out by the students. Again the students were surprised: "You need to think further than just the exercises from the book."

SICILY

from March 26
to October 29

maart		26						
april		9-16-23	2	30				
mei				7	14-21-28			
juni					4-11-18	25		
juli						2	9	16-23-30
augustus					27	20	13	6
september				17-24	3-10			
oktober		1-8-22			15			
type B 3 kamers	5 pers.	728	903	763	845	939	992	1061
	4 pers.	764	956	802	887	988	1046	1115
	3 pers.	823	1045	866	956	1070	1135	1204
	2 pers.	942	1222	994	1094	1235	1312	1381
type M 2/1 kamers	3 pers.	747	946	790	875	979	1035	1104
	2 pers.	828	1073	880	973	1097	1163	1232
type L 2 kamers	3 pers.	779	1003	821	912	1028	1093	1162
	2 pers.	875	1160	928	1028	1171	1249	1318

LS *MS* *HS*

The above data come from the Southern Europe Foundation travel brochure. When answering the questions, look only at the columns LS(low season), MS(middle season) and HS(high season).
The price per person is composed of the price of the flight and a portion of the price of the apartment.
How can you calculate the price of the flight and the apartment from the data given above?
Do this calculation for apartment B in high season.

The flight prices are the following:
LS fl. 584; MS fl. 677; and HS fl. 845.
Calculate using this information the price of the three different types of apartments during the various seasons (use maximum apartment occupancy) and show the results in the form of a matrix. For each season one row.

The travel organization has the use of 40 type-B, 30 type-M and 20 type-L apartments. Show this in matrix form.
Arrange both matrices so that the product matrix shows the entire proceeds when all apartments are rented for a week during LS, MS and HS.

Next year the prices will be raised by 10% during LS and by 20% during HS. MS remains the same.
With what matrix should the matrix be multiplied in order to create a new matrix showing next year prices?
If, however, during the entire year the prices of type-B rise by 10%, type-M remain the same and type-L rise by 20%, with what matrix will the matrix then need to be multiplied?

The results for this exercise came close to a disaster, but the students *liked* the problem and the discussion about the solution very much. It seems clear that *class 2c* exercises are less appropriate for a restricted-time written test.
For complex problems, like the traffic light problem and the Sicily problem - both part of a larger written timed test - one needs more time to read, to reflect, to mathematize, to interpret results.
Students may not be expected to perform all these activities in a short time in written form. This sets strict limits to questions asked and answers anticipated. There is not enough time to reflect on the model, to generalize, to integrate it with other models, and the students have no chance to show their creativity.

Almost all tests were meant to be criterion-reference tests, that is, tests used to estimate an individual's performance on the entire universe of which the test is a sample, or to classify a person's mastery of the subject tested. The standards are set before the test is administered. [8]
The materials being tested were, in this case, the materials as concretisized in the experimental texts. A major purpose of most educational tests is to help improve instruction. This aim, however, is not always attained. Certainly, when we restrict ourselves to tests of *class 1* and *2a and 2b* we will be testing only a very limited scope of the goals of Math A, and the feedback from the educational process will be minimal. But *class 2c* informed the teachers and students about essential goals of Math A. It was our *problem* that these *goals were not sufficiently identified* by the teachers on the evidence of the experimental material and that they found it *difficult to develop new 'class 2c' exercises.*

Let us return to the two questions posed in the beginning of this chapter.

The second was:

Do the exercises of the tests resemble the exercises in these materials, or are new contexts and new problems offered to find out how well the transfer to other problems has been mastered?

Our conclusion, from the analysis of the tests as given by the twelve schools, is that roughly 80% of the exercises look more or less like the exercises Only 20% of the exercises are the *class 2c* exercises that try to test the higher goals of Math A.
This suggests - as we indicated before - an answer to the first question:

In the restricted-time written tests the goals of mathematics A are operationalized in a limited way.
The reason is twofold as discussion with the teachers proves:
In the first place, the goals of mathematics are not manifest to the teachers from the experimental material and, even more important, it is very difficult to construct tests for Math A, especially because some goals cannot be operationalized appropriately by means of restricted-time written tests.
Before discussing the action to be taken to tackle this problem we would like to make some remarks on the classification. The classification was based on external factors but certainly not on quality. It is possible that a *class 2c* exercise may be very poor from the viewpoint of testing while, on the other hand, we have noticed some excellent *class 1* exercises. The following is an example of a typical *class 2a* exercise where we seriously doubt its value as a good test exercise:

Somewhere out in the middle of nature, where seldom anyone is to be seen, there stands a mysterious factory. Above the entrance hangs the sign "Côte d'Or". Whispered rumours say that the alchemist Ben Al-K'wasi is creating golden Christmas ornaments out of clay by means of a complicated procedure.
The 'young' ornaments are entirely clay.
After a year of maturing they turn silver and, after one more year they are old - but true gold!
If they don't break they will stay gold. If an ornament breaks, it dissolves forthwith into nothingness.

In 1983 the factory attic contained 217 clay ornaments, 128 silver ones and 70 gold ones.
By means of extremely uncommon earthly rays the silver and gold ornaments

can produce young: two silver ornaments are able to produce one young clay ornament.
In 1984 there were 288 clay ornaments and 98 gold ones.
In 1985 there were 230 silver ornaments.

Due to various causes, 70% of the golden ornaments breaks yearly:
a. Based on the above data, draw up the corresponding Leslie-matrix (for a period of one year). (Show the calculations!).
N.B.: round off all numbers to one decimal.
b. Calculate the missing data for 1984 and 1985.

This exercise may be successful in an appropriate classroom climate. But in general we would not recommend it.
It is an almost perfect example of a non-real-world context, which may irritate students. This imaginary context distracts rather than supports, and the teacher would have trouble finding the causes of the failure.

3 ON THE WAY TO 'BETTER'TESTS

To help the teachers construct restricted-time written tests we made tests from our collection available to other teachers by means of a series of publications under the title of '*Hewet and Tests*'.
By spring, 1985, as many as five such booklets had appeared in co-operation with the teachers of the experimental schools.

The second problem, identifying the goals, is part of the present study, with the prospect of selecting evaluation instruments to match the goals. Evaluation of students should have been assigned a more prominent role in the original plans. It should include the *full range* of the program's goals, including skills, problem solving, and problem-solving processes, according to the recommendations of the National Council of Teachers of Mathematics in the 'Agenda for Action' [5].

Other recommendations of this 'Agenda' deserve our attention as well when testing Math A students:
- Teachers should become knowlegedable about, and proficient in, the use of a wide variety of evaluative techniques.
- The evaluation of mathematics programs should be based on the program goals, using evaluation strategies consistent with these goals.

Aloon concludes from the recommendations that some additional concerns

should be met. [8]
We mention two of her points:

- Mathematics curriculum specialists and competent teachers should become more deeply involved in mathematics testing at all levels.
  They should be involved from the beginning, so that test objectives reflect the total mathematics program.
- Evaluation in mathematics should be concerned not only with minimal skills usually oriented to content but should also include cognitive competence. Ebel [9] characterizes cognitive competence as a combination of knowledge built from information by thinking and intellectual skills such as observing, classifying, measuring communicativity, predicting, referring, experimenting, formulating hypotheses, and interpreting data.

The limitations inherent to restricted-time written tests in particular have been discussed in '*Mathematics Counts*' as well:

- Examinations in mathematics which consist only of timed-written-papers cannot, by their nature, assess ability to undertake practical and investigational work or ability to carry out work of an extended nature.
  They cannot assess skills of mental computation or ability to discuss mathematics nor, other than in very limited ways, qualities of perseverance and inventiveness. Work and qualities of this kind can only be assessed in the classroom and such assessment needs to be made over an extended period.
  It is possible to go further. Not only do written examinations fail to assess work of the kind we have described above but, in cases in which they comprise the only method of assessment, they lead teachers to emphasize in the classroom work of a kind which is directly related to the type of question which is set in the examination. This means that, especially as the examination approaches but often also from a much earlier stage, practical and investigational work finds no place in day-by-day work in mathematics. [6]

The conclusion of the report coincides with our conclusion: The assessment needs to take into account not only those aspects of learning which can be examined by means of restricted-time written tests but also those which need to be assessed "in some other way".
An explorative study to find out "some other" possibilities is carried out in the following chapter. The *need* for such 'tasks' has emerged in the present chapter.

V ALTERNATIVE TASKS

1 INTRODUCTION

As they became more involved in constructing achievement tests - in October 1981 in the two schools phase - the Hewet team felt the tension between the goals of Math A and the restricted-time written tests. This tension was made clear in our previous chapter.

Up until then, written timed tests had been accepted as normal practice in Dutch schools. Other methods would unnecessarily disturb daily routine and the examination was also a written timed test. From our analysis of tests of the twelve schools it became clear that testing tends to emphasize the 'lower' behaviour levels, such as computation and comprehension. This is not a specific Math A problem. As Wilson states:

> "Mathematics teachers often state their goals of instruction to include all cognitive levels. They want their students to be able to solve problems creatively. But too much of their testing consists only of recall of definition, facts and symbolism." [1]

But in Math A the problem is even more serious because of its quite specific goals.
Mathematization, reflexion, inventivity and creativity are essential activities in Math A which are hard to test in the restricted-time written test.

The teachers in the experimental schools were hindered in several ways. In the first place, the goals of Math A had not been clearly stated, which made teachers rely heavily on the textbook materials. In the second place, no good exercises outside the booklets were available and only a few teachers created 'new' exercises as we noted in the preceding chapter. In the third place, the teachers were under heavy time pressure as there were hardly any provisions for teachers' participation in the experiment: preparations for teaching Math A took a great deal of time, compared with the old program. In the fourth place, the tradition of testing mathematical skills in Dutch schools did not inspire teachers to become involved with tests other than those in the textbooks.

During the experiment the goals of Math A showed up more clearly, as did the problem of assessment testing. The demand of the Cockroft-report that assessment (at 16^{+}) is to reflect as many aspects as possible, including those which need other means of assessment than restricted-time written tests, was

taken seriously by the Hewet team as well as by some teachers. [2]
The team, aided by teachers, tried to counter the problems as follows:

Firstly, an attempt was made to clarify the goals of Math A.
Secondly, and of more direct help to the teachers, exercises were sampled and distributed in order to confront teachers with new exercises. The effect was positive as well as negative: the pressure on teachers was softened, but at the same time teachers fell back on the new exercises rather than creating their own.
Thirdly, a discussion with the teachers was initiated concerning the validity and limitations of written timed tests, which eventually resulted in the present study discussed in this chapter.

The structure of this chapter is:
The *principles* behind the tasks and tests are presented in 2. Then four kinds of *alternative tasks* are described and discussed. In each case this is followed by a reflection on the principles.
As soon became clear, there was some concern among teachers about the 'objectivity' of scoring such rather complex tasks. In the case of one of the alternative tasks - the two-stage task - we made this a point for research. The total study concerned more than 160 students and 30 teachers from the twelve schools.

2 THE PRINCIPLES

Our principles behind the ideas of developing alternative 'tasks' are the following.

The *first* and main purpose of testing is to *improve learning*.
This principle - that we borrow from Gronlund [12] - is widely and easily underestimated in the teaching-learning-process. All too frequently we think of it as an end-of-the-unit or end-of-the-course activity whose primary purpose is to serve as a basis for assigning course grades. A properly designed test or task should not only motivate students by providing them with short-term goals toward which to work, but also by providing them with feedback concerning their learning progress.
Furthermore, more complex learning results such as the levels of understanding, application and interpretation are likely to be retained longer and to have greater transfer value than results at the knowledge level. This means that we should include measures of these more complex learning results in

our tests. In this way we provide the students practice reinforcing comprehension, skills, applications and interpretations we are attempting to develop.

The second principle is borrowed from 'Mathematics Counts': Methods of assessment (which are used) should be such that they enable candidates to demonstrate what *they know* rather than what they do not know. In traditional timed written tests most of the time we check what the students do *not* know. Students are asked a very specific problem which has, most of the time, one single solution. If the candidate doesn't know the solution, there is hardly any way to show what the candidate does know. A side effect is that a student may lose confidence, which should be avoided at all times. [2]

The third principle is that the tasks should *operationalize* the goals as much as possible. [3] Although the design of tasks that operationalize 'higher' goals is complex, we should not restrict ourselves to timed written papers, which by their nature cannot assess skills like mathematization, reflection, discussion of models, solutions or mathematization and creativity and inventiveness. This also means that we are not interested in the first place in the product (the solution) but in the *process* that leads to this product.
So we are in need of tests that do not only measure knowledge results. We need - according to Gronlund - tests that provide a freedom of response required for measuring certain complex outcomes. These include the ability to create, to organize, to integrate, to express, and similar behaviours that call for the production and synthesis of ideas.

The fourth principle is that the quality of a test is *not* defined by its accessibility to objective scoring. We accept that competent and independent judges may score differently - but *within* certain limits.

As the *'Handbook on formative and summative evaluation of student learning'* states:

> "No matter how well the items of a test match the table of specifications of intended outcomes, if any element of the scoring brings about inaccuracies in its application, then content validity is lowered. If any bias in the scoring causes the same relevant behaviour to be scored differently for different examinations, then both the validity and the reliability of the results are lowered.
> Objective scoring is considered especially important on summative tests. When one is doing formative evaluation, which takes place during the course, there are always opportunities for correction. However, *there is*

also the danger that the need for objective scoring overshadows the content of the actual test." [4]

In our opinion, certainly in formative evaluation, we should focus firstly on the content of the test and then try to maximize objectivity. During the last decade, most of the attention in the Netherlands was given to 'objective scoreable' tests, most often of the multiple choice kind. Contrary to a common belief it is hard, if at all possible, to design such tests to be foolproof.

Querelle [5] gives some examples of students that 'found' the correct answer by completely false reasoning. Nevertheless, test specialists are still promoting 'objectively scoreable' tests while overlooking features like:

- wrong reasoning leading to right answers;
- guessing right answers;
- encouraging students to remember, interpret and analyze the ideas of others, either teachers or textbook authors;
- training students in taking tests.

Sanders [6] argues that 'open' tests in general (multiple choice tests are 'closed') do not permit objective scoring. However, the use of answer-models improves the scoring objectivity. Full objectivity can never be attained. The author does not seem to be interested in an optimal mix between a good and fair *test* and a good and fair *score*.

At the school level we are dealing with open tests are not uncommon in mathematics education in the Netherlands.
In mathematics a relatively high degree of objectivity is obtained thanks to the nature of the discipline as it has been traditionally taught. De Jong collected (in 1980) tests of twenty students and had these scored by five teachers excluding the students' own teacher, in order to avoid bias. [7]

The teachers had a quite detailed answer model at their disposal. The eventual scores were as follows (fig. V.1):

Judge / Student	I	II	III	IV	V	largest difference
1	8.0	8.2	8.1	8.0	8.1	.2
2	5.6	5.7	6.3	6.3	6.3	.7
3	4.6	4.8	4.3	5.3	4.9	1.0
4	3.5	4.1	3.6	4.7	4.0	1.2
5	4.7	4.6	5.0	4.6	5.4	.8
6	8.6	9.0	8.5	8.9	8.9	.5
7	4.3	4.0	4.2	4.6	4.7	.7
8	5.7	6.1	6.4	6.6	6.2	.9
9	4.4	3.9	4.2	4.5	4.4	.6
10	3.7	4.1	4.4	4.6	4.0	.9
11	5.2	5.5	5.3	5.6	4.9	.7
12	6.4	7.3	7.3	7.8	7.9	1.5
13	6.4	7.0	6.6	7.8	6.8	1.4
14	6.0	6.5	6.7	6.6	6.6	.6
15	2.9	2.9	3.1	3.4	2.9	.5
16	2.3	2.3	2.4	2.9	2.2	.6
17	4.5	4.5	4.8	6.0	5.4	1.5
18	2.4	2.2	2.6	2.8	2.4	.6
19	6.0	5.6	6.0	6.4	6.3	.8
20	6.1	6.3	6.0	6.3	6.2	.3

Marks by five judges at a mathematics examination.

fig. V.1

This is much better than the scores of four teachers on an essay in the native language of ten students presented by Wesdorp (fig. V.2). [8]

Judges / Abstract	I	II	III	IV	largest difference
1	6.3	7.0	6.8	7.5	1.2
2	6.5	7.3	6.8	7.5	1.0
3	5.0	4.6	6.5	5.0	1.5
4	6.3	5.2	6.3	6.0	1.1
5	9.5	6.1	8.0	6.5	3.4
6	5.0	5.0	5.0	5.5	0.5
7	8.5	6.6	8.0	7.0	1.9
8	4.5	3.6	2.0	4.0	2.5
9	5.0	6.0	6.0	6.5	1.5
10	2.1	3.6	2.5	4.5	2.4

Assessment of ten abstracts by four judges.

fig. V.2

We will come back to these figures later, but for the moment we would like to stress our principle that the quality of a test is not in the first place defined by its accessibility to objective scoring.

Our *fifth and last principle* is that when developing alternative ways of evaluating students we should restrict ourselves to tests that can readily be carried out in school practice.
For this reason we must reject project-like tests, because they require too much activity 'on the road' and disrupt classroom practice for an extended period of time. We should offer students and teachers alternative tasks that can be realised at any time within school practice and which do not require disproportionate student or teacher activities, to be used in addition to, and as partial replacement of, restricted-time written tests.

Let us recapitulate the five principles:

1. Tests should improve learning.
2. Tests should allow the candidates to show what they know (positive testing).
3. Tests should operationalize the goals of the Math A curriculum.
4. Test-quality is not in the first place measured by the accessibility to objective scoring.
5. Tests should fit into the usual school practice.

The first four principles can be represented in the following way (fig. V.3):

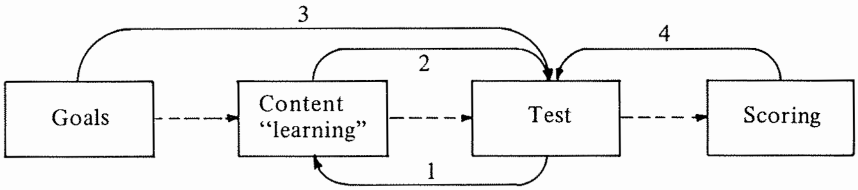


fig. V.3

The last principle is purely pragmatic in order to encourage teachers to use these tasks, and does not fit into the above schema.

There is little doubt that developing tests is a complex matter if we want to stick to our five principles. But we can at least try to do so as much as possible.

3 THE TWO-STAGE TASK

3.1 The two-stage task

We wish to propose a kind of test or task that combines the advantages of the traditional restricted-time written tests and the five basic principles as pointed out before: the *two-stage task.*

The characteristics of restricted-time written test as considered here, are:
- All students are administered the same test at the same time.
- All students must complete it within a fixed time limit.
- The test is oriented more towards finding out what students don't know rather than what they do know.
- Most of the time a lot of attention is given to the 'lower' goals: computation, comprehension.
- It is of an 'open' kind: it asks for short and long answers.
- Scores are as objective as they can be under these conditions (in mathematics).

These, then, are the characteristics of the *first* stage of our two-stage task.

Obeying our principles we add a *second* stage:
- The test is done at any convenient time at home.
- The amount of time spent on the test is almost unlimited - to be decided by the student.
- The test emphasizes 'what you know'.
- Much attention is given to the 'higher' goals: interpretation, reflection.
- The structure is open: long-answer questions and essay-type questions.
- Scoring can be difficult and not 'objective' - inter-subjectivity should be stressed.

The classification of questions used above is the one commonly used in the Netherlands by the National Institution for Educational Measurement CITO:
- Closed questions:
  the answers are scored by the computer.
- Open questions:
  - Open-ended questions which call for the respondent to answer by writing a statement which may vary in length.
  - Short-answer questions.
  - Long-answer questions.
  - Essay questions:
    a. Restricted response questions. [9]

b. Extended response questions.

The distinction between short-answer questions and long-answer questions is rather arbitrary. For mathematics a short-answer question is a question that calls the respondent to answer by means of a simple sentence, a number, a definition, a formula or a simple drawing or sketch. A long-answer question should be answered by means of a more complex sentence, a complicated calculation or drawing, or a proof with several steps.

The distinction between restricted-response and extended-response questions - though also arbitrary - is merely a convenient set of categories for classifying essay questions. The restricted-response question places strict limits on the nature of the answer to be given. The boundaries of the subject-matter content considered are usually narrowly defined by the problem, and the specific form of the answer is also commonly indicated. (E.g.: List, Define, Give Reasons). In certain cases the nature of the response is further limited by the use of introductory material or by the use of special directions. The advantages and disadvantages of this kind of essay questions are clear: the questions can be more easily prepared and scored, can be more directly related to specific learning outcomes: the students have less freedom to escape.
However, these restrictions limit the students' opportunities to demonstrate their ability to organize, to integrate and to develop essentially new patterns of response. The extended-response questions, on the other hand, are quite difficult to score, but provide creative integration of ideas and broad approaches to problem solving and reflection.

The two-stage task, inspired by ideas of Van der Blij, is a test with both open and essay questions. The *first* stage is carried out like a traditional timed written test. The students are expected to answer as many questions as possible at a given moment, and within a fixed time limit - say 45 minutes.
In principle, the students are free to tackle the questions they like during this time. The first half of the test is mainly open questions, the second half may include essay questions, so students may be expected to answer the open questions in the first place.

After having been scored by the teacher at home, the tests are handed back to the students, while the scores are disclosed, as well as the biggest mistakes (and only those).
Now the *second* stage takes place: provided with this information, the

student repeats the work at home ad lib: without restrictions and completely free to answer the questions as he chooses, whether one after the other, by way of an essay, or by any combination of these. After a certain time, say three weeks, the students turn in their work, and a second scoring takes place. This provides the teacher (and the students) with two marks: a first-stage and a second-stage one (fig. V.4):

Schema two-stage task:

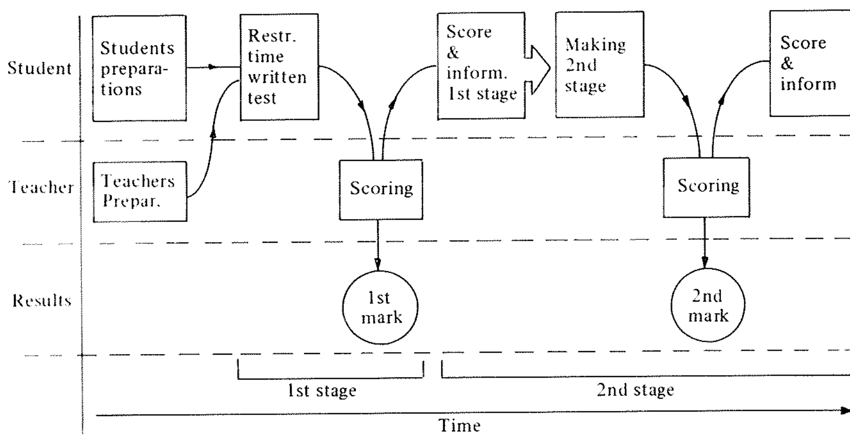


fig. V.4

Next we will discuss an example of a two-stage task while focussing on:
- The task: an example
- The scores (first stage)
- The students' second stage work and reactions
- The scores (second stage)
- Inter-subjective scoring of the second stage
- Teacher reactions.

An example

The following example is a test that tries to operationalize the goals of that part of the Math A curriculum that can be found in the booklet 'Matrices'.

Since this subject is treated in the first months of the course, there is virtually no possibility of integrating different subjects: the test is really a 'Matrices' test. The first questions of restricted-time written tests are very carefully posed, lest students get lost directly. For this reason the problem is already partly mathematized, which seems unavoidable in such timed tests. Later, questions are more open with the view on operationalizing higher goals.

TWO-STAGE-TASK

An example: The foresters problem.

A forester has a piece of land with 3000 Xmas trees. Just before Xmas he cuts a number of trees to sell them.

The forester distinguished three classes of length:

SMALL, MEDIUM, LARGE trees

The *small* trees have just been planted and have no economical value, the *medium* trees are sold fl. 10,- a piece and the large ones for fl. 25,-.

He has, just after Xmas 1000 small, 1000 medium and 1000 large trees.

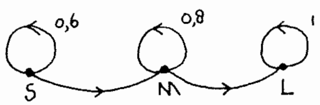
All these grow uneventfully till just before next Xmas.

From experiences of colleagues he knows approximately about the growth per year:

40% of the small trees becomes medium

20% of the medium trees becomes large

Or, in a GRAPH:



This graph may be represented by a GROWTH-MATRIX G:

$$
\begin{array}{cc}
 & \text{from} \\
 & \begin{array}{ccc} S & M & L \end{array} \\
\text{to} & \begin{pmatrix} .. & .. & .. \\ .. & .. & .. \\ .. & .. & .. \end{pmatrix}
\end{array}
\begin{array}{c} S \\ M \\ L \end{array}
$$

*1. Complete the matrix G.

*2. Calculate the composition of the forest just before the next Xmas (using G).

