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Beginning Pre-service Teachers' Approaches to Teaching the Area Concept: identifying tendencies towards realistic, structuralist, mechanist or empiricist mathematics education

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SUMMARY *This study, on approaches to mathematics education, aims to identify curricular frameworks that are robust across cultures and education systems. Lesson ideas of Irish and Dutch pre-service teachers were studied to identify the selections of: (1) the basic aspects of the concept of area; and (2) the real-life contexts, shapes, and manipulatives chosen to teach this concept. A two-dimensional framework emerged, relating preservice teachers' ideas to archetypal realistic, structuralist, mechanist, or empiricist approaches.*

RÉSUMÉ *Cette étude sur la façon d'aborder l'enseignement des mathématiques vise à identifier des programmes d'études qui soient consistants pour différentes cultures et méthodes. Des notions de leçons ont été étudiées pour identifier les choix d'élèves-enseignants irlandais et hollandais; (1) les aspects fondamentaux de la notion de surface; (2) les contextes de la vie réelle, les formes et les méthodes de manipulation choisies pour enseigner ce concept. Une configuration à deux dimensions s'est dégagée dans la manière d'aborder un sujet, permettant de classer les idées archétypales des élèves-enseignants en réalistes, structuralistes, mécanistes, ou bien empiristes.*

RESUMEN *Este estudio, centrado en los enfoques en la enseñanza de matemáticas, está orientado a identificar las estructuras curriculares y programas de estudios que resulten válidos a través de las diferentes culturas y los distintos sistemas educativos. Se estudiaron las propuestas de lecciones de maestros irlandeses y holandeses en formación inicial para identificar la selección de (1) los aspectos básicos del concepto de área; y (2) los contextos de la vida real, formas y objetos escogidos para enseñar este concepto. De este estudio surge una estructura bidimensional, que relaciona las ideas de los futuros maestros con los enfoques arquetípico realista, estructuralista, mecanístico o empírico.*

ZUSAMMENFASSUNG *Diese Studie zu Lehrmethoden der Mathematik hatte sich zum Ziel*

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gesetzt, ein Rahmenwerk von Lehrplänen zu identifizieren, die aufgrund ihrer Robustheit gleichzeitig auf andere Kulturen und Bildungssysteme angewendet werden können. Unterrichtsideen angehender irischer und niederländischer Lehrer wurden im Hinblick auf folgende Punkte untersucht: (1) das grundlegende Prinzip eines Flächenkonzepts und (2) lebensnahe Zusammenhänge, Formen und Anwendungsmaterialien, die zum Unterrichten dieses Konzepts gewählt wurden. Zum Vorschein kam ein zweidimensionales Rahmensystem, bei denen die Ideen zukünftiger Lehrer Methoden archetypischer Realisten, Strukturalisten, Mechanisten oder Empirikern vergleichbar waren.

1. The Context of the Study

In the last four decades, much debate has taken place about mathematics curriculum, and in particular about the philosophies of mathematics and mathematics education that underlie different curricula (Fehr & Bunt, 1961; Griffiths & Howson, 1974; van der Blij *et al.*, 1980; Hilding & Weinzweig, 1980; Howson, Keitel & Kilpatrick, 1981; Howson & Wilson, 1986; Ernest, 1991; Dossey, 1992; Schmidt *et al.*, 1997). The 'modern mathematics' approach, with its strong emphasis on mathematical structure, dominated the 1960s. It waned during the 1970s and, for some time, no one philosophy exercised as great an influence (Howson, 1991, p. 15). Latterly, however, pride of place has been given to 'realistic' mathematics education (Treffers, 1987; De Lange, 1987). Starting in The Netherlands around 1968 as a critical reaction to the 'modern mathematics' movement, especially in the USA, it has spread out via publications, international conferences and international projects. Its main characteristics—*mathematics as a human activity* and *the use of contexts as sources for learning mathematics*—can be recognised in reform movements in many countries, for example England and Wales (Cockcroft, 1982) and the USA (National Council of Teachers of Mathematics, 1989).