*3. After cutting medium and large trees and planting small trees the forester wants his starting population B (1000 of each) back. How many of each kind should be cut and planted?

*4. Cutting one tree costs fl. 1,-; planting one tree costs fl. 2-; what will be the forester's profit this Xmas?

The forester wonders whether the above strategy is the most profitable one. He considers two other strategies, so he has the choice of the following three strategies:

I. Cut after one year and plant as to get your starting population back.

II. Cut after two years and plant back as to get the start population.

III. Cut after one year the large trees only (leaving 1000) and replant the same number of small trees: repeat this the second year.

*5. Which of the above three strategies is the most profitable per year?

The forester considers to use fertiliser to have the trees grow faster. There exists a fertiliser that according to the producer, might lead to the following growth-matrix:

$$G_{*} = \begin{pmatrix} 0,6 & 0 & 6 \\ 0,4 & 0,5 & 0 \\ 0 & 0,5 & 1 \end{pmatrix}$$

*6. Explain why the trees grow faster with this model.

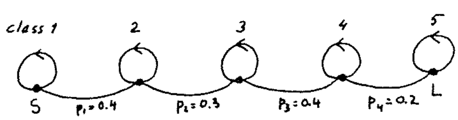
The forester likes to use the fertiliser but has some doubts whether it would n't be possible to get the start population B back after *each* Xmas. Because getting back the B-population is essential to him.

*7. Will it be possible to get back the start population B when the matrix G_{*} is of issue (after one year).?

*8. The forester decides to thin the fertiliser such that to get his B-population back every year he only has to cut large trees and to re-plant the same number of small trees.
Can you suggest any Matrix G_{+} that will have the desired effect?

*9. Using all information available, what would you advise the forester?

Another forester prefers *five* length-classes:

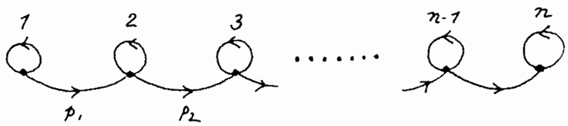


$$B = \begin{pmatrix} 1000 \\ 1000 \\ 1000 \\ 1000 \\ 1000 \end{pmatrix}$$

*10. Write an essay including:

- the growth-matrix in this situation
- the effect on the total population after one year
- the possibility to get B back
- if this impossible: change one of the entries of the matrix in order to make it possible
- the effect if a tree manages to grow in one year from class 3 to class 5

*11. Find the matrix for the general case:



How can you conclude from the matrix whether getting B is possible or not?

*12. What are the limitations of the model?

What refinements would you like to suggest?

*13. (see question *5) The third strategy was:

Cut after one year the large trees only (leaving 1000) and replant the same number of small trees; the same is done the next year(s).

What will be the effect of this strategy in the long run?

If we follow the classification of the questions as treated in our previous chapters, it is clear that we have no closed questions at all in this test.
This leaves us with thirteen open questions of different types:

- short-answer questions;
- long-answer questions;
- essay-questions:
  - restricted response;
  - extended response.

Although the distinction between these categories is rather arbitrary, we draw up the following table (fig. V.5):

question	short answer	long answer	R.R essay	E.R essay
1	x			
2	x	x		
3		x		
4		x		
5		x	x	
6		x		
7		x		
8		x	x	
9			x	
10			x	
11		x	x	
12			x	x
13			x	x

fig. V.5

Looking at this classification one may expect students to handle successfully the questions 1-8 during the first stage of the task. The remaining questions would seem to be more suited for the second stage.

This test was given to two groups of twenty students each. One group consisted almost exclusively of Math A students, the other group was mixed: students who had chosen *both* Math A and Math B.
Both groups had the same teacher and the students prepared themselves in the usual way.

3.2 Scoring the first stage

Scoring the first stage is considered relatively objective because, in this stage, we usually ask short-answer and long-answer open questions. The two-stage task involves essay questions which are considered difficult to judge. To see how 'objectively' the first stage can be scored we look at how successful the students are in answering questions from different categories.

For this reason we compare our table of classifications (page 188) with the success rates of the students (fig. V.6).

CLASSIFICATION

question	short answer	long answer	R.R essay	E.R essay
1	x			
2	x	x		
3		x		
4		x		
5		x	x	
6		x		
7		x		
8		x	x	
9			x	
10			x	
11		x	x	
12			x	x
13			x	x

SCORES

question	>75%	50-75	25-50	<25%
1	x			
2	x			
3	x			
4	x			
5		x		
6	x			
7	x			
8			x	
9				x
10			x	
11			x	
12				x
13				x

fig. V.6

Upon comparing these two tables it is clear that during the first stage the students concentrated on the first seven open short-answer and long-answer questions. This was indeed the intention of the first stage in order to insure a relatively objective score. The second stage should give the students the opportunity to show how well they answer essay-oriented questions.

The results per group were rather different, as was expected: the students of the mixed group had been taught mathematics for eight hours per week, compared to four hours per week for those who had chosen math A only.

The latter group scored an average of 5,9 with a S.D. of 1,6.
The mixed group scored 7,4 with a S.D. of 1,2.

3.3 The second stage

It is not easy to transmit a proper impression of the quality of the students' productions as delivered in the second stage of the two-stage test. The variety was enormous: some tasks were completed in a straightforward way, just by answering the questions and paying no attention whatsoever to lay-out and related topics. Other students turned in a veritable booklet, with colour illustrations, self-made computer programs, and typed-out or word-processed by computer.

Most students followed strictly the order in which the questions were posed and did not stray too far from these questions. Some wrote an essay in which all questions posed were answered and a number of students took the opportunity to show their own creativity in content-related questions.

To give an impression we have selected two examples.

The first is of a student who scored just below average at the second stage and just above average at the first stage. This work reflects more or less an average result. It is in its entirely without comment at this moment, though later on we will show teacher reactions to this work and the scores assigned to the work by different teachers.

Secondly, we present one student's answer to question 12 in order to show an example of an excellent answer by a student who did very well both in the first and the second stage. This is merely to show the possibilities of an essay question, compared with the shortcomings of open questions as posed in restricted-time written tests.

First example: A complete work

A lumberman owns a piece of land with 3000 Christmas trees. Just before Christmas a number of them are felled to be sold.

The lumberman has divided the trees into three categories:

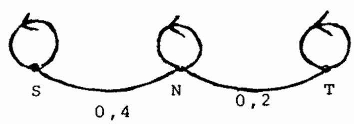
short, normal and tall trees.

The short trees are new plantings which are (hardly) worth anything, the normal trees cost fl. 10,-- each and the tall ones fl. 25,-- each.

Just after Christmas he has 1000 short trees, 1000 normal trees and 1000 tall ones. He lets these grow until the following Christmas. Felling them earlier would be a waste, since nobody wants a Christmas tree in the summer (not counting fake trees).

The lumberman knows from experience how fast the trees grow in the period of about a year:
40% of the short trees become normal trees:
20% of the normal trees become tall ones.

Or, in a graph:



Another way of indicating the tree growth is the Growth-matrix:

$$\begin{array}{ccc} & \text{From} & \\ & \begin{array}{ccc} S & \;N & \;T \end{array} & \\ \text{To} & \begin{pmatrix} 0{,}6 & 0 & 0 \\ 0{,}4 & 0{,}8 & 0 \\ 0 & 0{,}2 & 1 \end{pmatrix} & \begin{array}{c} S \\ N \\ T \end{array} \end{array}$$

The lumberman can now calculate how the forest will look just before the following Christmas, if he neither fells nor plants any trees.

$$\begin{pmatrix} 0{,}6 & 0 & 0 \\ 0{,}4 & 0{,}8 & 0 \\ 0 & 0{,}2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1000 \\ 1000 \\ 1000 \end{pmatrix} = \begin{pmatrix} 600 \\ 1200 \\ 1200 \end{pmatrix} \quad 3 \times \underbrace{3 \quad 3} \times 1$$

Just before the following Christmas there are 600 short trees, 1200 normal trees and 1200 tall trees. After felling the trees and planting young ones, the lumberman wants to have the same initial distribution of trees (1000 of each sort). He must, therefore now fell 200 tall trees and 200 normal ones and plant 400 young ones. He has calculated that felling one tree costs him fl. 1,-- and planting one new young one costs him fl.2-- and is curious what this first Christmas profits will be after deducting his costs.

Proceeds felled trees:	fl. 25,--x 200= fl. 5000,--	Costs:(200+200) fl. 1,-	
	fl. 10,-- x 200= fl. 2000,--		= fl. 400,--
	fl. 7000,--		+ fl. 800,--
			fl.1200.--

fl. 7000,-- fl. 1200,-- = fl. 5800,--

After regarding his profits the lumberman begins to wonder if this strategy is really so clever. He then considers two other strategies:

Wait two years before felling and replanting trees to regain the 1000 per sort. This would yield:

$$\begin{pmatrix} 0{,}6 & 0 & 0 \\ 0{,}4 & 0{,}8 & 0 \\ 0 & 0{,}2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 600 \\ 1200 \\ 1200 \end{pmatrix} = \begin{pmatrix} 360 \\ 1200 \\ 1440 \end{pmatrix}$$

fl. 25,--x 440= fl. 11000,--
fl. 10,--x 200= fl. 2000,--
fl. 13000,--

Costs: 640 fl. 1,--= fl. 640,--
640 fl. 2,--= fl.1280,--
fl.1920,--

fl. 13000,-- - fl. 1920,--= fl. 11080,- per two years
fl.11080,-- : 2 = fl. 5540,-- per year

Fell all the tall trees (until 1000 remain) after one year and replant the same number; and after the second year also only fell the tall trees (until 1000 remain) and replant the same number. this would yield:

600 short trees
1200 normal trees
1200 tall trees

200 of the tall trees would therefore to be felled and 200 new ones planted.

Proceeds year 1:	200	fl. 25,-- =	fl. 5000,--
Costs year 1	200	fl. 1,-- =	fl. 200,--
	200	fl. 2,-- =	fl. 400,--
			fl. 4400,--

$$\begin{pmatrix} 0,6 & 0 & 0 \\ 0,4 & 0,8 & 0 \\ 0 & 0,2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 800 \\ 1200 \\ 1000 \end{pmatrix} = \begin{pmatrix} 480 \\ 1280 \\ 1240 \end{pmatrix}$$

Proc.y.2	240x fl. 25,--	=	fl.6000,--
Costs Y 2	240x fl. 1,--	=	fl. 240,--
	240x fl. 2,--	=	fl. 480,--
			fl.5280,--

The average profit per year is: fl.5280,- + fl.4400,-- = fl. 9680,--
fl.9680,- : 2 = fl. 4840,--

After having compared these results the lumberman reaches the conclusion that his first strategy was the best one, namely, the one yielding the highest profits fl. 5800,--

He decides, therefore, to continue with this strategy.
He considers using a certain type of fertilizer which will help the trees to grow much faster.
Experiments show that the following growth-matrix is one of the possibilities:

$$G = \begin{pmatrix} 0,6 & 0 & 0 \\ 0,4 & 0,5 & 0 \\ 0 & 0,5 & 1 \end{pmatrix}$$

This matrix indicates the faster tree growth: the probalility that normal trees will have become tall ones a year later has increased by 0,3 (see $G_{2,2}$) and is now 0,5. Half of the trees which were of normal length a year ago now belong to the tall category. This signifies faster growth than the growth of the trees without fertilizer (the probability there was only 0,2).

The lumberman is extremely enthusiastic about the faster growth through the use of fertilizer, but doubts whether he can regain his original tree distribution by replanting young trees after every Christmas. That, according to him, guarantees a reliable business.

Well, his doubts were well-founded:

$$\begin{pmatrix} 0,6 & 0 & 0 \\ 0,4 & 0,5 & 0 \\ 0 & 0,5 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1000 \\ 1000 \\ 1000 \end{pmatrix} = \begin{pmatrix} 600 \\ 900 \\ 1500 \end{pmatrix}$$

The number of normal trees has decreased in a year's time by 100, and there are therefore 100 trees too few to meet with the requirement of 1000 trees in each category. The lumberman cannot compensate for this by planting new trees because he can only plant new short trees. So he must think up something else. He decides to dilute the fertilizer to the extent that each year he will only

have to fell tall trees in order to reestablish the original distribution (and replant them as well of course).
The growth-matrix, therefore, looks somewhat different:

$$G_{\bullet} = \begin{pmatrix} 0,6 & 0 & 0 \\ 0,4 & 0,6 & 0 \\ 0 & 0,4 & 1 \end{pmatrix}$$

In the case of this matrix, the probability that normal trees become tall ones has decreased by 0,1, but is still 20% more than when no fertilizer at all is used. The growth is somewhat slower, but the number of normal trees still reaches 1000. (0,4 + 0,6=0 1).
To our lumberman, it looks like this last strategy is the best one, namely, to use fertilizer but to dilute it enough so that the number of normal trees will not fall below 1000 but will reach their tall growth sooner. (Of course, he must fell and replant every year).
The lumberman seems to have solved the problem but there are still other factors involved which can influence the tree growth. Nowadays, for instance, we are faced with the problem of acid rain, which attacks trees and impedes their (normal) growth. It takes much longer for the trees to reach an appreciable height. In addition, the lumberman will on occasion be required to reduce his selling price, because he will otherwise not be able to get rid of trees which have been decimated by the acid rain.
After all, who wants a bedraggled spindly Christmas tree?

The climate must also be favourable to the lumberman. Severe frosts and droughts can impede the tree growth. A bad year, in which many trees die of frost or drought may require the lumberman to replant many more new trees than he had counted on. In this case, the distribution of the types of trees will have changed, since normal trees will already have reached their full height. In that case, the lumberman must take care that the number of normal trees does not exceed 1000.
It is quite probable that the sales and therefore also the profits may differ from year to year for a variety of reasons: due to may people taking winter holidays, to competition with other lumbermen, and to the increasing demand for artificial Christmas trees. But these factors have no effect o the tree growth.
Let us hope that this will prove a good year for the lumberman, with lots of good weather and a white Christmas. We can first of all

determine what can be done to deal with certain factors: in the case of bad weather, the young trees could be cared for longer in hothouses; the lumberman could join forces with other people to fight acid rain. That would help not only the general ecology but would also be to his personal advantage.

Let us now observe the methods of another lumberman.
He works with five classes of trees, but he, too, starts with 1000 trees per class.
These classes are all connected to each other, that is, all the new short trees keep growing until they have reached their full growth. The matrix to be applied to this lumberman is the following:

$$\begin{array}{c} \\ 1 \\ p1 \\ p2 \\ p3 \\ p4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & p1 & p2 & p3 & p4 \end{array} \\ \begin{pmatrix} 0,6 & 0 & 0 & 0 & 0 \\ 0,4 & 0,7 & 0 & 0 & 0 \\ 0 & 0,3 & 0,6 & 0 & 0 \\ 0 & 0 & 0,4 & 0,8 & 0 \\ 0 & 0 & 0 & 0,2 & 1 \end{pmatrix} \end{array} \cdot \begin{pmatrix} 1000 \\ 1000 \\ 1000 \\ 1000 \\ 1000 \end{pmatrix} = \begin{pmatrix} 600 \\ 1100 \\ 900 \\ 1200 \\ 1200 \end{pmatrix}$$

after 1 year the distribution of the trees into classes is as follows:

One sees immediately that 100 trees are missing in the third class. This is very annoying since 100 trees in this class would have to be replanted. But since only young trees can be planted, this signifies that the original distribution cannot be regained.
The effect of this matrix on the distribution of the trees can be seen quite clearly after one year. There are many short and tall trees, but the third class is no longer well represented. In order to retain 1000 trees in the third class, we will have to alter one P1.
The new matrix will be:

$$\begin{array}{c} \\ 1 \\ P1 \\ P2 \\ P3 \\ P4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & P1 & P2 & P3 & P4 \end{array} \\ \begin{pmatrix} 0,6 & 0 & 0 & 0 & 0 \\ 0,4 & 0,7 & 0 & 0 & 0 \\ 0 & 0,3 & 0,7 & 0 & 0 \\ 0 & 0 & 0,3 & 0,8 & 0 \\ 0 & 0 & 0 & 0,2 & 1 \end{pmatrix} \end{array}$$

Replanting is an important and necessary affair. If we take a look at what would happen if a tree grew so fast that it went, for instance, in one year from class 3 to class 5, we will see that, after a few years, all trees will belong to class 5. In this way, the lumberman will be able to supply trees very quickly, as long as he replants sufficiently. His initial tree population, however, will not be the same after one year.

The matrix for this lumberman can be notated for cases in general.

$$\begin{array}{c|c} & \begin{array}{cccccc} 1 & P1 & P2 & P3 & P4 & N \end{array} \\ \begin{array}{c} 1 \\ P1 \\ P2 \\ P3 \\ P4 \end{array} & \begin{pmatrix} 1-p1 & 0 & 0 & 0 & 0 & 0 \\ p & 1-p2 & 0 & 0 & 0 & 0 \\ 0 & P2 & 1-p3 & 0 & 0 & 0 \\ 0 & 0 & P3 & 1-p4 & 1-p_n-1 & 0 \\ 0 & 0 & 0 & p4 & pn-1 & 1 \end{pmatrix} \end{array}$$

It is evident from this matrix that the starting population can, in fact, be regained because the rate of growth is such that no classes are skipped over.

The rows must be added up which are more than or equal to 1, excepting the top row with the youngest trees which can be replanted.

This matrix is limited by the assumption that all trees will be sold and that no losses will be suffered from felled trees and all other trees.

It is possible toadapt the model with more refinement. We can introduce a sort of risk-factor to the matrix, which indicates how many trees are sold on the average each Christmas and how many trees die each year from various couses (such as the weather).

Let us again scrutinize the third strategy of the first lumberman: fell each year only the tall trees above 1000 and replant the same number. We have seen what can happen when trees grow so fast that they skip an entire class in one year.

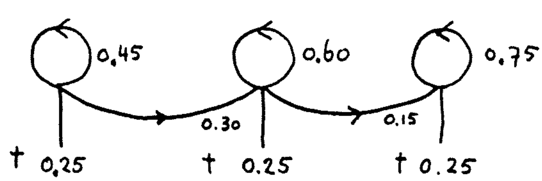
What would be the effect of this strategy in long run? Well, let us solve this by calculating with the initial matrix and a starting population of 1000 trees per class.

$$\begin{pmatrix} 0,6 & 0 & 0 \\ 0,4 & 0,8 & 0 \\ 0 & 0,2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1000 \\ 1000 \\ 1000 \end{pmatrix} = \begin{pmatrix} 600 \\ 1200 \\ 1200 \end{pmatrix} \begin{pmatrix} 0,6 & 0 & 0 \\ 0,4 & 0,8 & 0 \\ 0 & 0,2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 800 \\ 1200 \\ 1000 \end{pmatrix} = \begin{pmatrix} 480 \\ 1520 \\ 1240 \end{pmatrix}$$

$$\begin{pmatrix} 0,6 & 0 & 0 \\ 0,4 & 0,8 & 0 \\ 0 & 0,2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 720 \\ 1520 \\ 1000 \end{pmatrix} = \begin{pmatrix} 432 \\ 1824 \\ 1304 \end{pmatrix} \begin{pmatrix} 0,6 & 0 & 0 \\ 0,4 & 0,8 & 0 \\ 0 & 0,2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 736 \\ 1824 \\ 1000 \end{pmatrix} = \begin{pmatrix} 441.6 \\ 2188.8 \\ 1364.8 \end{pmatrix}$$

We can see that with this strategy we will eventually have a great number of normal trees and relatively few young ones. The forest will become enormous and the lumberman will eventually be forced to change strategies; otherwise he will have to fell normal trees for lack of space. So this is not such a good strategy after all.

The foresters' model has a number of restrictions:
- it assumes that all trees will be sold, so no cut-trees are left over → no losses
- one should consider the fact that due to the influence of competition the prices have to go down from a certain moment on; otherwise not all trees will be sold. Revenues per tree will decrease.
- furthermore one should take into account any disturbances in the growth process, not every year the trees will have the same growth-rate because of
  - insects
  - acid rain
  - draught etc.
- If, for instance one quarter of the trees are affected by acid rain the profits on those trees will be lost but there are still the cutting costs. The results for the forester will be:



- When there is a thunderstorm the forester will not use fertilizer because of the acid rain that pours down and results in denitrification of the soil. The fertilizer flows into the river and one gets many plants in the water. The lower plants will rot because of lack of oxygen resulting in CH_4- and H_2S-gasses. The forester has to take measures in order to prevent this resulting in lower profits.

Taking acid rain into account will result in:
The results from I (ƒ 5800.–) minus
$\frac{1}{4} \cdot T + \frac{1}{4} \cdot N \cdot ƒ 1,-$ (for cutting) + (1000 – the number of young trees to be planted). ƒ 2.– + costs to dispose sick trees.

We can calculate this with a matrix as well, growth matrix I:

$$I \quad \begin{array}{c} \\ S \\ N \\ T \\ \dagger \end{array} \begin{array}{c} \begin{array}{cccc} S & N & T & \dagger \end{array} \\ \begin{pmatrix} 0.45 & 0 & 0 & 0 \\ 0.30 & 0.60 & 0 & 0 \\ 0 & 0.15 & 0.75 & 0 \\ 0.25 & 0.25 & 0.25 & 1 \end{pmatrix} \end{array}$$

$\dagger$ = dead

I × B

$$\begin{pmatrix} 0.45 & 0 & 0 & 0 \\ 0.30 & 0.60 & 0 & 0 \\ 0 & 0.75 & 0.75 & 0 \\ 0.25 & 0.25 & 0.25 & 1 \end{pmatrix} \times \begin{pmatrix} 1000 \\ 1000 \\ 1000 \\ 0 \end{pmatrix} = \begin{pmatrix} 450 \\ 300+600 \\ 150+750 \\ 250+250+250 \end{pmatrix}$$

His population becomes:

- small trees 450
- normal trees 900
- large trees 900
- dead trees 750

felling costs ƒ 1.-
planting costs ƒ 2.-
disposing costs sick trees ƒ 2.-

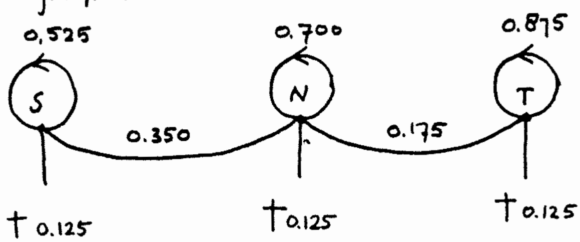
He has to fell 750 sick trees;
costs : 1x750 x 2x750 ƒ 2250.-
If he accepts that he will not get back his starting population B of normal and tall trees he will only have to plant 1000 - 450 = 550 trees, costing 550 x 2 ƒ 1100.-

ƒ 3350.-

He has no income, so he loses ƒ 3350.-

Let us assume that measures taken by the E.E.C decrease the effects of acid rain by 50%

The graph:



The matrix

$$I_* \quad \begin{array}{c} \\ S \\ N \\ T \\ \dagger \end{array} \begin{array}{c} \begin{array}{cccc} S & N & T & \dagger \end{array} \\ \begin{pmatrix} 0{,}525 & 0 & 0 & 0 \\ 0{,}350 & 0{,}700 & 0 & 0 \\ 0 & 0.175 & 0.875 & 0 \\ 0.125 & 0.125 & 0.125 & 1 \end{pmatrix} \end{array}$$

His population before last X-mas was :

$$\begin{pmatrix} 450 \\ 900 \\ 900 \end{pmatrix}$$ he plants 550 young trees ; the new population : $$B_* \begin{pmatrix} 1000 \\ 900 \\ 900 \end{pmatrix}$$

His population now (before X-mas)

$B_* \times I_*$

$$\begin{pmatrix} 0{,}525 & 0 & 0 & 0 \\ 0.350 & 0.700 & 0 & 0 \\ 0 & 0.175 & 0.875 & 0 \\ 0.125 & 0.125 & 0.125 & 1 \end{pmatrix} \times \begin{pmatrix} 1000 \\ 900 \\ 900 \\ 0 \end{pmatrix} = \begin{pmatrix} 525 \\ 350 + 630 \\ 157.5 + 787.5 \\ 125 + 112.5 + 112.5 \end{pmatrix}$$

The population :

- small trees 525
- normal trees 980
- tall trees 945
- dead trees 350

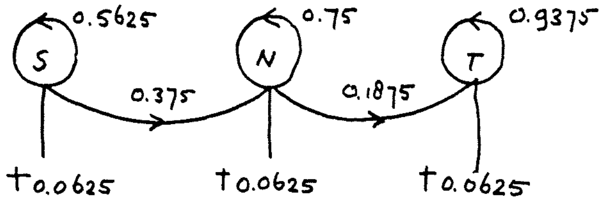
$$B_+ \begin{pmatrix} 525 \\ 980 \\ 945 \\ 350 \end{pmatrix}$$

He still makes a loss :

- for sick trees f 2.- × 350 + f 1. × 350 : f 1050.-
- for planting young trees (1000-525) × f 2.- f 950.-

f 2000.-

If the effects of acid rain decrease next year by another 50 % one gets :



$$I_+ = \begin{array}{c} \\ S \\ N \\ T \\ \dagger \end{array} \begin{array}{c} \begin{array}{cccc} S & N & T & \dagger \end{array} \\ \begin{pmatrix} 0.5625 & 0 & 0 & 0 \\ 0.3750 & 0.75 & 0 & 0 \\ 0 & 0.1875 & 0.9375 & 0 \\ 0.0625 & 0.0625 & 0.0625 & 1 \end{pmatrix} \end{array}$$

His population becomes $I_+ \times B_+$

- small trees	295	563
- normal trees	932	1085
- tall trees		1070
- dead trees		183

He fells some trees to sell :

income (1085 - 1000) × f 10.- + (1070 - 100) × f 25.- f 2600,—

costs 183 × f 2.- + f 1.- × 183 + f 1,- × (70+85) + (1000-563). f 2.- f 1578.—

income f 1022.—

After some time he will be able to cover all his losses.