However, curricular intentions—and especially those dependent on espousal of a particular philosophy—do not always translate into classroom practice (Robitaille, 1980; Westbury, 1980; Travers & Westbury, 1989). Their implementation depends, *inter alia*, on the extent to which teachers share the views encapsulated in the curricula. In this respect, the convictions, ideas and beliefs of pre-service teachers (PTs) are of particular relevance. On the one hand, they may indicate the extent to which intended philosophy has been realised in the classrooms from which the PTs have come; on the other hand, they reflect the PTs' degree of readiness for conveying the intended spirit of the programmes to their own classes in the future.

The beliefs which PTs bring to their courses have been the subject of a long-running discussion in the literature (Cooney, 1985; Broekman & Weterings, 1987; Ernest, 1989; Thompson, 1992; Wubbels, 1992a; Cooney, Shealy & Arvold, 1998). In this paper, it is intended to continue the discussion, particularly with reference to the approaches to mathematics education taken by some PTs from two countries with very different curricular traditions. The data presented here were collected in The Netherlands and Ireland, using the 'lesson preparation' method already described by van der Valk and Broekman in this issue of the *European Journal of Teacher Education*. They were then analysed using the tool of *didactical phenomenology* due to Freudenthal (1983). The aim of the work is to look for frameworks that might be robust across different cultures and education systems in describing teachers' approaches to teaching mathematics, with a view to considering applications of such frameworks in teacher education courses.

Section 2 of the paper outlines different approaches to mathematics education, characterising them as empiricist, structuralist, mechanist or realistic (Treffers, 1987). Because of its current importance, particular attention is paid to the realistic approach. In Section 3, the focus is on the concept of 'area', the topic addressed by the PTs in the study. In the context of this topic, research questions are posed and a tentative framework for describing teachers' approaches is set up. Section 4 describes the methods of collecting data from the PTs, and of analysing the data. Results are presented in Section 5; answers to the research questions are offered, and the appropriateness of the suggested framework is examined. The findings are discussed in Section 6, and conclusions and recommendations for teacher education are offered in Section 7.

2. Approaches to Mathematics Education

A frequently used dichotomy in classifying mathematics curricula is that of 'empirical' vs 'formal' (Kilpatrick, 1992; Fischbein, 1994). Thus, some curricula have opted for an *empirical* approach (for example involving environmental activities, such as cutting and measuring—that is, operations other than mental ones). Other have stressed the *formal* aspects of mathematics: mathematics seen as a structure (a complete building consisting solely of axioms, definitions and theorems) or a system of rules. However, this dichotomy is not sufficient to characterise realistic mathematics education. Following Freudenthal (1968), the realistic approach stresses the idea of *mathematics as a human activity*. The most important part of that activity is 'organising' or 'mathematising'. This mathematising has two components. *Horizontal mathematising* concentrates on 'reality': for instance, finding examples from real life with similar mathematical structures. *Vertical mathematising* focuses on developing mathematical structures (Treffers, 1987; van den Heuvel-Panhuizen, 1996).

2.1 Four approaches to mathematics curriculum

With the two components of mathematics as a human activity, four kinds of curricula can be characterised (Streefland, 1991):

1. *Empiricist* curricula contain much mathematising, but most of it is of the horizontal kind.
2. In the *structuralist* approach (mathematics as a structure), almost all attention is devoted to vertical mathematising.
3. In *mechanist* curricula, hardly any mathematising can be found.
4. The *realistic* approach stresses both the horizontal and the vertical component of mathematising.

The horizontal and vertical mathematising in realistic mathematics arise from adapting to the students' point of view: what should students do to 'reinvent' mathematics? For this, *contexts*—in particular, contexts from real life—are essential. They can have three roles. The first role is *context as an area of application*: applying mathematics to real-life situations. By applying their mathematics, students are prepared to use it in out-of-school situations. The second role is that of *context as a source of mathematics*: exploring mathematics in real life. The third is *context as a tool or a support* for vertical mathematising: helping students to develop mathematical structures.

Contexts figure in the other curricula also. In empiricist curricula, it is assumed that the relevant concepts are present in the contexts chosen, to be discovered by the students doing activities. In structuralist and mechanist curricula, contexts have two functions: to apply the abstract mathematics and to 'decorate' it *(context as a decoration)*. Both are used chiefly for purposes of motivation. For the decoration function, in particular, contexts do not play an essential role and can be dropped without affecting the mathematics required. The following example provides an illustration.

> A triangular garden has two sides at right angles to each other. One of these sides is 3 m long; the other is 5 m long. Calculate the area of the garden.

A further point about structuralist and mechanist curricula can be made here. The mechanists see mathematics as a system of rules; these rules are told to the students, who are supposed to verify and apply them to problems similar to examples previously used by the teacher. It is a well-known phenomenon that curricula which were meant to be structuralist appear to have been used in mechanistic ways in the classroom: perhaps because teachers have found out that direct vertical mathematising is too hard for their students, and that it is easier to tell them the 'correct' rules (van der Blij, Hilding & Weinzweig, 1980; Howson & Wilson, 1986).

2.2 Didactical phenomenology

For realistic mathematics education, a major question is how to find 'didactical structures' of mathematical topics suitable for the age group concerned. For this, a 'didactical phenomenology' (Freudenthal, 1983) of the topic is needed. This implies not only a description of the mathematical structure of relevant concepts, but also an unveiling of didactical aspects of the topic: that is, students' thinking with respect to the topic and the relevance of the topic to real life

> *as a way to show the teacher the places where the learner might step into the learning process of mankind. Not in its history but in its learning process that still continues ... (Freudenthal, 1983, p. ix).*

Making a didactical phenomenology of a mathematical topic can be done (and should be done, according to realists) from two perspectives:

- a mathematical phenomenology;
- a real-life phenomenology.

The aim of making a *mathematical* phenomenology is to explain the mathematical structure of the topic, and to point out the main steps students will have to take and the difficulties they may face. The aim of making a *real-life* phenomenology of a topic is to show what structures in real-life situations may give rise to a need for the mathematical aspects and/or the related mathematical methods, may promote or deepen students' understanding of the mathematical concept, or may form a suitable field of application. This phenomenology can be used to map the real-life structures—those appearing in the contexts—to the mathematical structure of a topic, showing in what respect real-life structures correspond to, or are different from, mathematical structures. In the case of corresponding structures, the real-life structure can be used to introduce the concept to students. In the case of differences, students' difficulties can be foreseen. Contexts that reveal differences between real-life and mathematical structures may be used for vertical mathematising, thus solving the difficulty.

The didactical phenomenological method also applies to some of the other approaches to mathematics education, but to a limited extent in each case. *Structuralists* focus on developing mathematical structures in students, and so confine themselves to the mathematical phenomenology. *Mechanists* focus on transmitting the mathematical system of rules to students. As the mathematical rules are well known, for them there is no need for a phenomenology. When structuralists and mechanists use real-life contexts, they do so as applications or as 'decorations' of mathematics to help in motivating students, as described earlier. *Empiricists* focus on the many interesting and instructive examples of mathematics that can be found in real life. They do real-life phenomenology to select good contexts (contexts as sources) for horizontal mathematising. To make steps in the vertical direction, empiricists may do some mathematical phenomenology as well. However, the two phenomenologies are not connected; empiricists do not use contexts as a tool for vertical mathematising, for arousing in students a need for more mathematical structures. Connecting the two is one of the characteristics of the realistic mathematics movement. A second characteristic is the idea of 'mathematics as a human activity', as mentioned above. It results in a search for a diversity of student activities: doing drill and practice as well as handling manipulatives and carrying out small investigations.

In realistic mathematics education, therefore, didactical phenomenology is used for

> starting from those phenomena that beg to be organised and from that starting point teaching the learner to manipulate these means of organising. Didactical phenomenology is to be called in to develop plans to realise such an approach. (Freudenthal, 1983, p. 32)

When this is applied to a topic such as 'area', it means that—rather than starting from the mathematical concept of area and looking around for contexts to concretise it—one can start with contexts by looking first for phenomena that might compel the learner to construct the mental object that is being mathematised by the area concept. Realistic mathematics is also characterised by its student-centredness because of the way it emphasises students' own productions:

> There is a large contribution from the students themselves to the course by their (mental) productions and constructions, which lead them from their own informal methods to the more standard methods. Constructing, reflecting, anticipating and integrating are fundamental functions of students' own productions. (Gravemeijer, 1994)

3. Research Issues and Operationalisation for the Concept of Area

The concepts 'empiricist', 'structuralist', 'mechanist' and 'realistic', as described above, are constructed to categorise curricula. The aim of our study is to investigate whether PTs' approaches at the start of their teacher education can be described with the help of this categorisation. For the purposes of this study, the categorisation is operationalised in terms of the aspects of a concept—specifically in the study, the concept of 'area'—mentioned by PTs in their lesson plans, the 'exemplary situations' they used as contexts and the student activities which they introduced.