3.4 Scoring the second stage

The teacher who had to grade the forty tests had no easy task. Not only were there many things to assess but he had neither an answer model, nor any suggestions at his disposal on how to grade fairly.
He chose the following procedure:
- read the complete work;
- mark positive and negative points;
- give a grade.

Examples:

Student A: + continuing growth well tackled;
+ graphs of stable population;
+ influence of acid rain calculated;
- minor errors in arithmetic.
Grade: 8.

Student B: - *continuing growth not well tackled;*
+ excellent lay-out;
- despite the correct conclusion, the argument for 13 is not correct;
+ different strategies when working with less concentrated fertilizer.
Grade: 8.

This example already shows that there is no one-to-one correspondence between the number of '+' or '-' signs and the actual grade. The total impression of the teacher was more or less 'adjusted' by the '+' and '-' conclusions.
This way of scoring would seem to lack objectivity. We will return to this problem later on.
Let us shift our attention to the actual results of the second stage and compare them with the scores of the first stage (fig. V.7).

	Stage I		Stage II	
	$\bar{x}$	S.D.	$\bar{x}$	S.D.
A-students	5.9	1.6	7.2	0.8
A/B-students	7.4	1.2	7.8	0.9

fig. V.7

One has to be very careful when drawing conclusions from this table.

Of course, when looking on the one hand at the differences between the first and second stage, and, on the other hand, at the A-group and A/B-group, one should not be surprised that the A-students made more progress in the second stage than did the A/B students: their lower results ($\bar{x} = 5.9$) made it much easier to get higher marks in the second stage. Certainly, we are not allowed to compare the results in such a superficial way. Both stages of the test operationalize different goals under different conditions. As noted earlier, in the first stage the students concentrated almost all their efforts on problems of the open question type.

In the second stage most attention was given to the essay-type questions. One should therefore not look at the second stage as a second chance for improving the results of the first stage. The two stages aim, in fact, at different goals. Only the context is the same and, in certain cases, the distinction between questions meant for the first stage and the second stage is arbitrary and left to the discretion of the students.
Only for those typical first-stage problems that were unsuccessfully handled in the first stage did the second stage really offer a second chance.

Nevertheless, one is tempted to compare the results of the first and second stage. For this purpose we have arranged the scores of the *first* stage from low to high (fig. V.8):

2	4^-	4^-	4^+	$4\frac{1}{2}$	5^-	5	5		9^-	9^-	$9\frac{1}{2}$	Mark 1^{st} stage.
↓	↓	↓	↓	↓	↓	↓	↓	....	↓	↓	↓	
$6\frac{1}{2}$	7	7	9	$7\frac{1}{2}$	$7\frac{1}{2}$	7	$6\frac{1}{2}$		9	10	10	Mark 2^{nd} stage.

fig. V.8

In this table we show only the extremes; the rest of the figures is represented in the following graphs (fig. V.9a,b,c):
- Fig. V.9a shows the correlation between the scores of the first and the second stages.
- Fig. V.9b shows the scores in a similar way but only for the girls.
- Fig. V.9c shows the scores for the boys.

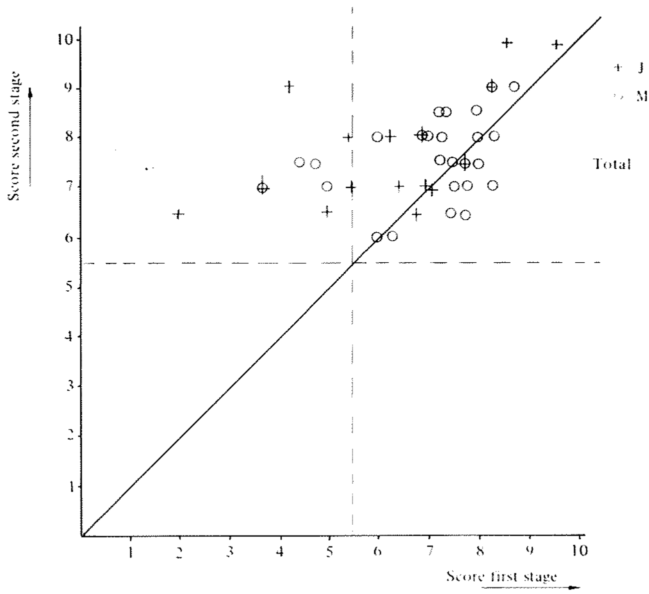


fig. V.9a

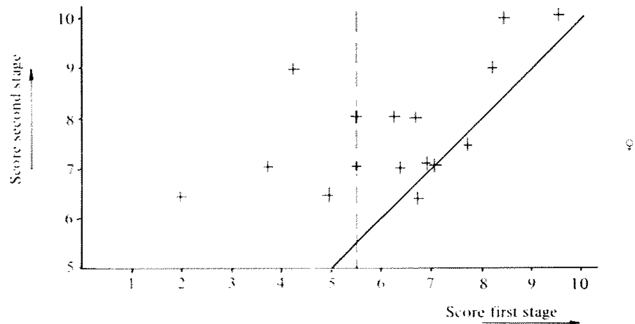


fig. V.9b

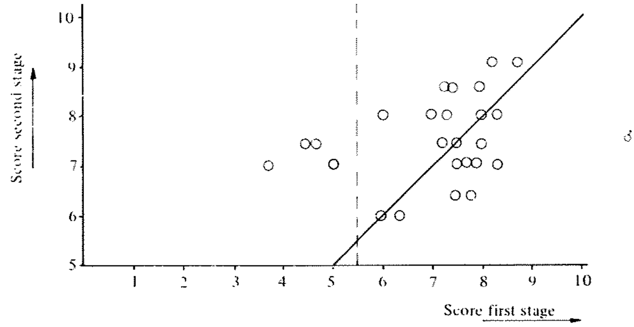


fig. V.9c

3.5 Discussion and reactions

Looking back at the scores of the first and second stages:

- There is a relatively wide spread in scores for the first stage: from very poor to excellent. In the second stage all students perform at least satisfactorily.
- Girls perform relatively poorer than boys in the first stage (given equal conditions: girls in the A group perform poorer than boys in the A group). At the second stage this difference seems to disappear. In fact, the best results were scored by girls. This is in accordance with experiences of teachers in other disciplines, and of Van Streun in his project. [10]
- Provided the scoring of the second stage proves to be more or less 'objective' it would seem mandatory to offer students tests of this kind in light of the different goals, different conditions and - not surprisingly - different results.

As one of the participating students stated:

> "Although I don't particularly like the second stage of this test because of the time necessary to complete this part, I consider it fair to offer students the possibility to manifest abilities which they are unable to in a restricted-time written test. For this reason I think we should be offered such a test once each quarter; preferably the test should aim at a higher level and integrate more subjects."

A special aspect was mentioned by another student:

> "Usually when a restricted-time written test is returned to us, we just look at our grade, and look whether it is in accordance with the number of errors and mistakes we made.
> In the case of the two-stage test one *learns* from performing the task: one has to study the first stage carefully in order to perform well at the second stage."

As pointed out earlier, reflection is an essential activity in Math A. Since it appears that at the second stage the students are *forced* to reflect on their first stage work, it seems that this goal is well operationalized at the second stage.

Per group the reactions were quite different. The group consisting of A students was unconditionally in favor of the two-stage test. Their arguments were much in line with the reactions just mentioned.
Not surprisingly, students who had scored poorly at the first stage were

amazed to see how well they did at the second stage. This phenomenon enhanced their self-confidence. A number of them had not succeeded - during the period we were following their performances - to improve their results in timed-written tests. This provides one more argument to promote two-stage tests as a complement to restricted-time written tests.

The reactions of the A/B-group were less favourable. Looking back at the results, and considering the rather high correlation between the first and second stage results, some students expressed their doubts about the need for a second stage.
Confronted with the results of some of their mates - with a big discrepancy between first and second stage - they concluded that it was 'fair' and 'justifiable' to offer tasks of this kind.
One should not be surprised that some students - and teachers - were suspicious of successes at the second stage after first-stage failures. The teacher tackled this problem by interrogating the students. From these discussions it became clear that the students really 'mastered' the subject and were able to explain orally their solutions.

Another point of discussion was that first-stage open questions could 'always' be graded as correct or incorrect, which is impossible for essay-questions.
Indeed, rather than being 'digital' the decision instrument looks 'analog': the answer looks reasonable, or very good, or rather poor. Although more or less familiar in other disciplines and daily life this kind of decision seems to be a new phenomenon in mathematics.

This remark led the discussion to the next question:
"How 'objective' can tasks like the second stage one be scored?"
Only one of the forty students was not entirely satisfied with the grade he received, while all other students were so.

The lack of objectivity in scoring tests like these is often used as an argument not to use this kind of evaluation. As we have pointed out earlier, this criticism neglects one aspect: it places the 'objectivity' of scoring above the goals and fairness of a test. Education becomes endangered by the shadow of preparing for objective tests and examinations. For this reason we wanted to find out how subjective the second stage scoring actually was.

3.6 Objective Scoring

3.6.1 Objective scoring

Second stage tasks of five students (showing a reasonable spread in quality) were sent out to fifteen teachers from the fifty-two Hewet-schools. The teachers were asked to grade the tasks.
The boundary conditions were the following:

- No information on the students.
- No information on the first stage results.
- No indication on how to score the tasks.

Most of the fifteen teachers considered their task rather unfair because of these boundaries and one even refused to grade the test under these conditions. All others, however, cooperated, albeit some of them reluctantly: "It took me a long time before I felt confident to start grading the tasks. Finally I felt I could do it. But the fact that I don't know the students means that my grading suppresses inevitable subjectivity."

Most teachers gave 'global' grades. They read the tasks, compared them and proposed rather rough grades.
Some teachers rounded off to whole numbers, but others to halves or even quarters. However, teachers who give such apparently accurate grades, immediately indicate that the grades are actually less accurate than one might be tempted to conclude from the numbers. The history of such grades is often as follows: The teacher first gives a rough indication of a grade, let us say 7-8. If he has three students in this bracket he decides about their order, and eventually ends up with the following scores:

7; 7.5; 8 or 7+; 7.5; 8-.

The complete list of scores:

	Teacher														
Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	6	5	6½	7	6^-	7	6	7	6½	8	7	6½	5½	7	5
2	7	8	7	8	6	7^+	8	6	7	7½	6	7^+	6½	7	6
3	8	6	7	8	6½	8^-	8	8½	7½	8½	6	7½	7	8	7
4	9	8	9	9	7½	8	8	9	8½	8^+	8	8^+	8	8	8
5	10	8½	9^-	9	8	8^+	9	9	10	9½	9	9^-	9	8	9

List of Results. fig. V.10a

The question arises: How serious is the discrepancy between the different scores? The answer to this question is related to other questions.
In the first place, one should bear in mind that this score is the *second* on this task. The *first* score is considered more or less objective.
In the second place, in the whole process of achievement testing the two-stage test (or similar ones) is only one in a row of otherwise restricted-time written tests. So the effect of the score of each of the tests is levelled off by the others.
In the third place, the lack of any hints on how to score this test may have influenced the spread of scores. This point was made already in the teachers' reactions but, as we pointed out, we intended to find out the 'objectivity' of scores under the most unfavourable conditions.

Assuming a beneficial effect from the availability of an answer-norm model we 'adjusted' the scores in the following way.
First we looked at the average score of the whole group (15 x 5 scores). This is 7.6 with a S.D. of 1.13. Then we adjusted the scores per teacher so that the five scores of each teacher showed an average of 7.6 and a S.D. of 1.13.
The average appreciation per teacher of the work differed in the following way (fig. V.11):

Teacher #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	ALL
$\bar{x}$	8.0	7.1	7.7	8.2	6.8	7.7	7.8	7.9	7.9	7.9	8.4	7.2	7.7	7.2	7.6	7.6
S.D.	1.4	1.4	1.0	0.7	0.8	0.5	1.0	1.2	1.2	0.7	1.1	0.8	1.2	0.5	1.4	1.13

fig. V.11

This resulted in a 'normed' score list:

Student															
1	6.0	5.8	6.3	5.8	6.5	6.0	5.5	6.8	6.3	7.0	7.4	5.9	6.0	6.2	6.0
2	6.8	8.4	6.9	7.3	6.5	6.6	7.8	5.8	6.8	6.1	6.4	7.0	6.9	6.2	6.8
3	7.6	6.7	6.9	7.3	7.2	7.8	7.8	8.2	7.2	7.9	6.4	7.4	7.4	8.5	7.6
4	8.4	8.4	9.1	8.8	8.6	8.5	7.8	8.6	8.1	7.4	8.4	8.5	8.4	8.5	8.4
5	9.2	8.8	8.8	8.8	9.3	9.1	9.0	8.6	9.5	9.6	9.3	9.2	9.3	8.5	9.2

fig. V.10b

Normed Results

Student 1 is the student with the poorest results and student 5 the best. Analyzing the tables one notices one particular outlier: (2,2).

We will encounter this score later when discussing the results of student #2. Furthermore we have circled the places of order violation. The large contribution of student #1 to violation of order may be explained by the fact that the purely mathematical content of this work was quite satisfactory, whereas the lay-out and appearance left much to be desired. As we will see, teachers disagree upon how much each of those aspects should contribute to the final result. It may be worthwhile to notice that student #1 was the only one among forty that was not satisfied with his teacher's score. This score had been a 6, which is supported by most teachers.

Let us study in some detail the scores of student #2. As indicated, the teachers had to grade this test under very unfavourable starting conditions, and this student showed the widest variety of scores (see also fig. V.15).
Here are some typical reactions to the task of student #2 showed on pages 193-198 (all scores on a 10 point scale):

- Very readable, good work. (8)
- Mathematically spoken without errors. (7)
  Lack of creativity.
- Very well presented. (7)
  Contents are good.
- Lay out: excellent. (8)
  Contents: good.
  Well balanced, much information on non-mathematical aspects.
  The end lacks long-term developments.
- Very readable. (7)
  Much attention to details.
- All students did very well. (8)
  This is about average in this group.
- I don't like non-quantative stories (6)
  as in exercises 9 and 12.
- I liked very much that she made a (8)
  complete 'story' of the task.

The last remarks reveal opposite views: the teacher who likes such 'stories' is positive about the work, whereas the one who dislikes them, is negative. This is no surprise, of course. It was anticipated that the grading of these tasks would stir up a discussion not only about 'objective scoring' but also about the goals of assessment testing, and the goals of Math A itself.
One reaction: "This is excellent material that contributes to the improvement of learning, but it is not what tests are for."
Another teacher stresses the fact that the essay-type question offers the

students the possibility to show their creativity and reasoning capabilities. This leads to rather big differences in student results, which makes scoring these tasks very difficult.
Another teacher 'complains' that with this kind of task it is hard to get 'unsatisfactory' results. But he admits that this might not be of great concern to us.
A reaction with regard to the task in relation to the goals of Math A:

> "In my opinion the goals of mathematics can be operationalized with a two-stage test. Grading these tasks is more difficult, however, than grading restricted-time written tests."

Let us return to the actual scores of the work of student #2. As indicated in the teacher reactions there is a discrepancy in the appreciation of the non-mathematical aspects of the work; it is larger here than with the work of the other students. This has certainly contributed to the spread of scores, the largest of the five students involved.

The scores were:

6 6 6 6 6.5 7 7 7 7 7.25 7.25 7.5 8 8 8

or, graphically (fig. V.12).

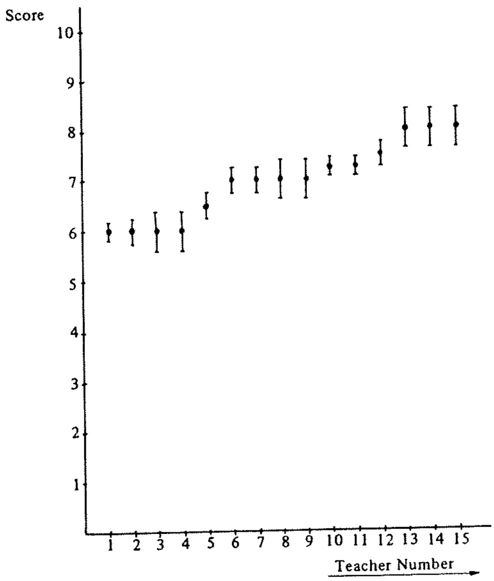


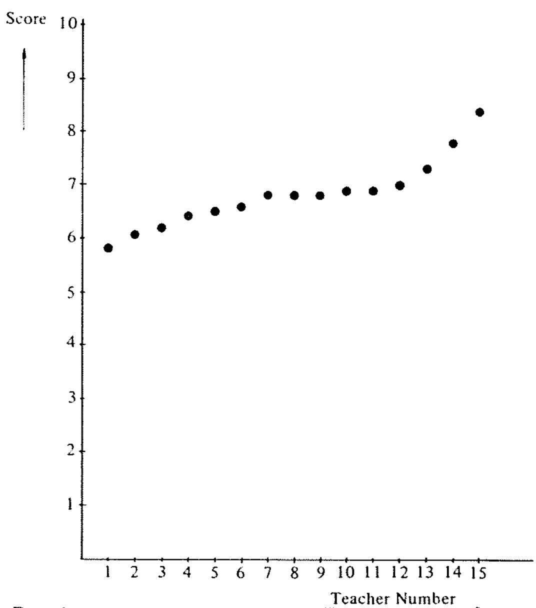
fig. V.12

Results student #2

If we adjust the scores of the work of student #2 in the way indicated before we get the following scores:

5.8
6.1 6.2 6.4 6.5 6.6 6.8 6.8 6.8 6.8 6.9 6.9
7.0 7.3 7.8
8.4

or, graphically (fig. V.13).

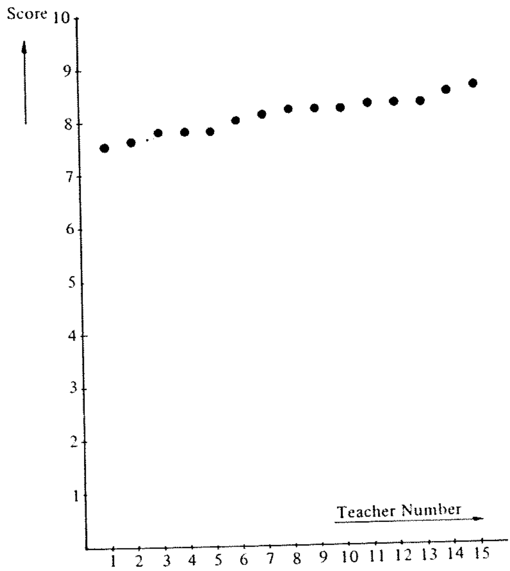


Results student #2 (normed)

fig. V.13

Looking at these 'normed' results one notices two grades (7.8 and 8.4) that don't seem to fit in the overal picture. That leaves us with thirteen scores relatively close together, and eleven differing less than 1.0 point.

The agreement between the 15 teachers when judging the work of the *best* of these five students is remarkable, especially when normed (fig. V.14):



Results of student #5 (normed)

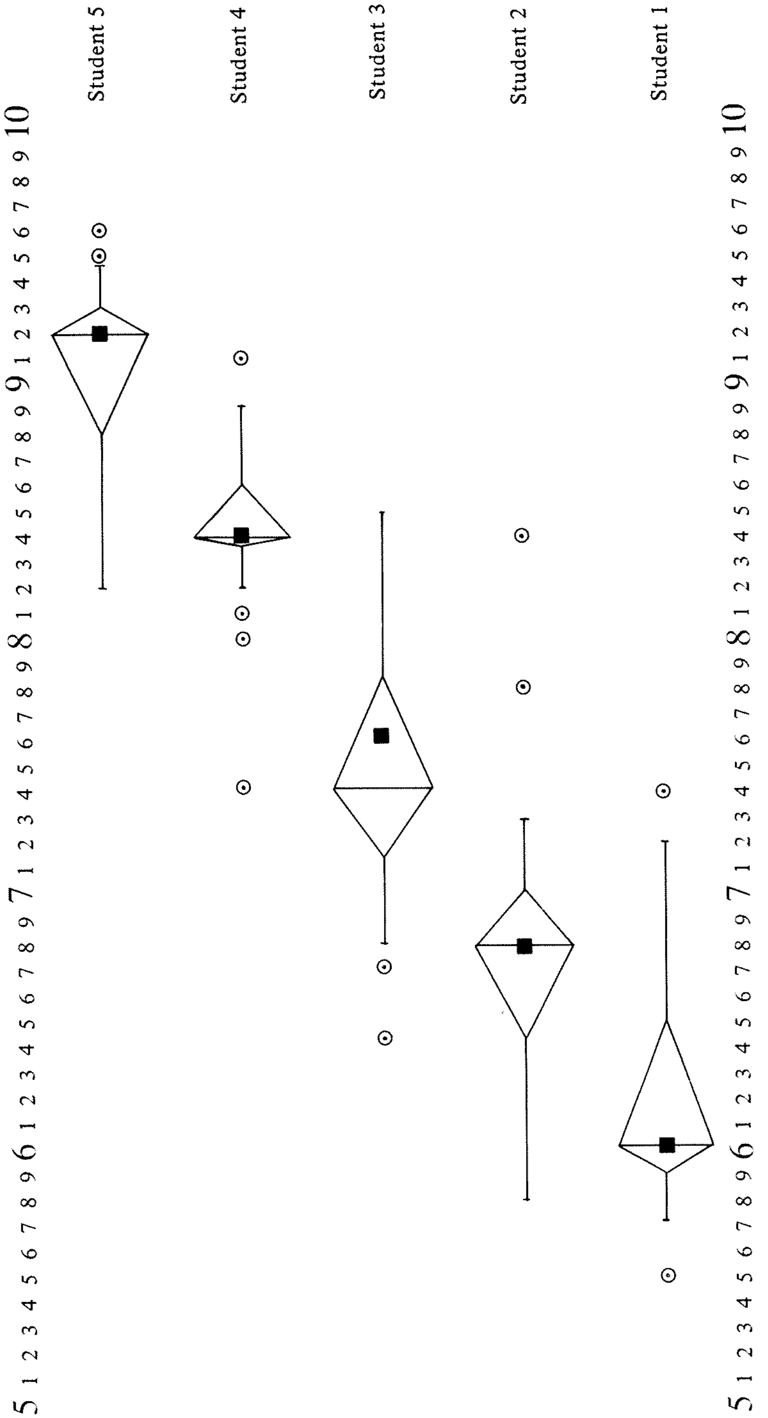
fig. V.14

In this case the rate of agreement seems to be high under the given conditions.
Student #2 and #5 represent the extremes, as we can see from our complete list of scores (fig. V.10a,b) and from the box-plot form (fig. V.15).
This graph shows the scores of all five students, and their spreads.
As indicated, student #2 shows the widest spread with two outliers: 7.8 and 8.4.
Students #4 and #5 show a small spread with relatively close outliers. One of the fifteen teachers involved in the scoring was the student's own teacher. He was in a different position from the other fourteen teachers, supposedly showing a bias for the students. This is not the case as can be seen in the box-plot graph. The actual score given by the student's own teacher is indicated by ■ and is the median of four out of five cases.

Among three control groups of five teachers, each scoring the work of five students, no significant differences were found compared to the results of the group we discussed.

3.6.2 Different views of teachers

It seems worthwhile to pay attention to the scoring differences caused by teachers who are more or less attracted by non-mathematical contents.



Box-plots of normed scores of five students. ■ Actual score by own teacher.

fig. V.15

The two variables:

- mathematical contents (1)
- non-mathematical contents (2)

are - considering the teachers' report - more relevant than others like:

- lay-out and general appearance (3)
- language (4)

If we assume that both variables (1) and (2) have the same weight we can make good use of a representation with parallel lines: equilateral lines as in the following figure (fig. V.16):

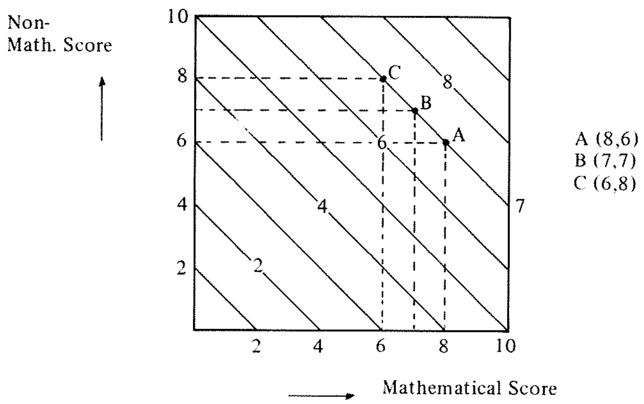


fig. V.16

From this nomogramme we can read that the score of 7 can be reached in different ways: an 8 for the mathematical content and a 6 for the non-mathematical content, or two 7's, or a 6 and an 8, respectively.
This nomogramme seems to represent the ideas of most teachers as long as the scores don't differ too much. But it would be too simple to attribute a 6 if the mathematical content is perfect (10) and the non-mathematical content a 2. In cases like this, one will be inclined to grade higher than 6. On the other hand, in a similar situation with a 2 for mathematical content and a 10 for non-mathematical content, one will tend to grade lower than the 6 of the first proposal. A more realistic assumption would lead to the following nomogramme (fig. V.17):

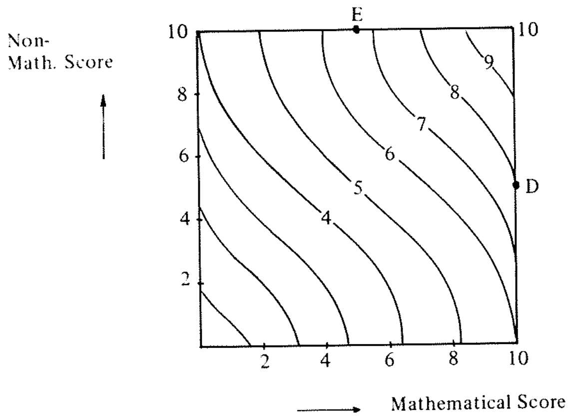


fig. V.17

Here the examples A(8,6) B(7,7) and C(6,8) lead to the same results under the first assumption, but we get differences in cases like D(10,5), to wit weight distribution 7.6 and for E(5,10) we get 6.5 as a result.
This is likely to represent the attitude of many teachers involved in grading the second stage of the two-stage test. But there are some exceptions:

"I don't like the non-quantative stories"

was a remark of one of the teachers.
His grading of the second stage was rather low compared to those of the other teachers. The lack of appreciation for non-quantative stories may have contributed to this fact. Teachers having this opinion have a different idea on grading, or so it seems.

The mathematical, quantative aspects are more appreciated than the non-mathematical ones, which leads to the following nomogramme (fig. V.18):

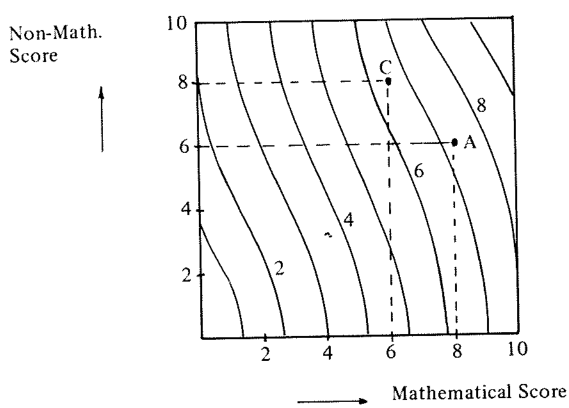


fig. V.18

Here not only rather extreme situations like D(10,5) and E(5,10) yield a score different from the original one. Also A(8,6) and C(6,8) lead to slightly different results.

A(8,6) gives a grade 7.5
C(6,8) gives a grade 6.5.

From the graph we see that scores of different teachers using the third weight distribution will be 'more objective' than those of teachers using the first two. Indeed, scoring strictly mathematical content seems to allow more objectivity than scoring non-mathematical and non-quantative results.
In our situation, however, most teachers seem to use a method similar to our second.

In the real class situation the discrepancy between the different teacher-policies does not necessarily lead to problems. The weight distribution used will be part of the didactical contract [11] between teacher and students. The students know in advance what the teacher expects from them.
The more explicit this didactical contract is made in the classroom, the fewer problems will arise when scoring open essay type questions. In our situation one should realize that only one of the fifteen teachers had a didactical contract with the students.

3.6.3 Intersubjectivity vs. Objectivity

Where the National Institution for Educational Measurement (CITO) is concerned, objective scoring means that the human factor does not play any role at all in scoring. To put it differently: the same work is scored the same by different judges. For open question tests this is unfeasible [6] even when using answer models, or a second judge.
In the case of our second stage scoring there is no way to effectuate an objective score. According to Gronlund and others the best way to check on the reliability of the scoring of essay answers is to obtain two or more independent judgements. [12]
If the results are to be used for important and irreversible decisions the pooled ratings of several competent persons may be needed to attain a level of reliability that is commensurate with the significance of the decisions being made.
Such pooled ratings however, may differ significantly as we have seen in paragraph 2: the scoring of the abstracts. (See page 182).
In other cases there may be a *large degree of agreement*, for instance in the case of students #4 and #5 (of the second stage). In this case we speak of *intersubjectivity*.

In the situation of the two-stage test we were fortunate to have the score of fifteen teachers: pooled ratings of fifteen competent people.
In the practical school situation, however, the scoring can be done by the teacher together with a colleague. That means two independent ratings.

In order to find out how much ratings by two judges will differ given the distribution of the ratings of the fifteen teachers from our study we could have computed this with standard techniques but we have chosen a different method:
We take from our sampled data 1000 times two marks (at random) from the 15 given marks of one student.
Next we take the (absolute) difference of these two scores.
For instance:
The computer chooses (at random) a student.
Let us assume: # 2.
Next, two scores are taken (at random):
let us assume: 7 and 6.5
The difference is in this case: 0.5.
This is repeated another 999 times resulting in the following tables (fig. V.19).

```
FREKW. VERSCHIL:
        0   180
     0.25   110
      0.5   176
     0.75   74
        1   263
     1.25   34
      1.5   69
     1.75   12
        2   71
     2.25   0
      2.5   7
     2.75   0
        3   4
     3.25   0
      3.5   0
     3.75   0
TOTAAL: 1000
GEM. VERSCHIL: 0.7725
STDEV. VERSCHIL: 0.606315718
```

```
FREKW. VERSCHIL:
        0   152
     0.25   274
      0.5   214
     0.75   154
        1   94
     1.25   52
      1.5   32
     1.75   16
        2   7
     2.25   2
      2.5   3
     2.75   0
        3   0
TOTAAL: 1000
GEM. VERSCHIL: 0.552
STDEV. VERSCHIL: 0.453509647
```

fig. V.19

Frequences of differences of 1000 drawings of two from the data.
Left: original data.
Right: normed data.

Cumulative graphs representing the tables give a good impression of the rate of agreement among teachers, or about how well intersubjectivity is reached (fig. V.20).

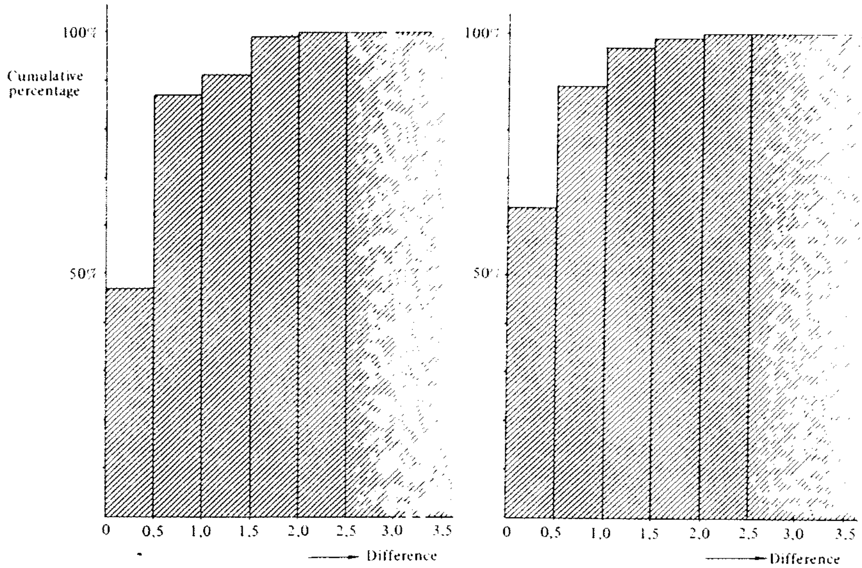


fig. V.20

Cumulative graphs of relative frequencies of differences of two drawings of two from the data.
Left: original data.
Right: normed data.

From the graphs it is clear that with the rough uncorrected material, two scores of a student's task lay within 1.0 point in 81% of the cases. In the case of the adjusted scores this number reaches nearly 90%.
One may also calculate the difference of the average of two scores and the average of all scores. The latter is considered to be the *correct* grade for a student.
In this situation we arrive at results similar to the above: roughly 90% of the average of two scores lies within a half point of the 'correct' grade.
This leads us to the following conclusion:
Given:
- the fact that there was here no answer model;
- the fact that the teachers didn't know the students;
- the fact that many teachers had no experience at all in grading these tasks;

we think that intersubjectivity is so large that it would seem 'fair' and 'objective' enough to advocate the use of these tests, considering the fact that the 'loss' of objectivity seems to be complemented by justice done to the students and the goals of Math A. There seems to be no doubt that the scores of the first and second teacher will be even more closely related if: the teachers cooperate within one school; have a clear didactical contract with the students about the value of the different aspects of the task; and are more experienced in using these kinds of tasks.

The role of the answer model is unclear. Although detailed answer models are in use in mathematics examinations in the Netherlands it is not yet certain whether these contribute to better results. Research carried out in the Netherlands showed *no* improvement in 'objectivity' when a rough model for scoring was replaced by a more detailed one in one case; but in another case it did. [6] We face the problem: the answer model is very precise, but is it correct? The first examination for Math A with a more or less detailed answer model stirred quite a lot of protest from teachers. Insight was underrated and pure quantative work was overrated. This again makes clear that objective scoring doesn't exist and will never exist in open questions.

3.7 The five principles

1. The tests should improve learning

The example given shows that the tasks contributed positively to the learning process.
New applications and concepts were introduced, students had the opportunity to introduce their own ideas, reflection was necessary between the first and second stage. Interaction took place between the two stages, outside advice was asked, students used the library to gather facts. There was ample opportunity for the students to show their creativity; students learned from the discussion afterwards.

2. The test should be such that candidates show what they know (positive testing).

Positive testing was optimal in the second stage, which will be clear from the two examples shown. Most students realized this very well - some of them by writing an essay rather than answering all the questions.

3. The test should operationalize the goals of the Math A curriculum.

From earlier discussions it will be clear that the problem lies in operationalizing the 'higher' goals. Some of these 'higher' goals were reasonably operationalized by this task. Mathematizing, problem solving, reflection, creativity, flexibility are important activities that take place at some moment during the second stage. Furthermore, the lower skills - computation, definition, etc - are present in the first and second stage.

4. The quality of a test is not *defined by its accessibility to objective scoring.*

It is clear that this task was never intended to be scored objectively. Nevertheless we feel that the results of this chapter show that intersubjectivity is reasonably high in the scoring of the second stage. Students recognized the problems when scoring tests like the second stage. But they were satisfied with their marks (except one) and felt that the 'lack of objectivity' should not prevent teachers from executing similar tasks.

5. The test should fit into the usual schoolwork.

All students spent more time on the two stages together than was usual on one restricted-time task. This was not considered to be unfair as the task replaced two restricted-time written tests. The regular school practice was hardly disrupted, although both making and scoring the second task took more time than usual. Teacher and students agreed that such a task should be held about once every quarter.

In the remainder of this chapter we will briefly discuss other examples of alternative tasks as carried out by Hewet experiment schools.

4 THE TAKE-HOME TASK

4.1 The take-home task

The first form of a task to be discussed in this way is the *'Take-home task'*. This task compares well with the second stage of the two-stage test that is it is an essay-type test.
The starting situation was similar to that of the two-stage test: twenty students from the second group of students at the first two schools had just completed the booklet *'Matrices'*. The booklet was covered in twenty-five hours. After some ten hours the students took a fifty-minute written test. After having finished the book the students were given a short description of a task that they had to do at home. They were allowed to:

- choose one out of five subjects;
- either work alone or in pairs.

The intended level of the task was definitely *higher* than covered in the book. The reasons for this were to get a reasonable picture of the possibilities and capabilities of the students when confronted with tasks at a somewhat higher level, a level that could be expected when students continued their study at college and university level.

This did not seem unfair to the team that carried out the experiments considering the fact that the students were given:

- a reasonable freedom of choice of subject;
- the possibility of cooperating with one another;
- the possibility of completing at home; that is, with the help of others.

The students were handed out short descriptions on the subject of the task.

An example of such a description:

Matrices and Coding

Secret Services often use codes to make messages incomprehensible to other people. Such a message must be difficult to decode. Matrices can be used fairly well to develop proper working codes.

For coding and decoding the computer is used.

The other subjects treated the following subjects:

- A puzzle-like problem using graphs and matrices.
- A shortest-path problem.
- A problem about the amount of leaves eaten by one sparrow-hawk during one year.
- A problem about composing Leslie-matrices.

All five tasks were developed by the team that was responsible for the experiment. The teacher had no role in the composition of the tasks.

All five subjects were chosen by one or more students; the Coding-task was the most popular.

Ten students worked alone, the other ten made up five groups of two.

The task itself consisted of some five pages of typewritten material with many open questions, more or less comparable to the forester's-problem of the two-stage task.

In this case it was also made clear to the students that they need not necessarily stick to the order in which the questions were asked. It was allowed to turn in an essay without indicating where to find which answer.

After some three weeks most students had turned in their work. Before we continue the report on the scoring and student reactions, we shall give some

4.2 The food problem of a sparrow-hawk

Simply stated, this problem concerns the amount of leaves needed to feed one sparrow-hawk during one year under the following conditions:

- a sparrow-hawk eats half a sparrow per day;
- a sparrow eats ten caterpillars per day;
- one caterpillar eats 0.5 g of leaves per day.

The following page is taken from the work of a girl who usually performed poorly in mathematics.
Her comments were:

> "To start this task was difficult. It took me quite a while to understand the problem. When I finally understood the problem correctly the actual work was not as hard and difficult as expected. It took me only two evenings to complete the work.
> I appreciated very much the fact that one was allowed to think about the problem for an extended period of time; ask for help from other people, or look for more information in other books.
> Usually I spend a lot of time on preparing a restricted-time written test; in my opinion a take-home task like this one does not take more time."

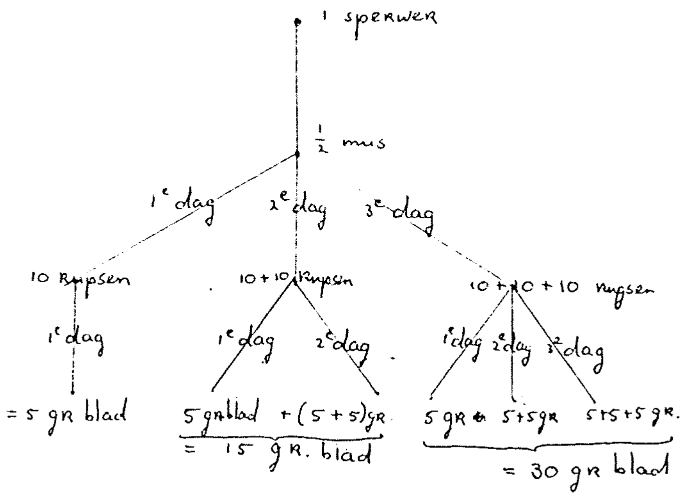
We choose the first page and the last page (page 1 and page 10) of her work, which was graded 7.5 by her teacher.
The first page shows the graph representing the problem for the first three days.

sperwer: sparrow-hawk	blad: leaves
mus: sparrow	dag: day
rups: caterpillar	

The last page shows the concluding result of the problem.
The matrix T^{365} shows in its lower-left entry the amount of leaves needed to feed a sparrow-hawk for 365 days.
It is also clear from this example that not only matrices and graphs play a major role in this problem but combinatorics as well.
The final result is quite a surprise to the students, which adds to the motivation. In this task the students were confronted for the first time with entry by entry matrix multiplication: students learned something completely new.

M4 Het voedselprobleem van de sperwer

Opg 1:



De sperwer eet een halve mus op de 3e dag. Deze mus heeft in de drie dagen 3 × 10 rupsen gegeten. De eerste dag at de mus 10 rupsen die 5 gr blad hadden genuttigd, de 2e dag at de mus weer 10 rupsen, die in twee dagen ook 2 × 5 gr blad hadden genuttigd, de derde dag eet de mus weer 10 rupsen die 3 dagen metelkaar 15 gr blad hadden genuttigd.

Na 3 dagen heeft de sperwer dus:

- 60 rupsen en
- 50 gr blad geconsumeerd.

7½

opg 17:

$$\underline{\underline{S}}^{365} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 365 & 1 & 0 & 0 \\ \binom{365+1}{2} & 365 & 1 & 0 \\ \binom{365+2}{3} & \binom{365+1}{2} & 365 & 1 \end{pmatrix}$$

$$\binom{365+1}{2} = \binom{366}{2} = \frac{366!}{2!\,364!} = \frac{365 \cdot 366}{2} = 66795$$

$$\binom{365+2}{3} = \binom{367}{3} = \frac{367!}{3!\,364!} = \frac{365 \cdot 366 \cdot 367}{2 \cdot 3} = 8171255$$

$$\underline{\underline{T}}^{365} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 365 \times 0{,}5 & 1 & 0 & 0 \\ 66795 \times 10 & 365 \times 20 & 1 & 0 \\ 8171255 \times 5 & 66795 \times 10 & 365 \times 0{,}5 & 1 \end{pmatrix} =$$

$$\begin{array}{c|cccc} & s & m & r & b \\ \hline s & 1 & 0 & 0 & 0 \\ m & 1825 & 1 & 0 & 0 \\ R & 667950 & 7300 & 1 & 0 \\ b & \underline{40856275} & 667950 & 1825 & 1 \end{array}$$

opg 18: Er is $\underline{40856275}$ gr blad na één jaar gegeten: tijd om één sperwer in leven te houden!!

4.3 Matrices and Coding

In this task the student is confronted with the problem of coding with the help of matrices, and of decoding with the inverted matrix. Actually, this was the first time that they were confronted with inverted matrices, and with the determinant of a matrix.

We have chosen as an example two girls' answer to question 7:

> 7. *Choose a word or a message, and a code matrix.*
> *Code your message. Exchange your matrix and coded message with your partner and decode each other's message.*
> *Use - if necessary - the computer.*

Als codeermatrix nemen we $C = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \Rightarrow D = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$

Natasja neemt als opdracht:

Garfunkel $\rightarrow \begin{pmatrix} G & A & R & F & U \\ X & L & E & K & N \end{pmatrix}$

In cijfers
$\begin{matrix} 6 & 0 & 17 & 5 & 20 \\ 23 & 11 & 4 & 10 & 13 \end{matrix}$

$\rightarrow$ coderen met C

$$\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 6 & 0 & 17 & 5 & 20 \\ 23 & 11 & 4 & 10 & 13 \end{pmatrix} =$$

$$\begin{pmatrix} 110 & 44 & 67 & 55 & 112 \\ 81 & 33 & 46 & 40 & 79 \end{pmatrix}$$

$\rightarrow$ decoderen met D:

$$\begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 110 & 44 & 67 & 55 & 112 \\ 81 & 33 & 46 & 40 & 79 \end{pmatrix}$$

Nicolette neemt als opdracht:

Beethoven $\rightarrow \begin{pmatrix} B & E & E & T & H \\ X & N & E & V & O \end{pmatrix}$

In cijfers
$\begin{matrix} 1 & 4 & 4 & 19 & 7 \\ 23 & 13 & 4 & 21 & 14 \end{matrix}$

$\rightarrow$ coderen met C:

$$\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 4 & 4 & 19 & 7 \\ 23 & 13 & 4 & 21 & 14 \end{pmatrix} =$$

$$\begin{pmatrix} 95 & 64 & 28 & 141 & 77 \\ 71 & 47 & 20 & 101 & 46 \end{pmatrix}$$

$\rightarrow$ decoderen met D:

$$\begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 95 & 64 & 28 & 141 & 77 \\ 71 & 47 & 20 & 101 & 46 \end{pmatrix}$$

$\begin{matrix} 6 & 0 & 17 & 5 & 20 \\ 23 & 11 & 4 & 10 & 13 \end{matrix} \rightarrow \begin{matrix} GARFU \\ XLEKN \end{matrix}$

$\begin{matrix} 1 & 4 & 4 & 19 & 7 \\ 23 & 13 & 4 & 21 & 14 \end{matrix} \rightarrow \begin{matrix} BEETH \\ XNEVO \end{matrix}$

The score given by the teacher for this work was 6.5.
The reactions of the two girls who did this task differed considerably.
One was more critical than the other:

> "I prefer an ordinary restricted-time written test because you feel more certain. You know exactly what you have to learn and what will be asked on the test. It takes a lot of time, considerably more than preparing ordinary tests. Besides that, I doubt if I learned more from doing a take-home task.
> One of the main problems - not only with this task but in general with Math A - is the fact that you are not always sure what the meaning of a question is.
> Nevertheless, I think one should carry out such tasks once a year. Other goals than just mastering the contents of the booklet are important, one has to be creative to get a high mark."

Her partner - both girls agreed that they had cooperated well in preparing the task - had rather different feelings.

> "I enjoyed doing this task. It was much more fun than preparing a restricted-time written test. You can work much better if there is no time pressure. You get ample time to prepare answers in your own way. It is also an advantage that you have to consult literature, in this way you learn new facts in a better way than just learning by heart from the book."

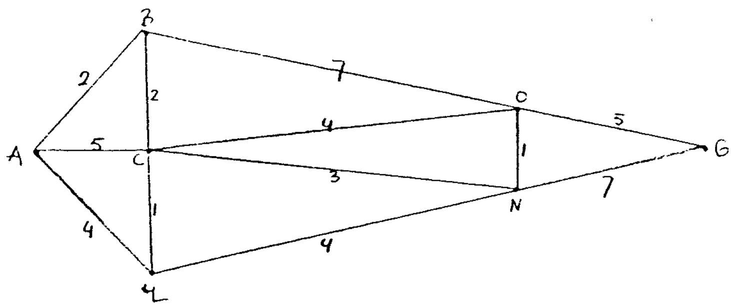
4.4 The shortest path problem

The starting point of this task is a problem found in the student text on *Matrices*.
It is the problem of how to find the cheapest routes from Holland to the United States deserts.
Examples studied were flights starting in Amsterdam or London, with final destination Las Vegas or Tucson.
One intermediate stop has to be made, either in Dallas, Los Angeles or New York.

This problem is solved with the shortest-path algorithm which is explained in the task description.

For this purpose, a simple example is included: find the shortest route in a wildlife preserve. The trip starts at the Apes (A) and ends with a visit to the Giraffes (G).
The girl whose work will be partly shown chose to write an essay, while she

indicated which questions were answered at a particular place.
She was known as a poor mathematics student, both in math in general and in Math A.



1* De éénstapsmatrix hiervan ziet er als volgt uit:

$$m = \begin{array}{cc} & \text{naar} \\ & \begin{array}{ccccccc} A & B & C & L & O & N & G \end{array} \\ \text{van} \begin{array}{c} A \\ B \\ C \\ L \\ O \\ N \\ G \end{array} & \begin{pmatrix} 0 & 2 & 5 & 4 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 7 & 0 & 0 \\ 5 & 2 & 0 & 1 & 4 & 3 & 0 \\ 4 & 0 & 1 & 0 & 0 & 4 & 0 \\ 0 & 7 & 4 & 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 4 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 5 & 7 & 0 \end{pmatrix} \end{array}$$

0 betekent: niet bereikbaar in één stap

2* Ik ga nu een nieuwe matrix opstellen m_1 met behulp van m. A doet niet meer mee de kolom van A vervang ik door nullen. De rij van A blijft onveranderd. De rest van de matrix blijft open.

$$m_1 = \begin{pmatrix} 0 & 2 & 5 & 4 & 0 & 0 & 0 \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \end{pmatrix}$$

3* het kleinste getal in de rij omcirkel ik, in dit geval is dat 2. d.w.z. punt B ligt het dichtst bij A.