In our research, the PTs had to design a lesson on the concept of area without referring to a textbook; details are given in Section 4. The lesson plans could then be investigated to see whether the PTs aimed at vertical and/or horizontal mathematising and/or at relating the two. Paying attention to vertical mathematising was operationalised as using many basic aspects of area: hence, in fact, as making an implicit mathematical phenomenology of area. Paying attention to horizontal mathematising was operationalised as the use of many real-life contexts and also manipulatives (for example, cutting out cardboard triangles or handling cubic blocks); hence, as making an implicit real-life phenomenology of area. Another factor that might have been included is the use of mathematical shapes (such as drawn rectangles and tetrahedra). However, shapes can be interpreted as models that differ in role and character, depending on their use (Gravemeijer, 1994); hence, their use would not be expected to distinguish the approaches. A hypothesis to this effect is formulated in Section 3.2.

3.1 Mathematical phenomenology

Having studied aspects of the mathematical concept of area used by the Dutch PTs in their lesson plans, van der Valk & Broekman (1997) made a mathematical phenomenology of area, listing all the basic aspects of the concept. To explore these basic aspects, they discussed and reviewed mathematics textbooks on area. To determine the importance of the aspects to mathematics education, they also studied the literature on student thinking and student difficulties about area.

The following basic aspects of area were identified.

a. *General idea of area:*
   1. area is two-dimensional;
   2. the area of a shape is determined by the closed contour of the shape.
b. *Isometric transformation*:
   1. the area of a shape does not change when the shape is moved or canted;
   2. two shapes have the same area if surfaces fit;
   3. the area of a flat shape does not change when its surface is curved; the area of a curved shape does not change when its surface is flattened (e.g. the surface of a cylinder).
c. *Conservation of area (division and addition)*:
   1. the area of a shape equals the sum of the areas of the parts into which it can be divided;
   2. two shapes form a new shape when a part of the boundaries are put together; and the area of the new shape is the sum of the original areas.
d. *Calculation:* the value assigned to the area of a flat shape
   1. involves the choice of a unit of area;
   2. can be determined by counting or approximating the number of units covering the shape;
   3. is dependent on the length, width and configuration;
   4. can be calculated using a rule or formula (in words or abstract notation);
   5. is determined by the sum of small parts (towards a Riemann summation, resulting in an integral).
e. *The area of spatial shapes can be specified or approximated by unfolding onto a flat plane and determining the area of the resulting net:*
   1. spatial shapes with flat sides;
   2. infinite small parts of curved spatial shapes.

If students lack knowledge of aspects of area, or if they misunderstand the aspects, then they can experience conceptual difficulties. Many such difficulties have been described in the literature. Examples include those resulting from choosing a non-square unit (Simon, 1995), 'area growth' (Friedlander & Lappan, 1987), confusion between area and perimeter (Hart, 1981; Reinke, 1997) and even visualisation of 'one half' (Watanabe, 1996). Here, four main student difficulties regarding area are described and related to the basic aspects.