4* Nu kijk ik in welke kolom dit getal staat. In dit getal is dat de tweede Nu vervang ik deze kolom door allemaal nullen en plaats ik het omcirkelde getal 2 vòòr de matrix op de juiste plaats. (De 2 blijft in de kolom)

In m_5 ga ik nu de andere 4 omcirkelen en de kolom door nullen vervangen:

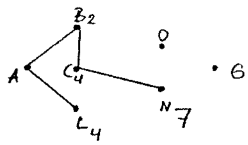
$$
\begin{array}{r l}
 & \\
2 & \\
4 & \\
m_6 = 4 & \\
 & \\
 & \\
 &
\end{array}
\begin{pmatrix}
0 & 2 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 9 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 8 & 0 \\
0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & & &
\end{pmatrix}
$$

Nu is 7 het laagste getal deze zit in de 6e kolom dus de 6e kolom wordt door nullen vervangen en in de 6e rij wordt er 7 bijgeteld, en vóór de matrix bijgeplaatst.

$$
m_7 = \begin{array}{r}
 \\
2 \\
4 \\
4 \\
 \\
7 \\

\end{array}
\begin{pmatrix}
0 & 2 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 9 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & & 0 & \\
0 & 0 & 0 & 0 & 8 & 0 & 14 \\
0 & 0 & 0 & 0 & & 0 &
\end{pmatrix}
$$

de graaf is ondertussen:



Het laagste getal is nu 8. Er zijn twee 8en in de vijfde kolom, d.w.z. dat het punt O, de olifanten, van twee kanten te bereiken is. n.l. vanaf C en vanaf N

$$
m_8 = \begin{array}{r}
 \\
2 \\
4 \\
4 \\
8 \\
7 \\

\end{array}
\begin{pmatrix}
0 & 2 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 9 & 0 \\
0 & 0 & 0 & 0 & 8 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 13 \\
0 & 0 & 0 & 0 & 0 & 0 & 14 \\
0 & 0 & 0 & 0 & 0 & 0 &
\end{pmatrix}
$$

→ 2 punten in dezelfde rij wil zeggen dat er vanaf het punt C twee wegen zijn, n.l. van C naar O van 8 km en van C naar N van 7 km

$$
m_9 = \begin{array}{r}
 \\
2 \\
4 \\
4 \\
8 \\
7 \\
13
\end{array}
\begin{pmatrix}
0 & 2 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 13 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

dus vanaf het punt met 8 is het 13 km om in het laatste punt te komen

De graaf staat op de volgende bladzijde:

The two pages from the students' work (out of a total of twelve pages) show the graph of the wildlife park and the initial development of the shortest-path-algorithm (matrix m_1).
The second page shows the further stages m_6 to m_9.
The score given by the teacher was 8 which later on, after consulting the experimenter, was upgraded to 8.5.
The reaction of the student:

> "During a whole week I tried to solve the problem by working on it for half an hour each day. After six unsuccessful efforts I suddenly understood the problem and it turned out to be not at all difficult.
> I did it all by myself and it was a lot of work, but more enjoyable than preparing a restricted-time written test. You're able to spread your work over a longer period of time, and you can write and rewrite it in order to produce the best solution and the best text.
> To me it seems difficult to score this work. With an ordinary test you just look at what's right and what's wrong. In the case of a take-home test most of what you turn in is more or less correct. Morever, I think that you should also take the lay-out of the task into account."

4.5 Scoring

The scoring of the take-home tasks was first carried out by the teacher without any support, and reviewed in a second round by the experimenter.
In eighteen (out of twenty) cases the grades differed less than 0.5 points and it was agreed to consider the teachers' score as definitive.
In two cases the scores were upgraded. 8 was upgraded to 8.5 and one 8.5 was upgraded to a 9.
The students had no comments on their grades which ranged from 5 to 9 with most scores around 6.5 and 7. The teacher found it remarkable that the three highest scores were attained by girls that usually did not score higher than 5.
This caused sceptical remarks from other students and some eyebrow raising by the teachers. It was decided that the experimenter should submit the three girls to a short interrogation to find out whether they really had understood their work. The result was most satisfying.

4.6 Reactions

After the test a discussion took place with the teacher and students.

From this discussion and the written reactions of the students we draw up the following table (fig. V.21):

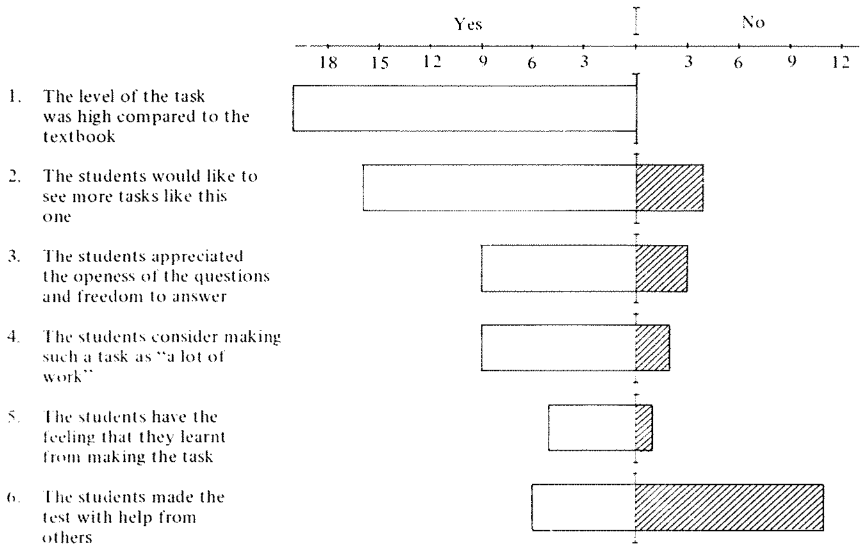


fig. V.21

The remark must be made that questions 3 to 6 were not explicitly asked, but implicitly answered by reactions.
The teacher's reaction was positive, with one critical note: he considered it almost impossible for a teacher to create a set of such tasks himself.

4.7 The take-home task and the five principles

1. The tests should improve learning.

It will be clear from the three examples given that new concepts and ideas were introduced in these tasks, and that the level was high. Most students showed positive attitudes toward these tasks. Some considered it unfair "because in a test you should ask questions similar to the textbook's." In other words: this girl expected only the lower goals to be operationalized, as is usually the case with restricted-time written tests.

2. The test should be such that the candidates show what they know (positive testing).

Positive testing was not as optimal as it could have been because of the

nature of the questions, which were almost all of the open question type and not of the essay type. Some students however, created more openness by writing an essay rather than answering all questions.
Most students experienced "positive testing" feelings, certainly when comparing the take-home task with the restricted-time written test.

3. The test should operationalize the goals of the Mathematics A curriculum.

The bottleneck in testing Mathematics A is how to operationalize the 'higher' goals. It is clear that some of these goals were reasonably operationalized by this task. Creativity plays an important role, but flexibility, reflection and mathematizing are also important factors. Some students who were good at restricted-time written tests had problems in making this test. On the other hand, the three best pieces of work were made by students who were poor in mathematics or, properly said, poor in making restricted-time written tests.

4. The quality of the task is not measured in the first place by the ability to score it in an objective way.

The scoring of these tasks posed no problems either to teachers or students although many students had the feeling that scoring these tasks was very difficult and maybe more 'unfair' than scoring ordinary tests.
The second scoring by the experimenter produced no significant differences. Looking back at our experiences with the scoring of the two-stage test this can hardly come as a surprise.

5. The test should fit into the usual schoolwork.

Although most students spent more time on the take-home test than on an ordinary written test, they did not have the impression that it disrupted their usual school practice as long as such a task was not given more than twice each quarter. Producing and scoring such a task is mainly problematic for the teacher, probably due to lack of experience. However, even unexperienced teachers did a reasonable good job in scoring. In fact, later on we will show how well teachers can agree as long as they are cooperating closely at one school.

5.1 The essay-task

The second form of a task to be discussed briefly is the open essay task. Descriptive statistics or, more precisely, graphic representation is a complex subject to be tested, especially if we look at the subject from the following point of view:

How to look critically at statistical material and representations;
how to use graphic representations in a correct way.

This means that a variety of intellectual skills and abilities are to be tested. This can be done by extended response essays which, however, suffer from certain flaws.

First, the limitations imposed by the relatively small number of questions that can be asked, which limits the number of achievements tested. A student who has concentrated his efforts on those particular areas will get a high score, while others who have devoted more attention to other areas will get a lower score.
In our opinion it would seem possible to counter this problem in two ways:
- the question(s) should cover a lot of areas;
- the essay may be written at home so that the 'ill-prepared' student can catch up.

A second shortcoming of extended response essays is that the writing and drawing abilities of the students tend to influence the scores.
Research has shown that even when teachers are told to disregard errors in punctuation, spelling, and grammar, they assign lower grades to papers containing such errors. [13]

Our own findings with the scoring of the second stage of the two-stage test show that if those factors play at all a role in essays in mathematics, the resulting differences in grades are not alarming.
Inconsistency of scoring essay-type tests has often been mentioned as a major problem. Our findings in Math A tend to show that inconsistencies are kept within reasonable boundaries. Nevertheless, we also explored the possibilities of an open essay task.

5.2 Migration

The example concerns an article from a newspaper on the problem of overpopulation in the Republic of Indonesia. As can be seen on the next page, this article contained a lot of numerical information. However, no use was made of graphic representation.
The question asked to the students was:

> "Rewrite the article on migration in Indonesia.
> Make optimal use of graphic representation."

INDONESIA INVOLVED IN MASS RESETTLEMENT

Many large countries have to contend with an uneven distribution of population on their territory, but nowhere is the problem so extreme as in Indonesia. On the island of Java - whose 132,000 square kilometers make up less than seven percent of Indonesia's ground surface of nearly two million square kilometers - live 92 million people, that is, 62 percent of the total population of 147 million.

This accounts for 'trans-migration' - the re-populating of large numbers of Javanese to other parts of the archipelago - having become a central if not dominant theme in the national development.
Transmigration entered the Indonesian language from Dutch. The Dutch colonialists began transmigratory activities in 1905. According to the Indonesian minister of labour and transmigration, Harud Al-Rasjid Zain, 30,569 families - 227,884 people altogether - were transfered from Java to Sumatra between 1905 and 1949. From 1950 to 1979 this figure rose to 204,425 families containing an average of five persons. The total number of transmigrants from Java has therefore reached only 1.2 million people in 75 years.

Question:
Is therefore the struggle against the overpopulation of Java not destined to fail or already lost?

Zain:
"Since the third five-year-plan (1974-'84) we have not only been active on a larger scale but have applied an integrated approach. We no longer merely transfer the people to another island, but link this transference systematically to regional developments in the outlying district concerned. Our target figure for transmigration during the third five-year-plan period is half a million families. During the fourth plan we

want to increase this figure to three-quarters or one million. Much depends on the experience now gained in field work."

Mistakes

Minister Zain says that initially a number of mistakes were made during the preparatory work. "Out of consideration for speed, we brought in the heaviest machines we had for clearing the jungle, until soil research showed that ripping out the trees damaged the topsoil to such an extent that it would delay construction considerably. Now we cut the trees - which are often 80 meters high and 2 meters in diameter - with a circle-saw just above the roots. The terrain is then prepared further by hand. This often takes a lot of time. The tempo of the settlements is entirely determined by the amount of cleared land available and by the infrastructure. Ten, twenty years ago we settled people, for instance, deep in the heart of Borneo. They produced a good crop but it was unsellable. We now have an integrated approach - the access must first be assured."

It is not yet clear which re-division of the population Indonesia eventually wishes to reach. A large map hangs next to Minister Zain's desk on which a population of 237 million is indicated for the year 2001, 139 million of which are on Java. Demographers, however, have set the optimum population of Java at 70 million.

Does this mean that tens of millions of Javanese will be resettled? "Yes, that is true", says Minister Zain. "Java's population is still growing at the rate of two per cent per year (...). We cannot jump right into the future to achieve the ideal size of population, but we can alleviate the problem."

He goes on to say that relieving Java is but the third goal in transmigration policy. The first two are increasing food production and developing the outlying districts.

Exodus

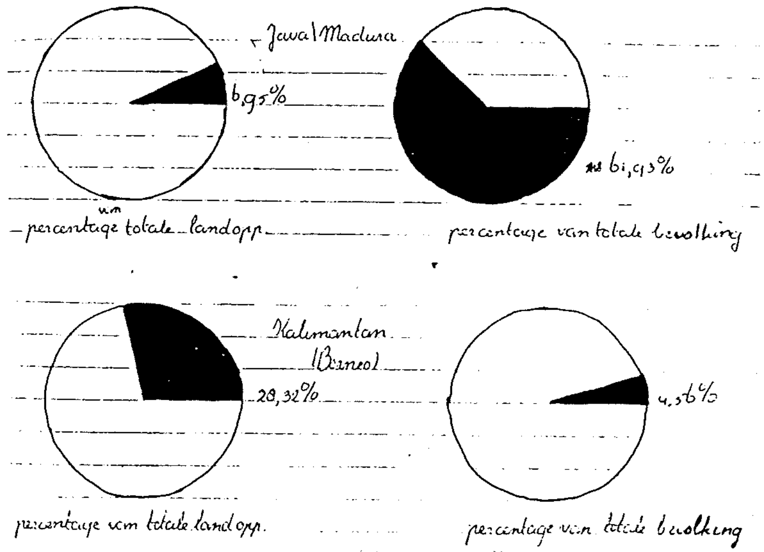
Aside from the systematic exodus from Java, there is still a stream of people going to Java, particularly to Jakarta. "But the scale now tips in favour of Java", says the minister. "Improvement of educational facilities on the other islands has slowed the trek to Java."

Region	Surface Area	Percentage of total area	Population in 1980 (millions)	Percentage of total population
Java/Madura	132.187 km^2	6.95	91.281	61.93
Sumatra	473.606	24.86	27.981	18.99
Kalimantan (Borneo)	539.460	28.32	6.721	4.56
Sulawesi (Celebes)	189.216	9.93	10.377	7.04
Bali	5.561	0.30	2.470	1.68
Irian Jaya	421.981	22.16	1.145	0.78
Other islands	142.558	7.48	7.406	5.02
TOTAL	1.904.569	100.	147.384	100.

The assignment was carried out in groups of two. The groups were formed by the students themselves. The results were satisfactory, which means that almost all modes of graphic representation were used, and even new ones were introduced.

Simple graphic representation turned out to be the most frequent. For instance, all students used circle-diagrams. But there were interesting differences in the way they were used.

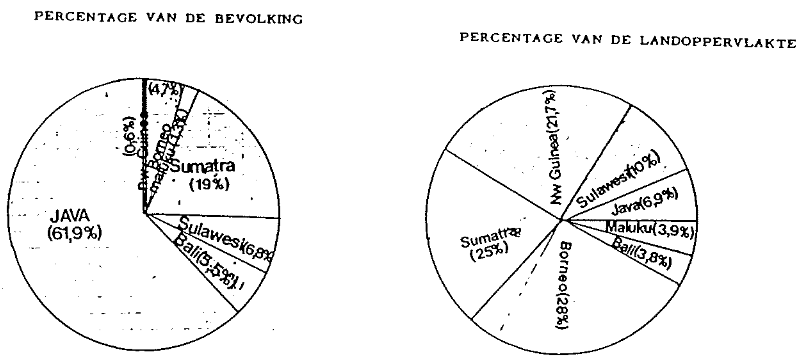
Some students drew *two* diagrams per island to show the uneven distribution of the population: *one* for the percentage of area, the *other* representing the percentage of population in relation to the whole of Indonesia.



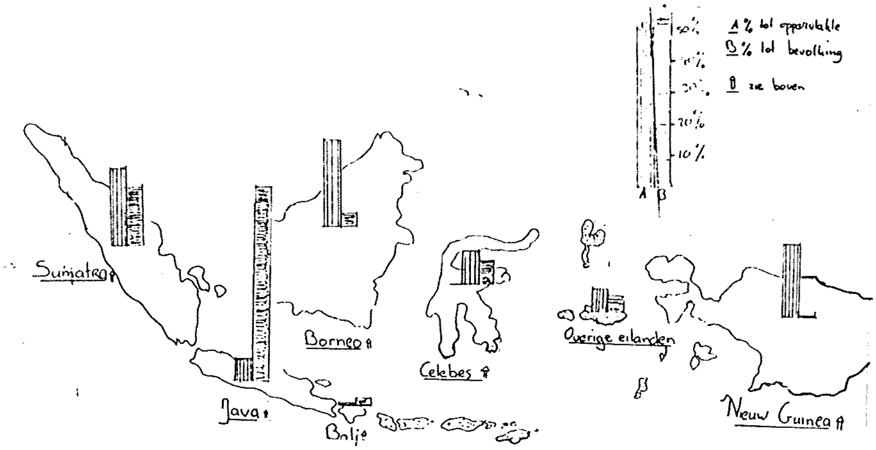
Others preferred to give only two diagrams for the whole of Indonesia.

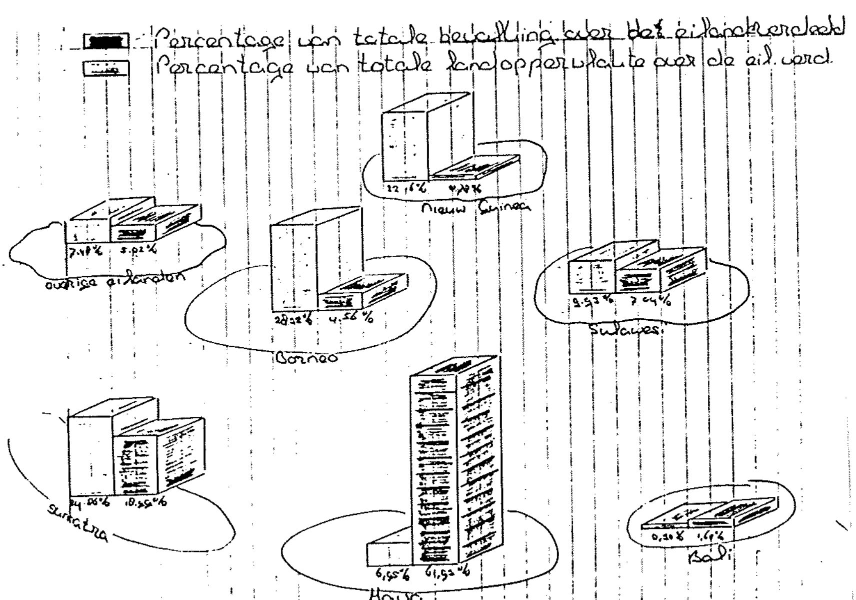
The first represents the distribution of the population over the respective islands, the other represents the size of the islands.

A classroom discussion made clear that both choices have advantages:



Two teams came up with very nice graphic representations to show the uneven distribution of the population over the islands:

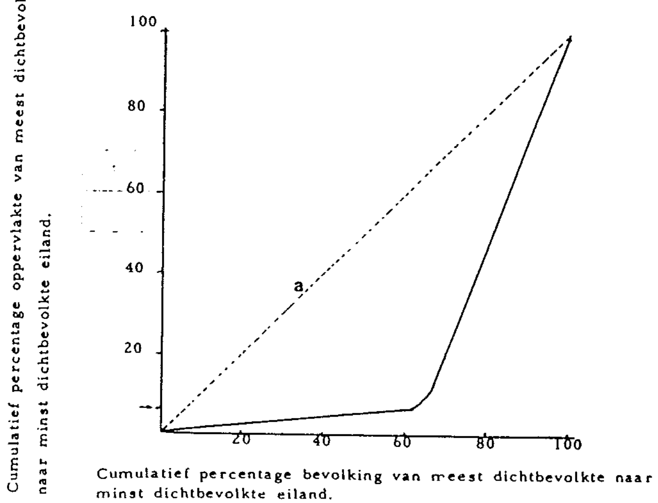




Lorentz curves were used to show the uneven distribution of the population.

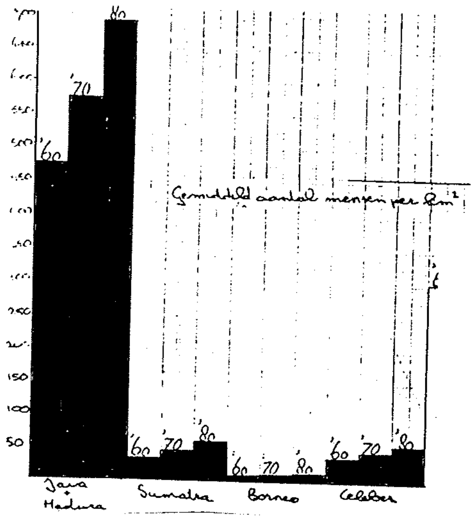
We zien hier dat het met de verdeling van de indonesische bevolking over de eilanden droevig gesteld is. Op Java (6,95 % van de totale land oppervlakte) woont alleen al 61,9% van de totale bevolking.

Figuur 1.

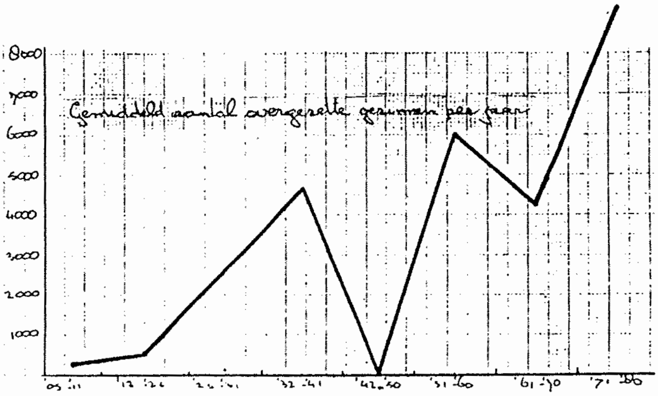


Cumulatief percentage bevolking van meest dichtbevolkte naar minst dichtbevolkte eiland.

Bar-diagrams were used also to illustrate the time-related problem.
The following graph represents the number of people living on one square kilometer on some of the islands:



The polygon was used to show the number of people that have migrated per year.



Not only did the students try to illuminate their essay by means of graphic representations, they also checked some of the assertions made in the article. Here we find a nice example of mathematization: recognizing the essential mathematics and finding out the relation between those aspects.
The essentials from the article:

"Java will have 139 millions inhabitants in the year 2001."

"The population of Java grows by 2% per year."

"Some 92 millions live on Java."

The students checked whether these statements were consistent, even though they were not familiar with exponential growth. The pocket-calculator proved to be very useful in this case.

5.3 Scoring

The scoring was done in the same way as that of the take home task: the teacher scored the tasks and the experimenter reviewed them thereafter.
Since the differences were less than 0.5 point, the original scores of the teacher became definitive.
The scores varied from 6 to 9.
The students had no comments on their grades.

5.4 Reactions

All students - and the teacher as well - reacted favourably to the task. They liked the subject of the task - "a very good topic" - and understood what they actually had learned.
Especially with such a subject as *'Graphic Representation'*, which has a rather weak mathematical content, it is not always clear to the students what they were expected to learn. In this respect the task was of great help.

5.5 The essay task and the five principles

1. The test should improve learning.

From the students' reactions it is clear that working on the Indonesia article was a highlight for most students, but moreover that they had learned a lot from it. They understood - in a quite creative way - the power of graphic representations and the need to reflect on the proper use of this tool.
Comparing the different possibilities and choices afterwards in a classroom discussion contributed even more to the learning process.

2. The tests should be such that the candidates show what they know. (Positive testing).

Easy extended-response tasks are very well suited to positive testing. This task was definitely more open than the two-stage test and take-home task where we were mostly dealing with open questions or, at best, restricted-response essay questions.
For the subject *'Graphic Representation'* the essay-form proved to be very successful. Not only were students given ample opportunity to show what they know, but they really put in a great deal of effort to show that they were able to use all kinds of graphic representation in a proper way.

3. The test should operationalize the goals of Mathematics A.

Mathematizing plays a major role in this task. Not only do the students need to find relevant mathematics in the text, they also have to find relations between different facts, and reflect on different aspects in order to decide how to rewrite the article. Decisions have to be taken on which facts to represent graphically and on what kind of graphic representation fits the purpose best.
On the other hand, it is clear that many 'lower' goals are not operationalized in this task.

4. The quality of the task is not in the first place measured by the ability to score it in an objective way.

It is a well-known fact that open essays are very difficult to score. However, the teacher found in this case that the positive effects of the task outweighed the difficulties of scoring the task. And, in his opinion and those of the students, it was more than worthwhile. Not only did they enjoy doing the task but it turned out that scoring the task was not so difficult as anticipated and the scoring of the teacher and the experimenter showed no relevant differences. (Twelve essays were scored: the two grades never differed more than 0.5 point.)

5. The test should fit into the usual schoolwork.

Although the students had widespread fears beforehand that doing this task would disrupt their ordinary schoolwork, they agreed afterwards that their usual work was hardly affected. They spent less time than expected on their essay so in this case we may conclude that such a test may fit into their usual schoolwork. The teacher, however, spent considerably more time on scoring than usual.

6 THE ORAL TASK

6.1 The oral task

The previously described examples: two-stage test, take-home test and essay do not have - certainly in mathematics - a long tradition in the Netherlands.
Our next example, however, does have a very long tradition. Until 1974 an oral examination took place as part of the examination - although in most cases it was possible to avoid this oral part if the results of the written examination were 7 or higher. It is unclear what the exact reasons were for having the oral part disappear from the mathematics exam. Economic and bureaucratic reasons may have played a role as the oral part of the examination had to be done in the presence of a government-appointed independent 'referee', most of the time a university professor in mathematics who had to be paid for services rendered.

In our opinion, however, the new Math A curriculum seems suitable for oral examination. The real meaning of a concept, the initial stages of mathematization and a discussion on the model are a few of the subjects that can be discussed more easily in an oral than a written form.
Oral tests do exist at this moment at different Hewet schools in different forms.

We mention three:
- An oral discussion on certain mathematical subjects that are known to the students in advance.
- An oral discussion on an article that was given to the students in order to be studied during twenty minutes prior to the discussion.
- An oral discussion on a take-home test (or similar alternative task) after the task had been completed by the student (and scored by the teacher).

We will here look at the first type, with the remark that the students had taken a written test on the same mathematical subjects a week before the oral test. In this way we could compare the results.

The organisational form of the test was the following:
- All students had twenty minutes.
- Present at the examination were:
  - the student
  - the teacher
  - an external independent examiner
  - an observer
- The teacher had some already prepared exercises to start with.
- The teacher starts the discussion.
- The second examiner joins the discussion whenever he likes.
- The scoring should be done - in this special case - by the teacher, examiner and observer; but the grades of teacher and examinator are decisive.
- Mathematical subjects, as presented in the booklets:
  - Matrices
  - Differentiation II
  - Linear Programming

In most cases - there were twenty students - roughly two problems were discussed. One aspect was very clear: the whole discussion was totally process-oriented. The solution, or product, was not important, and in many cases the process was stopped short of the solution by the examiners.
An advantage over all forms of written tests - including alternative tasks - is the fact that one is able to find out how much relevant information a student really needs to start solving the problem. In written tests one always presents the students with the same written information.
In the oral discussion the situation is quite different as will be shown, in particular, in the case of the question about the allies in the Second World War.

6.2 An example

The examiner starts by telling about the Germans and Italians later joined by the Japanese on one side and the allied forces on the other side. The question was posed as to how mathematics could serve to clear up this situation. All students' reactions and responses pointed either to matrices or to graphs. But some students proposed this at a very early stage, while others needed some more information and time, and were actually surprised by the fact that their proposal to use matrices or graphs seemed, indeed, a fruitful one.

Another interesting fact is that the level of the first questions differed substantially. Relatively low-achievers were approached at a very basic level. When they became confident at this level the examiners proceeded to a higher level; sometimes with surprising results.
Very well-performing students, however, were approached with a number of difficult questions that seemed appropriate to the expected performance level of the student. Also, in this situation, with some unexpected results.

Protocol 1 - student: female, poor in math A.

T(eacher):	What do you think is the most difficult subject: Matrices, Differentiation or Linear Programming?
S(tudent):	Differentiation. I would like to start with something different.
T :	I see, you prefer to start with something different.
S :	Because I'm not sure if I understand the subject.
T :	(Ignoring her objections). What actually is so useful in differentiation? Is it only a trick where you lower an exponent with one, and put the original one in front? What are you doing?
S :	Investigating functions.
T :	Investigating functions? That's right.
S :	For instance, profit functions. From the graph, by taking a point, you can see how large the profit is, and if the function rises how fast the profit rises ...
T :	And how, precisely, do you investigate that function?
S :	Well, you can look for the slope in a certain point and if the slope in that point has the value 6 then the functions rises steep by a factor 6. .... Later, after having discovered the formula: $l = 2L + \frac{1}{L} . 100$:

	the teacher wants the minimum of *l*.
E(xaminator):	Let's first look at the general idea. If you have any formula and they ask you for the minimum .....
S :	Than you look for the derivative.
E :	Aha.
S :	You put the derivative equal to zero.
E :	Aha. Would you mind doing that in this case?
S :	That's quite difficult. (Points at $\frac{1}{L}$. 100 in the formula)
E :	That's right. How about the first part of the formula?
S :	The derivative is just 2.
E :	Yes, that's right.
S :	So, that makes (writes down: $2L + L^{-1} . 100$ followed by: $2 - L^{-2} . 100$).
E :	How about a maximum or minimum?
S :	Equal to zero. (Writes down: $2+\frac{-1}{L^2}.100=0$)
E :	(Interrupting her efforts to find the proper value for L that she had already found intuitively). Yes, from now on it's only arithmetic. Let's continue with another subject.

(Teacher's reaction:

"Very surprised, I thought she was really a low-achiever in mathematics and that she especially had problems with differentiation.")

Protocol 2 - student: male, excellent in mathematics A

S :	Compared to last year's written tests I found the last one very easy. (He refers to last week's test).
T :	One of the subjects was matrices.
S :	Yes.

Then the discussion starts on what a Leslie-matrix actually represents.
This is followed by:

T :	Let us look at this Leslie-matrix L:

$$\begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$$

You know how to multiply matrices, right?

S : Yes.

T : If you compute L^2,
what is the meaning of this matrix?
Is that a Lesliematrix as well?

S : Let's have a look ....

T : Maybe you should compute ....

S : (Starts computing L^2):

$$L^2 : \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ \frac{1}{8} & 0 & 0 \end{pmatrix}$$

E : Yes, that's the product of the matrix with itself.
What's the meaning of that matrix?

S : That's quite a question.
Let's have a look ...
.... (silence)
mumbles: one generation multiplied by its own generation ...
.... (silence).

E : (Having noticed that the student is stuck).
Let us go back one step. To L.
How do you use L? We have seen the meaning of each of the entries of the matrix. But how do you use the matrix?

S : Well, usually you get a starting population and then
you can compute ....

E : Okay, let's take a starting population that we write down as

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

How do you get the population after one year?

(We are now back at a very basic level in order to see if that will make the task easier for the student; because of the obvious difficulties he has in interpreting L^2).

S : Writes down:

$$\begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

E : Okay, that's easy to compute, isn't it?
What will the result be? A number, a matrix, or ...?

S : Let's have a look. A three-by-one matrix will be the result.

E : Yes, a similar column will be the result.
Let's write down the result as

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

So we have:

$$\begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

S : Yes.

E : So, if we have a starting population of a, b, c ....

S : Then this (points to p, q, r) is the population next year.

E : Yes, yes, yes.

S : In percentages .... (?)

E : (Tries again)
If you put another Leslie-matrix in front of the one we have.
Then we need to put one in front of p, q, r as well.

S : Yes (while writing down:)

$$L \cdot L \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = L \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

E : Okay, let's compute the left side.

S : (carries out the multiplication).

E : What kind of a result do we get?

S :	Next year's population.
E :	Aha.
S :	So this is time zero, this is time one and this is time two.
E :	Let's return to my initial question.
S :	(Still looking at the result of the multiplication) So this should be time t is two.
E :	What should be the meaning of the square of a Leslie-matrix?
S :	Well, multiplication by itself is always a square.
	....

(Here we stop the discussion that went on without a positive result for the student. The answer was actually given by E while S was trying to understand what was going on).

6.3 Scoring

Scoring was done in the following way: the teacher and examiner each gave a grade. In most of the cases the difference was less than one point of the ten point scale. To double-check the observer also gave a grade that in all cases fitted into the interval bounded by the previous two grades. Intersubjective scoring proved to be satisfactory this way; this was confirmed by the results of two other schools from the twelve schools.

If we compare the results of the *oral* and the *written* part of the task we get roughly the following picture:

Oral	higher	(> 0.5 points):	10	students
	same		5	students
	lower	(> 0.5 points):	4	students

As we noticed before with other forms of alternative tasks, the spread in marks is smaller with alternative tasks than with restricted-time written tests. In restricted-time written tests grades as low as 2, 3 and 4 are to be found regularly. These grades are hardly ever given in alternative tasks. Even a 5 is seldom given. Some teachers have trouble with this fact. As one teacher remarked: "I was unable to find any of the works unsatisfactory (i.e. a score of 5 or lower). I wonder if I should consider this as a major drawback for this kind of task, or if I should be happy that math education seems so fruitful. But it will definitely have its effects on the status of the discipline."
In the next section we will return to the differences in grading restricted-time written tests, take-home tasks and oral tests.

6.4 Reactions

We would like to start with the reactions of the two students whose exam protocols we just presented.

First the girl:

> "I am in favor of an oral task. When I get stuck with some detail in a restricted-time written test I usually get very nervous. This makes it impossible to solve the rest of the test.
> With an oral test a hint can be given in order to complete the rest of the problem successfully.
> In this way I can try to prove that I really *do* understand the subject matter."

The reaction of the boy that was not so successful at the oral discussion:

> "I am personally not really in favor of oral examinations.
> In my opinion, time was too restricted. There was not enough time to think things over because keeping your mouth shut may give the impression that you are stretching the time.
> Another drawback is the fact that you cannot do much computation.
> Finally, I don't like people watching and observing me."

The general impression of the group was positive. Positive aspects mentioned explicitly, are:

- more questions on insight and theory and on mathematizing; less on arithmetic and computation;
- good atmosphere makes you feel relaxed;
- because of hints you will 'never' get stuck;
- no stress on details, but more general questions.

Negative points mentioned are:

- pressure due to lack of time;
- pressure due to presence of 'officials';
- no computation.

All students agreed that there should be a place for oral tasks during the learning process. They agreed on the fact that the difference in results proves that it is 'fair' to include other means of assessment testing than the written tests.

6.5 The five principles

1. The test should improve learning.

In the case of the oral task this is certainly not the strongest point from the student's point of view. But for the teacher the information can be of great value. The oral examination gives the teacher an opportunity to find out students' weak points and the reasons for them. The feedback on the learning of his students and his own teaching is at times impressive. The test may actually improve teaching.
For the students, a positive point is the possibility of learning more about their capabilities in the field of mathematizing, theorizing, reflection and interpretation. But time pressure and the fixed time are drawbacks that are all too familiar to the students from ordinary test practice.

2. The test should be such that the candidates show what they know (positive testing).

Compared to take-home tests and essays the oral tasks score lower on this point. There is no way to avoid certain questions, no time to think about a problem somewhat longer. But, on the other hand, the student can be helped when he gets stuck.
In this way he can show what he knows after all as indicated by the girl's reaction: "I could show that I really understand the subject matter."

3. The test should operationalize the goals of Mathematics A.

It is clear that one will usually try to operationalize the higher goals of Math A. Arithmetic and computational skills are very well suited for restricted-time written tests; (some of them maybe even by multiple choice tests). But mathematization - both horizontal and vertical - can be tested by an oral test, although time restrictions limit the possibilities. Also the interpretation of and reflection on results are goals that seem suited for operationalization by an oral task.

4. The quality of the task is not in the first place measured by the ability to score it in an objective way.

A weak point is the scoring of oral tasks. This was at least the assumption when we started out, based on the facts and stories we had heard about pre-1974 oral examinations. Bitter struggles between teacher and official external examiners took place at that time, although in only a few cases.
In our observations - however limited (a total of 60 students) - we have not seen any significant discrepancies. This may be due to the fact that the exa-

miners agreed on the goals of Mathematics A.
It seems more than likely that teachers with fundamental objections to Mathematics A would have disagreed strongly and that such teachers should neither teach Mathematics A nor examine it.

5. The test should fit into the usual schoolwork.

It is not always easy to include oral tasks in the usual schoolwork because of organisational problems. For other disciplines, however, there seem to be ample opportunities for oral tasks - especially for the languages. The internal school examination during the final year seems particularly suited for this purpose.

7 COMPARING RESULTS FROM DIFFERENT TASKS

At one of the twelve schools an internal examination took place half way through the year. The internal examination consisted of three parts:
- a restricted-time written test
- a take-home task
- an oral test.

A total of 28 students and two teachers took part in this internal examination. Afterwards, the teachers agreed that this combination certainly does disrupt ordinary classroom work. Both teachers had to work very hard to score the three parts of the examination. Nevertheless, they were very enthusiastic about the potential of such a combination of tasks. But a three-stage task was just one stage too many, it seemed.
The unique combination makes it worthwhile to compare:
- the results of the different stages
- the results of the two teachers
- the results of this task with those of the other tasks discussed previously.

How did the teachers reach their scores?

Restricted-time written test
These were graded by the student's own teacher only. There were only rough guidelines, such as:

Exercise 1:

question a)	3 points		
question b)	2 points		
question c)	5 points	Total	10 points.

Take-home task
The sample of 28 tasks was divided into two equal parts, which gave each teacher fourteen tasks to score: some of them were of his own students, some of students of his colleague.
There were no criteria for scoring.

Next, the teachers exchanged their fourteen tasks and graded them again, which resulted in two scores for each student. The two marks were averaged, resulting in scores such as: 6.0; 6.5; 7.0; 7.5 etc.

The final scoring stage consisted of a comparison of tasks with the same marks.
The question posed:
"Is this task with mark 7.5 really as good as another one with the same work?"
So the teachers tried - together - to order the tasks with the same mark. This led to results like 7.3; 7.5; 7.6 instead of 7.5; 7.5; 7.5

Differences between the two initial scores were well within the limits as described in our chapter on the scoring of the two-stage test.

Oral task
The oral tasks were carried out by both teachers. Using the protocol a first indication of the score was given directly after *each* examination.
Next, each teacher independently scored all students after the total examination (proposal). Together they arrived at the final score (final mark).

Table of results of oral examination

Student	Initial Teacher		Proposal Teacher		Final	Student	Initial Teacher		Proposal Teacher		Final
1	3	3.5	3.5	3.5	3.5	14	7.5	7.5	7.5	7.5	7.5
2	4.5	4.5	4.5	4.7	4.7	15	7.5	8.0	7.8	7.9	7.8
3	4.5	5.0	4.6	4.7	4.7	16	8.0	8.5	8.0	8.1	8.0
4	5.0	5.5	5.1	5.2	5.2	17	8.0	8.5	8.0	8.1	8.0
5	5.0	5.5	5.1	5.2	5.2	18	8.0	8.5	8.0	8.1	8.1
6	5.0	5.5	5.4	5.5	5.5	19	8.0	8.5	8.2	8.2	8.2
7	5.5	6.0	5.6	5.6	5.6	20	8.0	8.5	8.3	8.4	8.4
8	5.5	6.0	6.0	6.0	6.0	21	8.0	8.5	8.4	8.4	8.4
9	6.0	6.5	6.4	6.5	6.5	22	8.5	8.5	8.5	8.5	8.5
10	6.5	6.5	6.5	6.5	6.5	23	8.5	8.5	8.5	8.6	8.5
11	6.5	7.0	6.8	6.9	6.9	24	8.5	8.5	8.6	8.7	8.6
12	7.0	7.0	7.1	7.0	7.0	25	8.5	9.0	8.8	9.0	9.0
13	7.0	7.5	7.3	7.3	7.3	26	9.5	9.5	9.5	9.5	9.5

It is clear that also in this case intersubjectivity is prominent. The initial differences are already within 0.5 point and the proposals after reflecting on the results as a whole are even within 0.2 points.
In this particular case we should notice that the teachers cooperated *very* closely at all times.

Comparing the results
We will now compare the results of the: restricted-time written task, take-home task and oral task. (See table).

	R.T.W.T.	T.H.T.	ORAL
1	8.5	7.6	8.0
2	4.5	7.1	4.7
3	5.2	5.0	7.0
4	9.1	9.5	9.5
5	5.0	9.2	7.8
6	8.7	9.5	7.5
7	5.3	7.7	8.5
8	4.1	8.8	5.5
9	4.7	7.6	8.4
10	3.5	7.1	6.0
11	4.2	7.7	8.6
12	8.6	9.5	9.0
13	3.7	7.6	5.2
14	3.9	8.0	-
15	4.8	8.5	7.3
16	5.1	8.5	5.6
17	7.9	8.5	6.5
18	9.6	8.5	5.2
19	4.0	5.0	6.9
20	8.0	9.5	8.1
21	7.1	7.5	8.5
22	3.1	7.6	3.5
23	7.4	7.5	8.5
24	8.0	7.4	8.4
25	7.1	9.5	8.0
26	5.1	7.4	6.5
27	5.9	8.5	8.2
28	7.8	9.5	4.7
	$\bar{x} = 6.1$	$\bar{x} = 8.0$	$\bar{x} = 7.1$
	$\tau = 1.9$	$\tau = 1.2$	$\tau = 1.5$

$\rho = 0.49$

$\rho = 0.14$

$\rho = 0.42$

Comparing the results by means of box-plots we get:

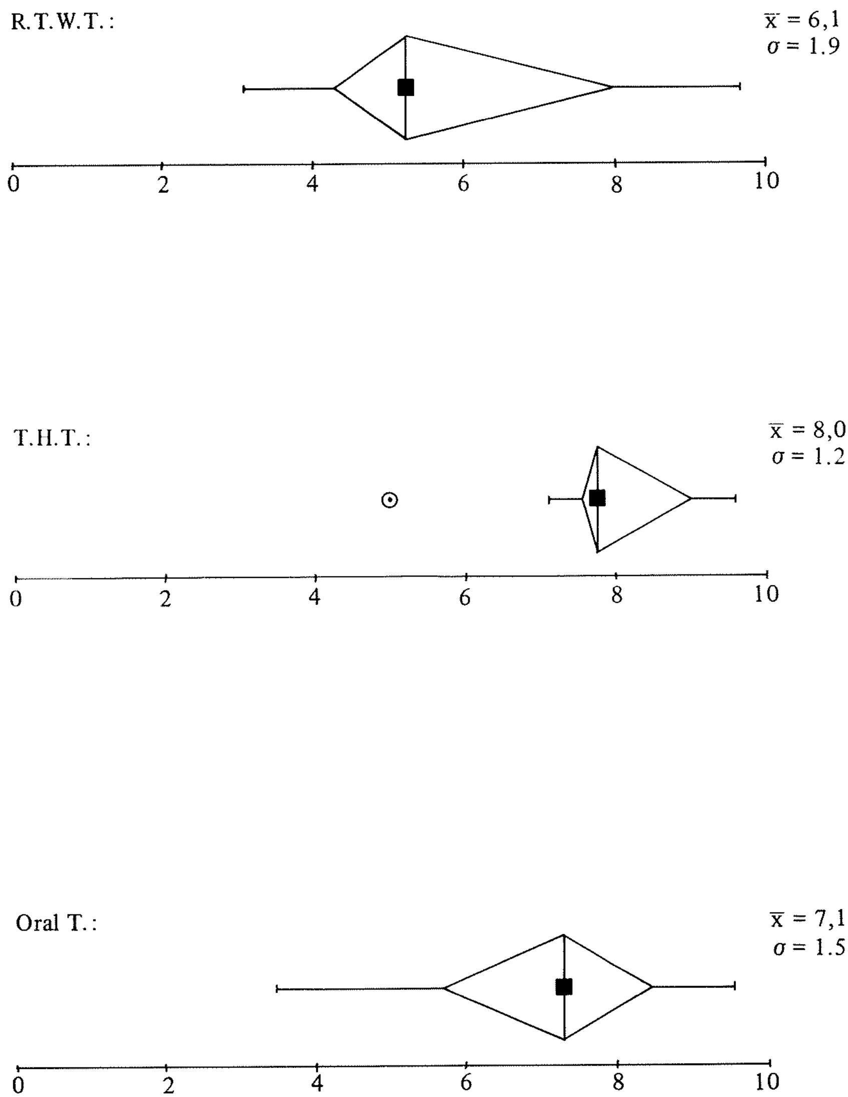


fig. V.22

and the following correlation matrix:

	rtwt	tht	oral
rtwt	1.00	0.49	0.42
tht	0.49	1.00	0.14
oral	0.42	0.14	1.00

Realizing that our sample consists of only 28 students we see the following:

- From restricted-time written test to oral test to take-home task we see a increasing average (6.1 → 7.1 → 8.0) and a decreasing spread (1.9 → 1.5 → 1.2).
- Correlation between the three stages is low which seems to indicate that we are actually measuring different dimensions.

To see if these conclusions from the small sample really hold we will now compare the results of all 140 students involved.

First we will look at each of the three different tasks: restricted-time written test, take-home test and oral test.
(The essay task is excluded).

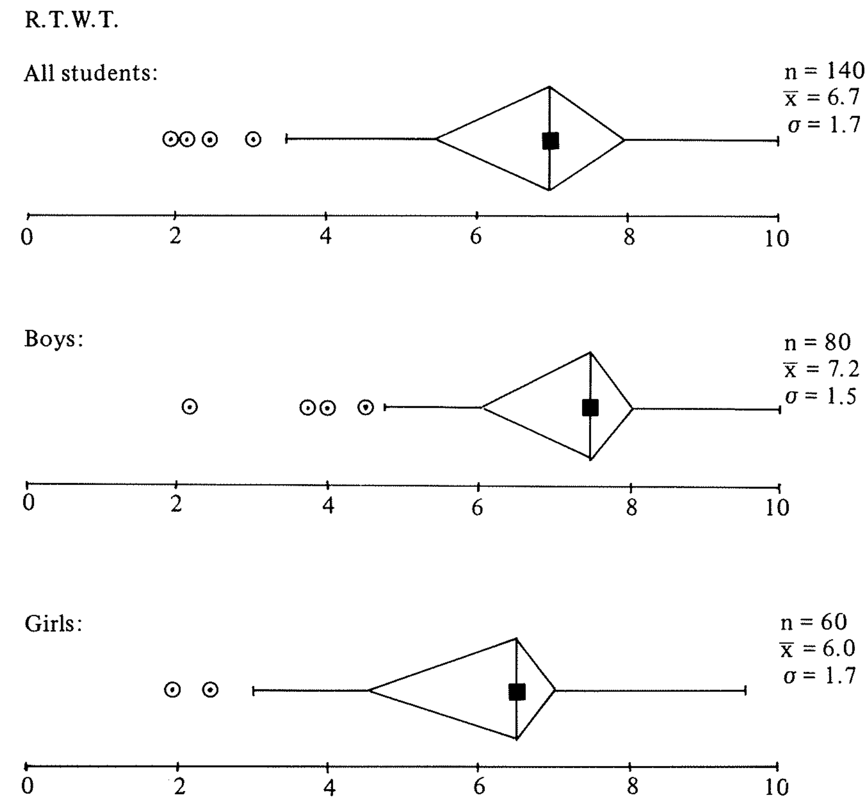
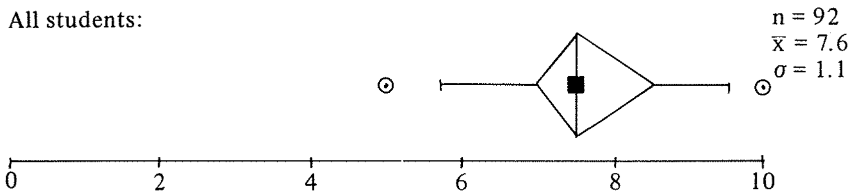


fig. V.23

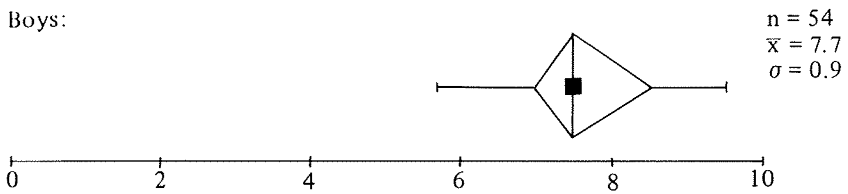
Conclusion: Boys perform considerably better than girls on restricted-time written tests.

T.H.T.:

All students:



Boys:



Girls:

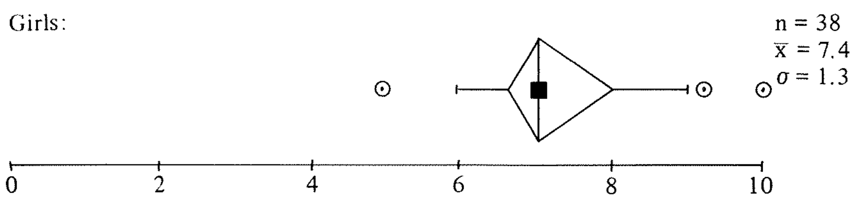


fig. V.24

Conclusion: Boys and girls perform more or less at the same (high) level.