- A first main difficulty has to do with measuring area. Students can measure the area of a flat shape by covering it with (a grid of) unit squares. However, this activity, at first sight simple, is essentially a complicated one. Students have to have an idea of isometric transformation (aspect b, moving, fitting shapes) and of course of the need to choose a unit of area (aspect d.1). Moreover, knowledge of aspect —addition (adding all units together) and division (if some unit squares lie partly outside the contour)—is also essential for measuring.
- A second main difficulty is distinguishing area and surface. For making this distinction, it is necessary to know that area is a number, connected with a unit (aspect d), and that this number does not change when the surface is curved, moved or canted (aspect b.1). A student with this difficulty may think that the area of a cardboard rectangle, the sides of which have different colours, depends on which side is uppermost.
- A third main difficulty is distinguishing area and perimeter. This distinction needs understanding of the aspect a.2 (contour) and d (amount of area). Students with this problem may say that a square of 1 metre by 1 metre has an area of 2 metres (the sum of length and width) or 4 metres (the perimeter). Other students may suppose that shapes with the same perimeter have the same area and vice versa.
- A fourth difficulty concerns the surface area of spatial shapes, such as cubes. Students may think a cube does not have area, because they do not understand the addition aspect, needed for summing the area of the sides. Making a net may solve their problem. However, if they have no understanding of the isometric transformation aspect (canting the sides of the cube into a plane), they will see no relation between the net and the area of the cube.

These examples show that for 'manipulating with rules' (aspect d.4)—an important aim of education on area—there is a fundamental need for knowledge of some of the other basic aspects, in particular the isometric transformation and conservation aspects and the choice of a unit of area. It is worth noting that PTs who are very capable mathematicians may well find these aspects so obvious that they do not make them explicit to their students. Reflection on student difficulties can be expected to promote finding the basic aspects of area and vice versa.

Altogether, therefore, in order to teach the area concept well and to promote vertical mathematising (for example, by exploiting student difficulties), teachers must have a good understanding of the basic aspects of area. Focusing on many aspects of area may indicate an approach that stresses vertical mathematising. Thus, our first research question is:

> Question 1. *Which of the basic aspects of the concept of area were used by (most of) the PTs?*

3.2 Real-life phenomenology

Traditionally, most examples or exemplary situations offered to students are determined fully by mathematics itself. In modern education, these still have a place, of course, although no longer an exclusive one. With respect to area, the 'mathematical contexts' appear most clearly in the form of (drawn) standard shapes such as rectangles, triangles, circles or cubes.

In real life, area is a concept that is often used. Moreover, many of the conceptual structures of area in real life can be used in mathematics without any change; so most of the basic aspects mentioned above apply to the real-life concept of area. That is why, in realistic mathematics education, the mathematical concept of area is elicited and developed by many examples from real life. However, some specific contexts do not apply fully to mathematics. For example, in carpet shops, carpet is sold by the metre. However, a 'metre' of carpet from one bolt may yield more carpet than a 'metre' from another bolt, because the bolts may have different widths. This kind of context may result in a difficulty that can be used productively for vertical mathematising, to introduce students to a new basic aspect: in this case, the square metre as a better unit. In other cases, for example dealing with the area of curved surfaces, problems met are too complex at a certain stage, so they are not (yet) considered.

It should be noted that some real-life contexts for area have been used frequently in textbooks. A good example of a suitable context is paving a courtyard with square paving-stones. In this case a 'natural' unit (the paving-stone) is suggested, resulting in the number of paving-stones being the area of the courtyard. Other contexts are covering the floor with carpet and covering a wall with wallpaper.

As well as the mathematical and the real-life examples, there is a third category that combines aspects of contexts and of standard shapes. This involves manipulatives such as cardboard figures suitable for cutting and gluing, cut-outs of spatial shapes such as cubes, tetrahedra, cylinders, etc. On the one hand, these manipulatives often have the form of mathematical standard shapes. On the other hand, folding, cutting and gluing are very much real-life manipulations. Therefore, the use of manipulatives plays an important role in realistic mathematics, as it may do in other kinds of mathematics education.

The number of real-life contexts and manipulatives can give an indication as to whether or not, and to what extent, PTs tended to make a real-life phenomenology. Thus, in particular, the use of many manipulatives and contexts from everyday life may reveal a realistic way of mathematising. By contrast, as indicated earlier, the number of mathematical shapes used is not expected to distinguish between the approaches. So the following research question was asked:

Question 2. *How many real-life contexts and manipulatives were used by the PTs?*

The argument about shapes leads to the formation of a *hypothesis: the number of shapes used by a PT is not related to the number of contexts and manipulatives used.*

3.3 A framework for describing teachers' approaches

Measures derived in addressing the two research questions—the number of aspects, and the number of contexts and manipulatives, used by individual PTs—provide the two dimensions of a possible framework with which to categorise PTs' approaches to mathematics education. The framework can be represented as a graph in which, for a given PT, the *number of aspects* used in the lesson plan (taken as a measure of vertical mathematising) is plotted against the *number of contexts and manipulatives* used (taken as a measure of horizontal mathematising).