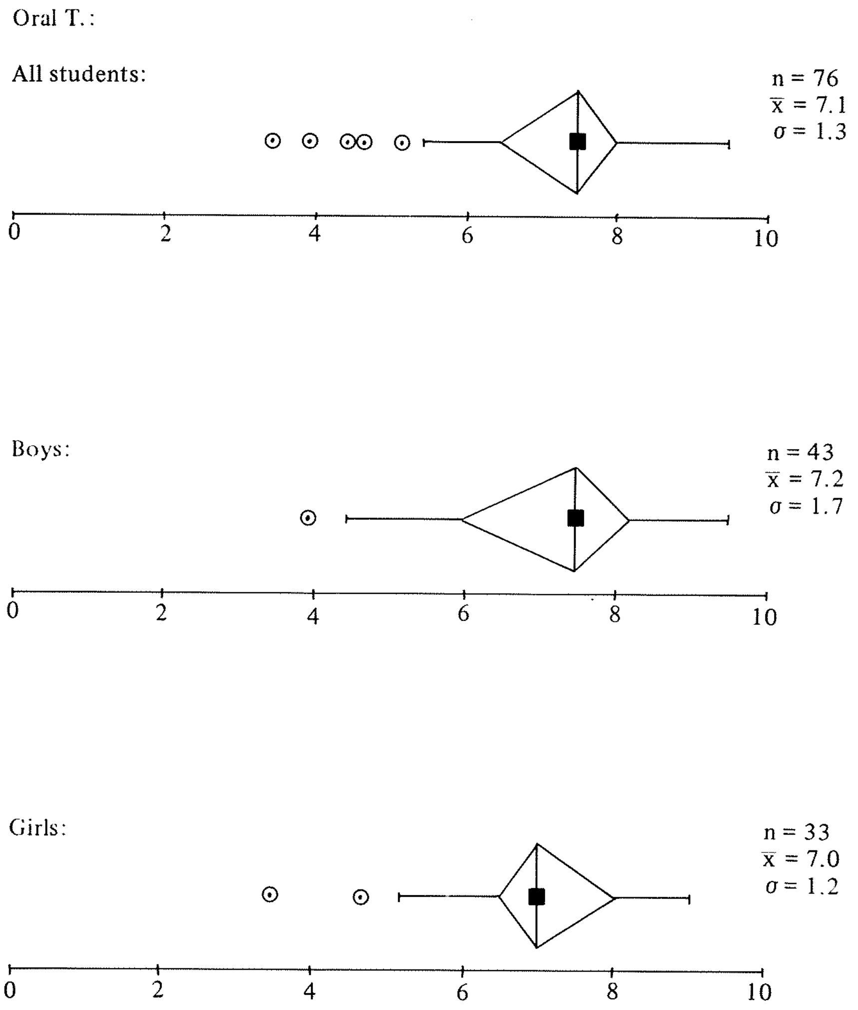
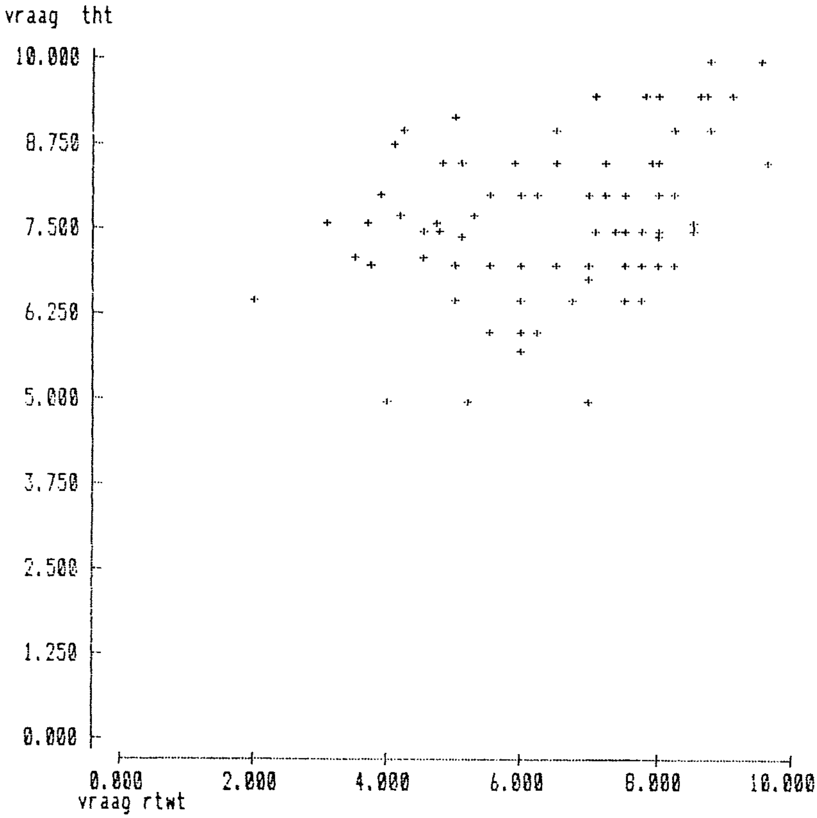


fig. V.25

Conclusion: Boys and girls perform more or less the same at a level that lies between the restricted-time written test and the take-home task. This trend also became clear in the case of the three-stage task. (See page 253).

Finally we will compare the results
R.T.W.T. - T.H.T.
and
R.T.W.T. - Oral.

R.T.W.T. - T.H.T.:

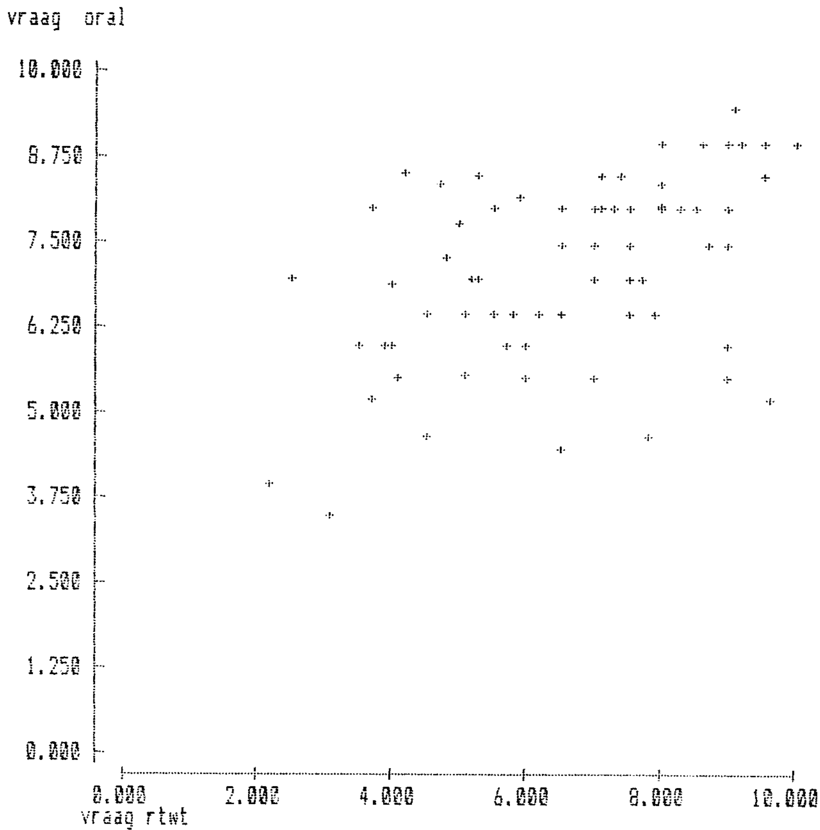


Scatter diagram of:
horizontal: restricted-time written test
vertical: take-home task
$\sigma = 0.33$ $\sigma_{boys} = 0.32$ $\sigma_{girls} = 0.33$

fig. V.27

The diagram shows clearly that the scores for the take-home tasks are considerably higher than for the restricted-time written test.
Furthermore, there is a low correlation between the two marks, which leads to the conclusion that we are not testing one-dimensionally, but that we are operationalizing other goals when adding the take-home task to our teaching practice.

R.T.W.T. - Oral



Scatter diagram of:
horizontal: restricted-time written tests
vertical: oral task
$\sigma = 0.47$ $\sigma_{boys} = 0.44$ $\sigma_{girls} = 0.54$

fig. V.28

The diagram shows that the scores for oral tasks are somewhat higher than for the restricted-time written tests. Correlation is still low (around 0.5) but somewhat higher than the correlation between the written tests and the take-home task (0.3).

8 LOOKING BACK

Let us start by recalling the conclusions of the previous sections:

1. Girls perform less well than boys on restricted-time written tests.
2. Girls perform more or less the same as boys on oral-tasks or take-home tasks.
3. From the above, one is tempted to advise more oral and take-home tasks in order to offer girls fairer chances.
4. Oral-test results have a somewhat higher correlation with restricted-time written test results than do take-home tasks.
5. Students perform best with take-home tasks. The constructive and productive aspect seems to offer students a fair chance to show their abilities (creativity, reflection, etc.). Positive testing is at its optimum in this way.

As we have pointed out, we started our explorative study on alternative tasks because of the fact that the restricted-time written tests as carried out by the teachers did not meet the intentions and goals of Mathematics A. Not only did the teachers stay very close to the exercises in the book, but when they did not they ran into trouble because of time-restrictions encountered with the timed tests.

Mathematics A is strongly process-oriented; the mathematization process needs time to develop, time to reflect, time to generate creative and constructive thoughts.
These 'higher' goals are not easily operationalized with timed tests.

On the other hand, it is our opinion that the tests or tasks should play a more constructive and productive role in the learning process. Formative tests in particular are well suited for improving the learning process.
Furthermore, we should try to offer the students ample opportunities to show their abilities. In timed tests we usually notice negative testing.

In our efforts to find other ways of assessment testing we should not be hindered by the strict rules of objective or even mechanical scoring. Too often, the influence of those rules has a very negative effect on the way of testing.

Finally, teachers should be given the opportunity to carry out these tasks without disrupting schoolwork too much; such tasks should be developed by a central institution with the help of the teachers.

The total impression gained from our observations is represented in the following table:

	R.T.W.T.	ORAL	2ST	T.H.T.	ESSAY
IMPROVE LEARNING	-	0	+	+	+
POSITIVE TESTING	-	0	+	+	++
OPERATION OF GOALS LOWER	++	0	+	+	0
OPERATION OF GOALS HIGHER	-	0	+	+	+
RELIABILITY OF SCORE	++[1)]	+[2)]	+[2)]	0[2)]	0[3)] (?)
FITS IN SCHOOLPRAC	+	0/-	0/+	0	0

Notes:
1) objective scoring with answer model
2) intersubjective scoring
3) insufficient data.

Of course, there is no clear-cut boundary between the different classes like '+', '0', '-' in the above table. Nevertheless, we think that the table gives a fair impression of the relation between our five basic principles and the different forms of alternative tasks and the restricted-time written test.

Our explorative study not only shows the potential of alternative tasks in classroom practice, but it also points out that girls perform less well on restricted-time written tests. Other forms of testing gave girls a better opportunity. We noticed this already when describing our experiences with the two-stage test, but the impression was reinforced when analyzing the results from other students from different schools.
The creative constructive aspect of the take-home task, or the two-stage test (or essay), seems to especially appeal to girls. Most of the very best results came from girls.

Students seem to like the idea of *producing* something - not only mentally - as part of the process of conceptual mathematization. They learn by producing; it forces them to reflect on their own learning process.
Positive testing is testing by production. Noting this aspect one can only regret that the experimental student material did not offer more moments for students to 'produce'.

In our opinion, teachers and authors should be advised to introduce more 'productive' activities into the textbooks and the classroom.
Alternative tasks can fulfill a very important role in this aspect. Teachers should not shy away from these tasks because of problems with scoring.
Teaching should be aimed at learning, and tests and tasks should be part of the learning process and not merely a means to obtain scores.

VI EPILOGUE

1 LOOKING BACK AT THE HEWET PROJECT

This study began with a description of the Hewet *project*. The experiences as reported in Chapter III concern the *Math A program* within the experiment. This means that we have restricted ourselves to reactions and experiences concerning the Math A program as such and have not dealt with aspects concerning the project as a whole. The following are a few of these missing aspects: the time schedule of the 2-10-40 stage, the contents of the teacher training courses, the 'all schools' stage, the (lack of) facilities for the experimental schools, the (lack of) teacher information, the regional differences as they appeared during information meetings, the relation we had with authors of commercial teams, the commercialization of the experimental materials. Nevertheless we would like to look back on the project for a moment in order to give some recommendations for future projects.

Basically, the structure of the Hewet project is sound. There are, however, many aspects that deserve special attention and need improvement:

- The Ministry of Education should have acted more promptly during the period 1970-1980.
- The Ministry of Education should shift its attention from the *structure of* projects to their *contents*.
- The Ministry of Education should have created more favourable conditions for the teachers of the experimental schools: one extra hour per teacher per week for the teachers of the ten schools is not much.
- The Ministry of Education should have followed the advice of the Advisory Commission to grant an extra year in order to evaluate the experiences of the forty schools. The nationwide introduction could have been delayed until 1986.
- The Ministry of Education should not have terminated the project on January 1st, 1986 - just months after the nationwide introduction at all schools. The most risky part of the project lies between August 1985 and May 1987, or May 1988 when the first exams take place. There are *no facilities at all* to support teachers and students during this period.
- The Team needed some direct support from teachers - for instance, two teachers with ten task-hours per week. In the present configuration the Team was continuously overloaded and had to skip important tasks.
- More meetings and conferences should have been planned, especially to benefit teachers of the forty schools and the schools as a whole.

- More information should have been provided to teachers, counsellors and university teachers. More attention should have been paid to the dissemination of information.
- Evaluation of the project, which has been poor, should have been taken more seriously.
- The problems of achievement-testing and of examinations, which were underestimated, should have been foreseen.
- The knowledge and experiences acquired by teachers and students of the experimental schools should have been transferred more efficiently to the next round.
- Measures should have been taken to stimulate sound curriculum development in the future.

Aside from this, future innovation projects might benefit from the strong points of the structure of the Hewet project.
- Stepwise experiments to try out student materials.
- Teacher training based on preceding experiments.
- A coherent flexible team with no fixed roles: all activities are carried out by all members.

We will come back to the structure of the project when discussing achievement-testing in the next section.

2 CONCLUSIONS: MATH A AND ITS TESTING

Chapter II provides an analysis of the Math A curriculum as concretized by the experimental materials. This 'a posteriori' description serves not only the team members by sketching a rough framework for instruction theory, but also the teachers.
As our research showed (Chapter IV) the goals of Mathematics A were not clear to most teachers. This in turn led to problems with achievement testing. This now will come as no surprise if we reflect on the nature of Math A.
Its process-oriented character makes it - a priori - very difficult to operationalize its goals in restricted-time written tests.

Our explorative study of alternative tasks was started because of the positive experiences with Math A in the experimental schools (Chapter III) at the one hand, and by the problems with achievement-testing on the other hand: the difficulties with operationalizing the goals of Math A as well as the teacher's problems with appropriate test design. This had to be confronted with the

fact that restricted-time written tests are common in the Netherlands. Especially in formative testing, we as pointed out, we should consider achievement tests as part of and contributing to the learning process.
Furthermore, we should offer the students the opportunity to show what they know, rather than what they do not know.

Being aware of the fact that many teachers do not consider open tasks to be scorable in an objective way, we studied the possibilities of scoring them in an intersubjective way. The results of this part of the study led us to the conclusion that the advantages of alternative tasks more than outweigh the disadvantages. Certainly given the fact that the boundary conditions for the teachers were minimal, the relatively small differences in scores are remarkable.

It is an interesting outcome of the study that girls perform rather poorly on restricted-time written tests when compared to boys. This difference disappears almost completely in the case of the alternative tasks that were part of this study.

The best possibilities are offered by a combination of restricted-time written tasks, take-home tasks (or combined in one two-stage task) and oral tasks, which clearly measure different aspects of learning.
In order not to disrupt the schoolwork, alternative tasks should be given only twice or three times a schoolyear, preferably of different types.
At least one of the three internal examinations in the final year should be an 'alternative exam'.

It will come as no surprise that we are concerned with the exam as well. This is still in the form of a restricted-time written test and, therefore, a less desirable form for the Math A examination. Especially if we notice the already clear trend towards more numerical answers instead of interpretation, reflection and creativity.
It can only be regretted that the oral examination disappeared years ago. The introduction of the Math A curriculum makes it necessary to reconsider oral examinations, not in the last place to offer girls better opportunities.

This leads to the following conclusion: The structure of the Hewet project (although basically sound) should have been adjusted so that, following the five years of experiments, a publication will appear which describe in some detail the goals of the program. Not only in a general way and concretely as far as the mathematical content is concerned; the publication should also

describe 'education in action': experiences with experimental student materials, with tests and tasks, with detailed studies of alternative tasks, with a clear description of process-oriented goals (including examples from classroom-work) and indicate the different levels attainable.
If it had existed, this publication could and should have served as a guideline for the remainder of the schools to follow, for the designers of tasks and exams, and for the authors of text books.
If the above points had been taken into consideration, the Hewet project would have been an excellent paradigm for innovation in the future.

SAMENVATTING

In augustus 1985 werd een nieuw curriculum in Nederland geïntroduceerd voor de hoogste klassen van het voortgezet onderwijs: Wiskunde A.
Dit programma was bedoeld voor die studenten die later op de universiteit psychologie, sociale wetenschappen, economie e.d. willen studeren.

HET HEWET PROJECT (Chapter I, pp. 5)

Voorafgaand aan de introductie van het nieuwe programma vond een uitgebreid experiment plaats: het Hewet project.
Gedurende een periode van vijf jaar werden experimenten uitgevoerd; eerst op twee, later op twaalf en tenslotte op 52 scholen (zie p. 12). Deze experimenten waren er in de eerste plaats op gericht de inhoud van het vak Wiskunde A vorm te geven. Daarbij speelden het ontwerpen van leerlingteksten, het observeren in de klas, het discussiëren met leerlingen en docenten, het informeren van docenten en het opzetten van nascholingsactiviteiten een grote rol.
De sterke punten van het Hewet project waren de stapsgewijze experimenten om het studentenmateriaal uit te testen en de scholing van de docenten die sterk leunde op de ervaringen opgedaan tijdens het experiment en plaatsvond vóór introductie van het nieuwe curriculum op de scholen.
Zwakke punten waren zonder twijfel de randvoorwaarden voor de scholen: er werden weinig of geen faciliteiten geboden, de tijdsdruk was te groot, er was onvoldoende gelegenheid tot scholing.
Andere tot nadenken stemmende zaken: er heeft vrijwel geen evaluatie plaatsgevonden en er is geen enkele 'follow-up' voor het experiment; de scholen werden na de introductie van het nieuwe programma aan hun lot overgelaten.

ZWAARTEPUNTEN

Het zwaartepunt van deze studie ligt niet bij het Hewet project als zodanig maar bij het vak Wiskunde A.
Twee punten staan daarbij centraal:
- Wat is Wiskunde A?
- Hoe ontwikkel je goede toetsen voor Wiskunde A?

WISKUNDE A (Chapter II, pp. 23)

De eerste vraag kan worden afgedaan met een beschrijving van de puur wiskundige inhoud van het programma (p. 23).
Alhoewel deze opsomming het begin vormt van Hoofdstuk II zegt deze weinig over de werkelijke inhoud van het vak Wiskunde A.

De inhoud van het vak dient bezien te worden in het licht van twee uitgangspunten:

- Het vak dient gebruikt te worden is disciplines zoals psychologie, sociale wetenschappen, economie etc.
  (Wiskunde als gereedschap.)
- De manier van behandelen dient te motiveren; deze motivatie zou mede ontleend kunnen worden aan het feit dat de wiskundige kennis als relevant dient te worden ervaren.

Een analyse van de experimentele leerlingmaterialen gekoppeld aan de klasseobservaties leert dat Wiskunde A een sterk proces georiënteerd vak is. Eén van die processen is het mathematiseren. Daartoe worden leerlingen problemen aangeboden uit de 'real world'. Deze problemen dienen georganiseerd, gestructureerd, gevisualiseerd te worden.
Dat proces heet mathematiseren.
Daarin kunnen verschillende aspecten benoemd worden. Zo herkennen Treffers en Goffree horizontale- en verticale mathematisering. Bij horizontale mathematisering ligt de nadruk op het vertalen van het probleem uit de 'real world' naar de 'symboolwereld', zoals Freudenthal het interpreteert. Het herkennen van de wiskundige essenties en de relevante gegevens, het verschematiseren of visualiseren, het ontdekken van regelmatigheden of verbanden zijn typische horizontale activiteiten. De veelal daarop volgende verdieping binnen de wiskunde kan als verticaal mathematiseren worden gekenmerkt.

De 'real world' speelt in de Wiskunde A een dubbelrol. Deze dubbelrol leidt tot een andere tweedeling in het mathematiseren.
Allereerst vormt een 'real world' het startpunt van ieder subcurriculum. Al mathematiserend ontwikkelen de studenten op interactieve wijze wiskundige concepten, waarbij reflectie op het eigen mathematiseringsproces essentieel is. Deze fase van mathematiseren wordt in deze studie als conceptueel mathematiseren aangeduid (zie p. 63).

Nadat de concepten zijn geabstraheerd en geformaliseerd kunnen zij worden

toegepast in nieuwe 'real world' problemen. Op deze wijze worden de concepten versterkt en de 'real world' van de student bijgesteld (p. 39). Hierbij spreken we van toegepast mathematiseren.
Aan de hand van vele voorbeelden worden de verschillende soorten en fasen van mathematiseren duidelijk gemaakt en wordt de rol van de context daarbij nader bestudeerd (zie p. 76).

Apart wordt aandacht besteed aan de kritische houding die de student ten aanzien van gebruik en misbruik van wiskunde zou moeten bezitten (zie p.88).

Van de overige zaken die in de beschouwing van Wiskunde A nog een rol spelen noemen we het feit dat Wiskunde A zoals vormgegeven in de experimentele leerlingmaterialen past in de definitie van 'Realistisch wiskunde-onderwijs', dat zich met name door contextgebruik en mathematiseren onderscheidt van het tot voor kort zo gangbare structuralistische en mechanistische wiskunde-onderwijs (zie p. 93).

ERVARINGEN (Chapter III, pp. 124)

Na de uitvoerige beschrijving van de gedachten achter Wiskunde A volgt een inventarisering van de ervaringen van docenten en leerlingen die tijdens het experiment werden verzameld. Hieruit blijkt dat veel leerlingen vinden dat de gestelde doelen worden bereikt: men vindt het vak zinvol omdat het aangeeft hoe nuttig wiskunde is; veel leerlingen ervaren het feit dat naar inzicht wordt gestreefd als positief en docenten beamen dat er in het algemeen gemotiveerd wordt gewerkt. Tevens werd tijdens de nascholingscursus duidelijk dat veel docenten die aanvankelijk nogal terughoudend tegenover Wiskunde A stonden, later gematigd positief tot positief reageerden (zie p. 141).

TOETSEN (Chapter IV, pp. 163)

Het proces-karakter van Wiskunde A, alsmede het gebrek aan duidelijkheid omtrent de essentie van het vak bracht veel docenten in moeilijkheden met name bij het ontwerpen van toetsen. Dit werd niet alleen duidelijk uit de reacties zoals weergegeven in hoofdstuk III, maar vooral uit de resultaten van een onderzoek naar de proefwerken zoals die op de twaalf experimenteerscholen afgenomen werden.

Veel opgaven weken weinig af van de opgaven in het boek met als gevolg dat wél kennis, maar nauwelijks vaardigheden werden getoetst. Of anders gezegd: wel produkt- maar geen procestoetsing (zie p. 163).
Dit feit vormde de directe aanleiding tot het exploratieve onderzoek naar andere toetsvormen dat het tweede zwaartepunt van deze studie vormt (zie pp. 178).

ALTERNATIEVE TOETSEN (Chapter V, pp. 178)

Bij het ontwikkelen van alternatieve toetsen voor Wiskunde A stonden de volgende vijf principes centraal (zie p. 179).
- Toetsen moeten bijdragen aan het leerproces.
- Leerlingen moet de kans gegeven worden te laten zien wat ze weten - niet wat ze niet weten.
- De doelen van Wiskunde A moeten zoveel mogelijk geoperationaliseerd worden.
- De kwaliteit van de toets wordt niet in de eerste plaats bepaald door de mogelijkheid tot objectieve scoring.
- De toets moet redelijk binnen de schoolpraktijk passen.

Toetsen die vervolgens worden beschreven zijn:
- *De twee-traps toets* (pp. 184)
  Bij deze toets wordt de leerlingen eerst gevraagd de toets te maken als een proefwerk. Nadat het werk beoordeeld is mogen de leerlingen vervolgens het werk thuis verder uitwerken, waarna een tweede cijfer gegeven wordt.
- *De thuistoets* (pp. 222)
  Een toets die door één of meer leerlingen thuis (met hulp) gemaakt kan worden en veelal op een hoog niveau ligt.
- *De essay toets* (pp. 234)
- *De mondelinge toets* (pp. 242)

Talrijke voorbeelden maken duidelijk dat deze toetsen in veel opzichten beter aansluiten bij het vak Wiskunde A dan proefwerken. Het is ook duidelijk dat er heel andere doelen worden geoperationaliseerd dan gebruikelijk, doelen die het hart van Wiskunde A vormen: mathematisering, reflectie, creativiteit, kritische beoordeling etc.