It was pointed out in Section 2 that the four different approaches can be characterised by their differing emphases on vertical and horizontal mathematising. According to operationalisation used in this framework, therefore, position on the graph describes the curricular approach taken by the PT. Thus, for example, a PT with a small 'horizontal' co-ordinate and a large 'vertical' co-ordinate is characterised as having a structuralist tendency, whereas a PT with a large 'horizontal' co-ordinate and a small 'vertical'

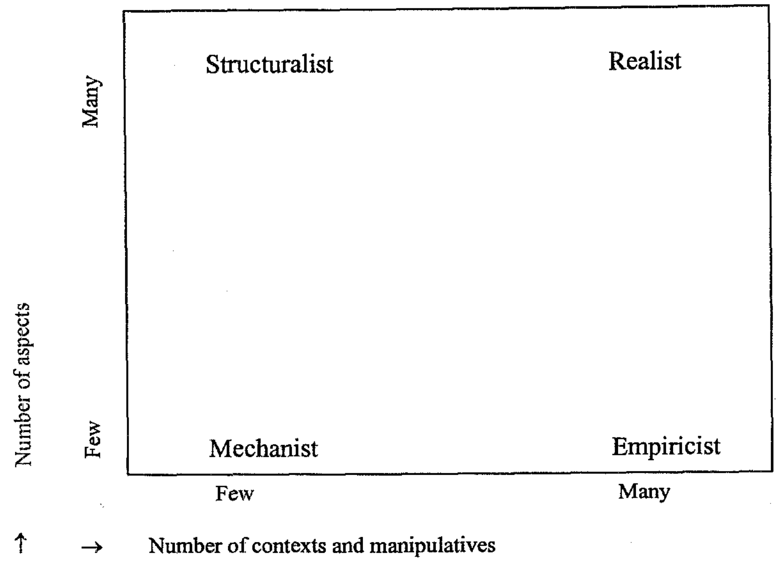

FIG. 1. The framework used in the study.

co-ordinate is characterised as having an empiricist tendency. The archetypal positions associated with the four different approaches are shown in Fig. 1.

Hence, a further issue for the research, directly addressing the chief aim of the paper, is to investigate the usefulness and applicability of the categorisation of PTs' views given by this framework.

4. Data Collection and Methodology

4.1 Subjects of the research

The empirical work described in this paper was carried out by members of the 'Oslo Mathematics Group', a group of teacher educators who first came together in 1995 at the ATEE Conference, held in Oslo (Berenson *et al.*, 1997). The group was interested in finding out what ideas—about mathematics and about teaching it—PTs brought with them into their teacher education courses. Each member therefore arranged for a small number of volunteering prospective teachers in his or her country (five or six in each country) to prepare a lesson, which they might teach to a mixed-ability group of upper primary or lower secondary students.

The volunteers were provided with many resources (but not school textbooks), and given an hour in which to plan their lesson; each volunteer then described his or her lesson to an interviewer or interviewers. This 'lesson preparation method', devised by the project group, is discussed in more detail elsewhere (see van der Valk & Broekman, 1999). A specific topic—the concept of area—was chosen as the subject of the lesson.

Data collected from two of the groups are used in this paper. The *Dutch* prospective teachers were mathematics students in their fourth year, just prior to graduation, and therefore had a fairly thorough understanding of mathematics. They had opted for a two-month orientation course in teacher education in order to find out whether they wanted to make a career in secondary teaching (Wubbels, 1992b). Most were in their early 20s; one was over 30 years of age. By contrast, the *Irish* students were in their first year of a bachelor's degree course that would qualify them as primary teachers, teaching children aged roughly 4–12 years. They were several years younger than their Dutch counterparts (the mean age being just 19), and—though they had taken mathematics throughout their schooldays—were not specialists in the subject. They had experienced one term of a mathematics teaching methods course, which focused on activity methods and the use of manipulatives, but had not undertaken any teaching practice (Oldham *et al.*, 1996). The groups are therefore distinct from one another in several respects.

The national curricula which they had experienced are also very different. As indicated earlier, the *Dutch* have been the pioneers in realistic mathematics education; some of the prospective teachers may have experienced such education in their schooldays, while others may have encountered more traditional approaches (Wubbels, Korthagen & Broekman, 1997). The *Irish* secondary curriculum, from which the young PTs had just emerged, was heavily influenced by the 'modern mathematics' movement of the 1960s. While no longer as dominated by algebraic structures as it was in the immediate aftermath of that time, it is still firmly structuralist in intention, though the implementations experienced by the Irish PTs may well have been more mechanistic (Travers & Westbury, 1989; Oldham, 1989, 1992). The philosophy underlying the Irish primary school curriculum also owes something to structuralist ideas, but perhaps more to theories of Piagetian developmentalism and discovery learning (Ireland, Department of Education, 1971), and is perhaps best described as empiricist with structuralist tendencies. Again, some implementations may have been more mechanistic.

4.2 Data collection and analysis

The data were collected in the manner agreed upon in the Oslo Mathematics Project, by requiring PTs to design a lesson about area and then interviewing them on the lesson (see van der Valk & Broekman, 1999). The PTs in the different countries were given the following task:

> *Make a plan for a forty-minute lesson about the concept of area, to be taught to a mixed-ability group of [specified grade level] students. You have one hour to plan the lesson, using whatever materials you need that are in the room. You will then be interviewed and asked some questions about your lesson.*

The specified grade level varied from country to country, depending on the local curriculum and the level at which the PT would ultimately be teaching; but the intended target audience was aged 11–13 years. The materials provided in the preparation room included those found in many mathematics classrooms: for example, a blackboard, an OHP, paper and scissors, grid transparencies, and spatial shapes such as cubes and tetrahedra. There was some variation between countries, reflecting the typical availability of different materials locally. However, in all cases, no textbook was supplied, in order to prevent the overruling of PTs' own ideas on teaching area by ideas taken from the book.

The Dutch and Irish data were analysed with respect to the two research questions posed above. With respect to research question 1—'*Which of the basic aspects of the concept of area were used by (most of) the PTs?*'—it was agreed that a PT would be deemed to have used the aspect if s/he either mentioned it when describing the lesson to the interviewers or referred to it in the subsequent discussion. The following transcript of one PT's (Lien's) description of her lesson illustrates such use of aspects:

> First, they will have to draw a flat rectangle, and then they will have to cut it perpendicularly in the middle. Then I would ask what is the area of the one half and of the other half. And then I would give them a rectangle and would have them cut it in the middle, so that they can experience as well that the parts fit on each other.

Lien used the isometric transformation aspect b.2 by making students fit the parts of the rectangle on each other. She did so for the sake of introducing the division aspect c.1: the division of a rectangle into two equal parts, both equal to half of the original rectangle. The use in this case is *explicit*. An example of *implicit* use of an aspect can be found, for instance, when the unit aspect (d.1) is used without introduction. This is illustrated by another excerpt from Lien's transcript as she explains a student difficulty:

> I think, for instance, that they have to realise if the area of ... with a grid you know, if that is 7 by 5 then the area is 35, and then exactly 35 pieces of 1 by 1 fit into it.

In this excerpt, the number-of-units aspect (d.2) is explicit, but the choice of unit (d.1) is implicit, so d.2 was regarded as a part of the mathematical phenomenology made by this PT, but d.1 was not. In analysing the data for each PT, the *number of aspects used explicitly* was counted.

With respect to research question 2—'*How many real-life contexts and manipulatives were used by the PTs?*'—it was agreed that a PT intended to use a real-life context if, in the prepared lesson plan, verbally or by drawings, s/he translated a real-life situation or object into mathematics, or applied the mathematics to a real-life situation. The following is an example of the use of real-life contexts by one of the PTs, Sheila:

> For the first 10 minutes of the class, I'd have conversations with the class about, you know, if they have had any experience of area and maybe somebody had carpet put in their house, and had heard the term square feet or square metres.