De vrees dat dergelijke open vormen van toetsen zouden kunnen leiden tot weinig objectieve scores werd ook onderzocht (pp. 209).
Vijf werkstukken van leerlingen werden ter beoordeling aan vijftien docenten

opgestuurd. Alhoewel deze docenten geen enkele steun hadden in de vorm van informatie over de studenten, of de beschikking hadden over een antwoordmodel, bleek de spreiding van de beoordelingen niet zeer groot. De mate van intersubjectiviteit was redelijk.

Aan de andere kant werd duidelijk dat het ontwerpen en beoordelen van dergelijke toetsen zeer tijdrovend is. Het is daarom dat wij ervoor pleiten dat dergelijke toetsen mede door deskundigen worden ontwikkeld en aan docenten ter beschikking worden gesteld.

De conclusie lijkt gewettigd dat de positieve effecten van 'alternatieve' toetsen ruim opwegen tegen de negatieve en dat de toetsen een vaste plaats verdienen binnen het onderwijsproces. Daarbij komt nog het feit dat met name meisjes lijken te profiteren van de mogelijkheden die alternatieve toetsen bieden (pp. 264).

De zorg voor het vak Wiskunde A weerspiegelt zich in de zorg voor het landelijk examen. Het examen voldoet in vrijwel geen enkel opzicht aan de principes die wij ons bij het ontwikkelen van goede toetsen voor Wiskunde A hebben gesteld. Gevreesd moet worden dat het examen in de huidige vorm een slechte invloed zal hebben op het vak Wiskunde A zoals in deze studie beschreven.

RESUME

En août 1985 fut introduit aux Pays-Bas un nouveau curriculum pour les classes supérieures de l'enseignement secondaire: Mathématiques A.
Ce programma était fait pour ces élèves-là qui après le baccalauréat voulaient faire des études de psychologie, sciences sociales, économie etc.

LE PROJET HEWET (Chapitre I, pp. 5)

Avant l'introduction du nouveau programme a eu lieu une expérience étendue: le projet Hewet.
Pendant une période de cinq ans on a fait des expériences; d'abord à deux écoles, plus tard à douze et enfin à 52 écoles (voir p. 12). Ces expériences avaient premièrement pour but de donner forme à la discipline de Mathématiques A. Aussi, la création de textes pour l'élève, l'observation dans la classe, la discussion avec les élèves et les professeurs, l'information donnée aux professeurs et le commencement d'activités de recyclage ont-ils joué un grand rôle.
Les points forts du project Hewet étaient les expériences graduelles pour tester le matériel d'enseignement et la formation des professeurs, qui était fondée sur les résultats pris pendant l'expérience et qui eut place avant l'introduction du nouveau curriculum dans les écoles.
Les points faibles du projet étaient sans doute que très peu de facilités étaient offertes aux écoles: la pression du temps était trop grande, il y avait trop peu d'occasion pour la formation.
D'autres choses qui invitent à la réflexion sont: presque aucune évaluation a eu lieu et l'expérience ne connaît aucune continuation. Après l'introduction du nouveau programme les écoles ont été abandonnées à leur sort.

POINTS CAPITAUX

Le point capital de cette étude ne concerne pas le projet Hewet en soi, mais plutôt la discipline des Maths A.
Voici les deux questions principales:
- Qu'est-ce que c'est que les Mathématiques A.
- Comment développer de bons épreuves pour Maths A.

MATHEMATIQUES A (Chapitre II, pp. 23)

On peut répondre à la première question en donnant une description du contenu purement mathématique du programma (p.23).
Bienque une telle énumération forme le début du Chapitre II, celle-ci dit peu de choses de la véritable matière de Maths A.

Il faudrait prendre les deux points de départ suivants:
- La matière doit servir des disciplines telles que la psychologie, les sciences sociales, l'économie etc.
- La façon dont on enseigne doit motiver l'élève.

Cette motivation pourrait aussi s'emprunter au fait que la connaissance des Maths doit être considérée comme pertinente. Une analyse des matériels expérimentaux pour l'élève ainsi que les observations en classe nous apprend que les Maths A est une discipline de plusieurs processus. Un de ces processus est la mathématisation. A cet effet on présente aux élèves des problèmes pris dans le 'monde réel'. Ces problèmes doivent être organisés, structurés, visualisés.
Ce processus s'appelle mathématiser.
Dans la mathématisation on peut distinguer des aspects différents. Ainsi Treffers et Goffree reconnaissent une mathématisation horizontale et verticale. Dans la mathématisation horizontale il s'agit de transposer le problème du 'monde réel' au 'monde des symboles', comme l'interprète Freudenthal. Reconnaître les essentiels mathématiques et les données pertinentes, schématiser ou visualiser, découvrir des régularités ou des rapports, sont typiquement des activités horizontales. L'approfondissement dans la mathématique qui y suit en général, cela on peut appeler mathématisation verticale.

Le 'monde réel' joue dans les Maths A un double rôle. Ce double rôle entraîne une autre division dans la mathématisation.
Tout d'abord le 'monde réel' constitue le point de départ de tout subcurriculum. Tout en mathématisant les étudiants développent de façon interactive, des concepts mathématiques; dans ceci la réflexion sur le propre processus de mathématisation, est une condition essentielle. Dans cette étude, on appelle cette phase de la mathématisation: mathématisation conceptuelle (voir p. 63).

Après avoir abstraits et formalisés les concepts, on peut les appliquer dans les nouveaux problèmes du 'monde réel'. De cette façon, les concepts sont

renforcés et le 'monde réel' de l'étudiant est ajusté (p. 39). On appelle ceci: mathématisation appliquée.
A l'aide de multiples exemples, on rend claires les différentes sortes et phases de la mathématisation et on étudie de plus près le rôle du contexte (voir p. 76).

Une autre chose qui demande l'attention, c'est l'attitude critique que l'étudiant devrait avoir quant à l'usage et l'abus des Maths (voir p. 88).

Parmi les autres aspects qui, dans la contemplation des Maths A jouent un rôle, nous relevons le fait que Math A, telle que la matière a été conçue dans les matériels expérimentaux pour l'élève, elle est conforme à la définition de 'enseignement réaliste des maths' qui, par l'usage de contextes et la mathématisation, se distingue de l'enseignement des Maths structuraliste et mécaniste, un enseignement qui, très récemment, était usuel (voir p. 93).

EXPERIENCES (Chapitre III, pp. 124)

Après la description détaillée des pensées derrière les Maths A, suit un inventaire des expériences de professeurs et d'élèves, rassemblées pendant l'expériment. Il en résulte que beaucoup d'élèves sont d'opinion qu'on a atteint les buts visés: on trouve la matière utile parce qu'elle montre l'utilité des Maths. Bien des élèves sont contents du fait qu'on a recherché la clarté et les professeurs confirment qu'en général on travaille de façon motivée.
En outre on a pu constater pendant les cours de recyclage, que beaucoup de professeurs qui au début étaient assez réservés vis-à-vis des Maths A, plus tard ont réagi plus ou moins positivement (p. 141).

LES EPREUVES (Chapitre IV, pp. 163)

Le caractère-processus des Maths A et aussi le manque de clarté en ce qui concerne l'essentiel de la matière, posaient des problèmes aux professeurs surtout là où il fallait composer des épreuves. On a pu constater ceci non seulement par les réactions comme on en trouve dans le Chapitre III, mais surtout par les résultats d'un examen des épreuves telles qu'elles furent présentées aux élèves des douze écoles expérimentales.
Bien des problèmes posés dans les épreuves ne différaient guère des problèmes dans le livre. Ainsi on pouvait tester les connaissances mais à peine les aptitudes. En d'autres mots: les épreuves testaient le produit mais

non pas le processus (voir p. 163).
Ce fait a résulté directement à la recherche explorative d'autres formes d'épreuve, ce qui constitue le second point capital de cette étude (voir p. 178).

EPREUVES ALTERNATIVES (Chapitre V, pp. 178)

Pour développer des épreuves alternatives pour Maths A nous sommes partis des cinq principes suivants:
- Les épreuves doivent contribuer au processus d'études.
- L'élève doit pouvoir montrer ce qu'il sait et non pas ce qu'il ne sait pas.
- Les buts de Maths A doivent être opérationalisés autant que possible.
- La qualité de l'épreuve ne se détermine pas en première place par la possibilité de marquer objectivement.
- L'épreuve doit cadrer raisonnablement dans la pratique de l'école.

Les épreuves ensuite décrites sont:
- *L'épreuve en deux phases* (pp. 184)
  L'élève doit d'abord faire l'épreuve à l'école: après que celle-ci a été jugée, l'élève peut ensuite élaborer le travail à la maison, après quoi lui est donné une seconde note.
- *L'épreuve faite à la maison* (pp. 222)
  Une épreuve qui peut se faire par un ou plusieurs élèves à la maison (avec assistance) et qui le plus souvent est à un niveau élevé.
- *L'épreuve-essai* (pp. 234)
- *L'épreuve orale* (pp. 242)

De nombreux exemples rendent évident que ces épreuves-ci correspondent mieux, à bien des égards, à la matière des Maths A que les épreuves traditionnelles. C'est évident aussi que d'autres buts sont opérationalisés que d'habitude, des buts qui constituent le coeur des Maths A: mathématisation, réflexion, créativité, jugement critique etc.

La crainte que de telles formes ouvertes de tester les élèves pourraient aboutir à des marques peu objectives, a été examinée (pp. 209).
Cinq épreuves d'élèves furent envoyées, pour être jugées, à quinze professeurs. Ceux-ci n'avaient aucune information sur les élèves, ils n'avaient non plus à leur disposition un modèle-réponses. Malgré cela, les différences entre les quinze jugements n'étaient pas très grandes; la mesure d'intersubjectivité était raisonnable.

De l'autre côté on a constaté que le temps qu'on met à composer et à juger de telles épreuves est énorme. C'est pour cette raison que nous sommes d'opinion qu'il vaut mieux que de telles épreuves soient développées par des experts et mises à la disposition des professeurs.

La conclusion semble légitime que les effets positifs des épreuves 'alternatives' compensent largement les effets négatifs et que les épreuves méritent une place permanente dans le processus de l'enseignement.
Ajoutons le fait que surtout les filles semblent profiter des possibilités qu'offrent les épreuves alternatives (pp. 264).

Le soin pour la discipline des Maths A se reflète dans le soin pour l'examen national. Cet examen ne répond, à presque aucun égard, aux principes que nous nous sommes posés pour le développement de bonnes épreuves pour Maths A. Il y a lieu de craindre que l'examen dans la forme actuelle ait mauvaise influence sur la discipline des Maths A comme décrite dans cette étude.

RESUMEN

En agosto de 1985 se introdujo en los Países Bajos un nuevo programa para los últimos años de la enseñanza media: Matemática A.
Este programa fue pensado para aquellos estudiantes que siguieran carreras universitarias relacionades con sicología, ciencias sociales, economía, etc.

EL PROYECTO HEWET (Capítulo I, pp. 5)

Previo a la introducción del programa se realizó un extenso experimento: el Proyecto Hewet.
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Los puntos más positivos del Programa Hewet fueron los experimentos graduales para verificar la validez del material de estudio y los cursos especiales para docentes que se basaron en gran parte sobre las experiencias recogidas durante la etapa experimental previa a la introducción de la nueva asignatura.
Aspectos negativos fueron sin duda la situación existente en los colegios:
las facilidades ofrecidas fueron muy pocas o nulas, la falta de tiempo fue muy grande, no hubo suficiente oportunidad para el dictado de cursos a docentes.
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Al respecto, son de importancia fundamental las dos cuestiones siguientes:
- ¿Qué es Matemática A?

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MATEMATICA A (Capítulo II, pp. 23)

La primera pregunta puede contestarse con una descripción del contenido puramente matemático del programa (p. 23).
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El contenido de la asignatura debe observarse tomando como base los siguientes puntos de referencia:

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Un análisis del material experimental unido a las observaciones hechas en las clases, nos demuestra que Matemática A es una asignatura con una gran orientación hacia los procesos. Uno de estos procesos es la transformación en términos matemáticos. Con ese fin se plantean a los alumnos problemas que se presentan en el mundo real. Los mismos deben ser organizados, estructurados y visualizados.
Esto es lo que se denomina proceso de transformación en términos matemáticos.
Al respecto pueden mencionarse diversos aspectos. Treffers y Goffree describen procesos de tipo horizontal y vertical. En los casos de procesos de transformación en términos matemáticos de tipo horizontal se pone énfasis en la traducción de los problemas del mundo real al lenguaje del mundo de los símbolos, tal como lo interpreta Freudenthal. El reconocimiento de esencias matemáticas y de datos relevantes, la esquematización o visualización, el descubrimiento de regularidades o relaciones, son actividades típicamente horizontales. La consecuente profundización, dentro de la matemática, se puede considerar como un proceso vertical.

En Matemática A el mundo real cumple una doble función que conduce a otra bifurcación en el proceso de transformación en términos matemáticos (matematización).
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respecto a la esencia de la asignatura, condujo a menudo a situaciones problemáticas a muchos docentes, principalmente en el diseño de ejercicios. Este se manifestó claramente no solo por las reacciones, tal como se describen en el capítulo, sino también por los resultados de una investigación de los exámenes tomados durante el experimento de los doce colegios.
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EXAMENES ALTERNATIVOS (Capítulo V, pp. 178)

Para el desarrollo de exámenes alternativos para Matemática A, se tuvieron en cuenta los siguientes puntos principales (ver p. 179):
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- Debe darse a los alumnos la posibilidad de demostrar lo que saben y no, lo que no saben.
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Los exámenes que a continuación se describen son:
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Un gran número de ejemplos demuestran que estos exámenes se adecúan mejor, en muchos sentidos, que las pruebas en general, a la asignatura Matemática A. Queda también en claro que de este manera se cumplen

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RESUMEN

En agosto de 1985 se introdujo en los Países Bajos un nuevo programa para los últimos años de la enseñanza media: Matemática A.
Este programa fue pensado para aquellos estudiantes que siguieran carreras universitarias relacionades con sicología, ciencias sociales, economía, etc.

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En Matemática A el mundo real cumple una doble función que conduce a otra bifurcación en el proceso de transformación en términos matemáticos (matematización).
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de manera interactiva, para lo cual es esencial la reflección sobre el propio proceso de matematización.
En este estudio esa fase del proceso de matematización se denomina matematización conceptual (ver p. 63).
Luego de haber abstraído y formalizado los conceptos, estos pueden ser nuevamente aplicados a los problemas del mundo real. De esta manera se refuerzan los conceptos y se adapta el mundo real del estudiante (ver p. 39).
En este caso hablamos de matematización aplicada.
Con la ayuda de varios ejemplos se aclaran las diversas clases y fases del proceso de matematización. Asimismo se profundiza el estudio de la función del contexto (ver p. 76).

Separadamente se dedica atención a la actitud crítica que el alumno debe tener con respecto al uso y abuso de la Matemática (ver p.88).

De los demás asuntos relevantes para la consideración de Matemática A nombraremos el hecho que Matemática A, tal como se presenta en el material experimental se corresponde con la definición de 'educación matemática realista' que, principalmente debido al uso del contexto y a la matematización, se diferencia de la, hasta hace poco usual, educación matemática estructuralista y mecanicista (ver p. 93).

EXPERIENCIAS (Capítulo III pp. 124)

Luego de la extensa descripción de las ideas que soportan Matemática A, sigue un inventario de las experiencias de docentes y alumnos recojidas durante el experimiento. De ello se deduce que gran cantidad de alumnos consideran que se alcanzan los objetivos planteados: la asignatura es considerada significativa ya que demuestra cuan útil es la matemática; muchos alumnos consideran positivo que los esfuerzos se dirijan hacia una mejor comprensión general, lo que es corroborado por los docentes que opinan que en general se trabaja de manera motivada. Al mismo tiempo quedó claro, durante los cursos para docentes, que varios de ellos, en principio reacios a Matemática A, luego se manifestaron positivos o moderadamente positivos (ver p. 141).

EJERCICIOS (Capítulo IV, pp. 163)

El carácter de proceso de Matemática A, junto a la falta de claridad con

respecto a la esencia de la asignatura, condujo a menudo a situaciones problemáticas a muchos docentes, principalmente en el diseño de ejercicios.
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Un gran número de ejemplos demuestran que estos exámenes se adecúan mejor, en muchos sentidos, que las pruebas en general, a la asignatura Matemática A. Queda también en claro que de este manera se cumplen muchos otros objetivos que lo que se haría normalmente, objetivos que constituyen en núcleo de Matemática A: formulación en términos matemáticos, reflección, creatividad, juicio crítico, etc.

Se analizó también la posibilidad de que exámenes de este tipo pudieran conducir a puntajes poco objetivos (p. 209).
Se presentaron a la consideración de quinze docentes cinco exámenes. A pesar de que estos docentes no contaban con antecedentes de los alumnos ni con los resultados correctos para el exámen, las evaluaciones coincidieron bastante. El nivel de intersubjetividad fue satisfactorio.

Por otro lado se constató que el diseño y la evaluación de este tipo de exámenes insume mucho tiempo. Es por eso que abogamos por exámenes preparados por expertos y puestos a disposición de los docentes.

La conclusión parece corroborarse, ya que los efectos positivos de los exámenes alternativos superan con creces sus aspectos negativos y por lo tanto los exámenes merecen un lugar dentro del proceso educativo. A esto debe agregarse el hecho que, principalmente las mujeres parecen sacar provecho de las posibilidades que ofrecen los exámenes alternativos.

La atención prestada a la asignatura Matemática A se refleja en la atención que se presta al exámen nacional. Este exámen no cumple de ninguna manera con los principios que nos hemos propuesto para el diseño de exámenes apropiados para la asignatura Matemática A. Debe mencionarse que el exámen nacional, en su forma actual tendrá una influencia negativa sobre a asignatura Matemática A tal como ha sido descripta en este estudio.

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NOTES

1. At the time of the experiments primary education in the Netherlands starts at the age of 6 and normally ends at the age of 12.

   Secondary education consists of 4,5 or 6 years, usually called 1th grade, 2nd grade, ... 6th grade.

   For this publication we have chosen for an unified numeration of grades.

   1st grade secondary education = 7th grade in this publication (children of 12 years of age).

   4th grade secondary education = 10th grade in this publication (children of 15/16 years of age).

   The Hewet project concerns grades 10 through 12: the last three years in secondary education.
2. V.W.O. = Voorbereidend Wetenschappelijk Onderwijs

   means: pre-university secondary education (7th through 12th grade).

   H.A.V.O. = Hoger Algemeen Voortgezet Onderwijs

   means: pre-polytechnic secondary education; this is a seperate school-type, running from 7th through 11th grade.
3. Team members:

   Martin Kindt, Jan de Lange

   assisted by Ellen Hanepen for organisational and administrative matters.

   Initially assisted by Guus Vonk for the automatic data processing part of the curriculum. He was later replaced by Heleen Verhage who contributed in the first place by developing software.
4. Commission members:

   F. van der Blij, W.E. de Jong, J. van Dormolen, J. van Lint, W. Molenaar, W.C. Riel, R. Ouwerkerk-Dijkers.

   W.E. de Jong was later replaced by B.J. Westerhof.

   In the last year B.J. Westerhof was replaced by W. Kleijne.
5. The two schools:

   Lorentz Scholen Gemeenschap, Haarlem;

   Liemers College, Zevenaar.
6. When we write 'examination' we mean the nationwide national examination: all students take the same exam at the same day. In this case this nationwide examination was restricted to the two schools; later to the twelve and fifty-two schools and from 1987 on it covered: all schools.
7. School Mathematics Project.
8. In one of the alternative tasks to be discussed in chapter V students also encounter the 'natural' multiplication: entry by entry.

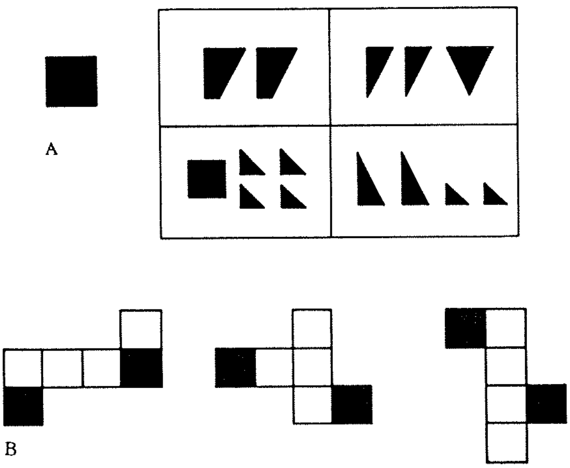
9. As early as 1844 Wigan arrived at the conclusion that the two (right and left) hemispheres are specialized to fulfill somewhat different tasks. [95]

Later experiments confirmed his results: in virtually all right-handed human beings and in about half of the left-handed, the parts of the brain associated with speech are primarily located in the left cerebral hemisphere.

However, investigators had difficulty in characterizing the differences. Some speak of a verbal-non verbal distinction, based on the just mentioned research.

Other research points to visuo-spatial abilities in the right hemisphere. Results of an impressive study carried out among over 200 patients with brain damage showed that patients with damage to the right hemisphere did consistently more poorly on non-verbal tests involving the manipulation of geometric figures, puzzle assembly, completion of missing parts of patterns and figures, and other tasks involving form, distance, and space relationships.

Two visuo-spatial tests are:



Visuo-spatial tasks. A. Which boxed set(s) can form the square on the outside? B. If you fold these patterns into cubes, in which cube(s) will the dark sides meet at one edge?

Descriptions like:

Left Hemisphere	*Right Hemisphere*
Verbal	Nonverbal, visuo-spatial
Sequential, temporal, digital	Simultaneous, spatial, analogical
Logical, analytical	Gestalt, synthetic
Rational	Intuitive
Western thought	Eastern thought

seem to be based on experimental evidence.

As Nobel laureate Sperry summarized in 1975: [96]

> "Repeated examination during the past ten years has consistently confirmed the story lateralization and dominance for speech, writing, and calculation in the (disconnected) left hemisphere."

Due to the fact that the cognitive skills that one may characterize as analytical or logical are strongly related to the left hemisphere, mathematics is often characterized as a left hemisphere activity.

As Gardner [97] notes:

> "Here the patient (with right hemisphere damage) exemplifies the behaviour associated with the brilliant young mathematician. This highly rational individual is ever alert to an inconsistency in what is being said, always seeking to formulate ideas in the most airtight way."

The question is if there are any consequences for education, and more specifically in math-education. Joseph Bogen has been an avid proponent of developing what he calls "appositional thinking" in school. [104]

"Propositional" describes the left hemisphere's dominance for speaking, writing and calculation. Bogen coined "appositional" te refer to the information processing of the right hemisphere in well lateralized right-handers.

In Bogen's view, society has overemphasized propositionality at the expense of appositionality.

10. According to Zajonc [98] a basis for intuition is suggested by his recent study indicating that feeling and thinking are separate processes. He argues that affective responses are primary and shows that in some cases, affective judgement occurs before cognitive analysis.
11. Lawler [105] states that non-uniformities of the mind can be based on other models of division than on simple 'hardware' division (right versus left brain halves); the *functional* division of mind in service of the body parts is more significant than physical lateralization of hemispheres. A central role in his theory is played by *microviews*: internal cognitive structures built through interacting with such microworlds and reflecting that fragmentary process of knowing. A microview (also called microworld) organises the knowledge of a certain domain.
    For instance, young children may possess various numerical worlds - a counting world, a money world, a paper-and-pencil calculations world - which initially may be clearly separated from each other.

The microviews are not governed from above by a central authority and different microviews can be connected with each other in order to co-operate rather than to compete.

Within the connections made between microviews a kind of hierarchy develops: microviews are constituted, connected and integrated while the relations of one layer become the thinking object of the next thanks to the active reflective engagement of the learner. A strikingly similar genesis as that of the Van Hiele levels.

Bauersfeld has extended Lawlers "microviews" to "domains of subjective experiences". [106]

A domain of subjective experience includes not only the cognitive dimension of knowledge but also emotional and motor experiences; they are, however, also determined by the domain specificity of the subject matter, cumulatively ordered and competitive.

They cannot be divided in separate domains of lower and higher sorts of learning or into different experimental fields of knowledge, feeling, willing. The learner determines his learning in the following way, according to Bauersfeld:

> "The subject does not form concepts as internal images or through copying other people's activities.
>
> The subject actively constructs meaning and relations through social situations of interaction and negotiation."

From this description and our analysis of Mathematics A it may not come as a surprise that we agree wholeheartedly with Treffers [107] when he states:

> "The kind of instruction Bauersfeld is aiming at, strongly emphasizes: subjective experiences and the development of comparing and connecting domains by means of reflection, incited by interactive processes. We may rightly conclude that the model of domains of subjective experiences generally leads to realistic mathematics instruction."

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