In this excerpt, the real-life context is "carpet in the house". Discussion took place as to whether "square feet or square metres" also referred to the "carpet" context; a decision was made that it did. Contexts that were used more than once were counted only once, except when a second or subsequent occurrence was used to introduce or apply a different aspect of area.

Similarly, agreements were reached on counting manipulatives. A manipulative that was intended to be used only by the teacher and not by the students (for example, the manipulative "net" in the following excerpt from another PT, Greet) was not included.

> I ... did not know for sure whether they know the net of a cube, for example, but if not, I want to show it. And make a drawing of the net of a cube.

To address the hypothesis posed in Section 3.2—*the number of shapes used by a PT is not related to the number of contexts and manipulatives used*—representations of shapes were also counted in the way described above. Different forms of the same standard shape, for example rectangles of different sizes, were counted as different shapes.

In order to check the reliability of the coding, each set of data was examined by two coders. One of the Dutch authors took the main responsibility, coding the Dutch and Irish data. Independently, the other Dutch and the Irish author coded the data from their own countries. Interpersonal reliabilities for the codings were calculated for the

TABLE I. Intersubjective reliability scores for the Dutch and the Irish groups

Reliability scores (%)				
	aspects	contexts	manipulatives	shapes
Dutch PTs	94	100	94	98
Irish PTs	89	92	89	94

number of sub-aspects, contexts, manipulatives and shapes (see Table I). Discussions took place to resolve discrepancies. In most cases, discrepancies appeared to be result of minor differences in interpretation of (say) an aspect, or of failure to notice use of (say) a shape. After discussion, consensus was reached in all cases.

The issue of validity was also addressed. It arises with regard to the operationalisation of vertical and horizontal mathematising in terms of number of aspects and number of contexts and manipulatives, respectively: hence, with regard to the framework provided for interpreting results as tendencies towards the four approaches to teaching mathematics. The transcripts of individual PTs were therefore examined critically, using protocol analysis (Klaassen & Lijnse, 1997), for evidence of views supporting or failing to support these interpretations.

5. Results

5.1 The research questions and the hypothesis

Results with respect to the use of individual aspects and their component sub-aspects, and the use of contexts, manipulative and shapes, classified by country, are given in Table II.

From this table, it can be seen that the answer to research question 1 is that the sub-aspects used by most (six or more) of the teachers are a.1, c.1, c.2, d.1, d.2, d.3 and d.4. The 'calculating' aspects (aspects d) are particularly well represented. Next come the division and addition aspects (c)—those dealing with the conservation of area—and also the basic idea that area is two-dimensional (a.1). Looking at the two national groups, it can be seen that the Dutch PTs made less use of a.1, d.1 and d.3; on the other hand, they were the only ones to mention b.3, and made much more reference to the aspects e. At the level of individuals, there was considerable variation. This is not shown in Table II, but some impression can be gained from Fig. 2 (see p. 36).

Table II also presents the answer to research question 2 regarding the extent of use of contexts and manipulatives. The PTs used on average 4.6 contexts (0 as a minimum to 9 as a maximum) and 3.5 manipulatives (0 as a minimum to 8 as a maximum). The ranges were very similar for the two national groups.

As regards the number of shapes, the mean is 10.5, and the variation is considerable (from 4 to 35). However, the hypothesis that the number of shapes used by a PT is not related to the number of contexts and manipulatives used must be rejected. On the contrary, there is a tendency for the number of shapes to be similar to the total number of contexts and manipulatives: in fact equal, plus or minus 3, with two exceptions where more shapes were used.

TABLE II. The number of sub-aspects, contexts, manipulatives and shapes used by the Irish and Dutch groups

Country	Aspects															manipul-	
(N = 5 + 5)	a1	a2	b1	b2	b3	c1	c2	d1	d2	d3	d4	d5	e1	e2	contexts	atives	shapes
Ireland	5	1	1	2	–	4	3	5	5	5	5	–	1	–	23 (0–9)	19 (0–7)	39 (4–11)
Netherlands	1	1	1	2	2	5	4	2	4	1	5	1	4	1	23 (0–9)	16 (1–8)	66 (7–35)

Note: a1 refers to sub-aspect a.1, etc. The numbers in parentheses give the range for the number of contexts, manipulatives and shapes used by individuals.

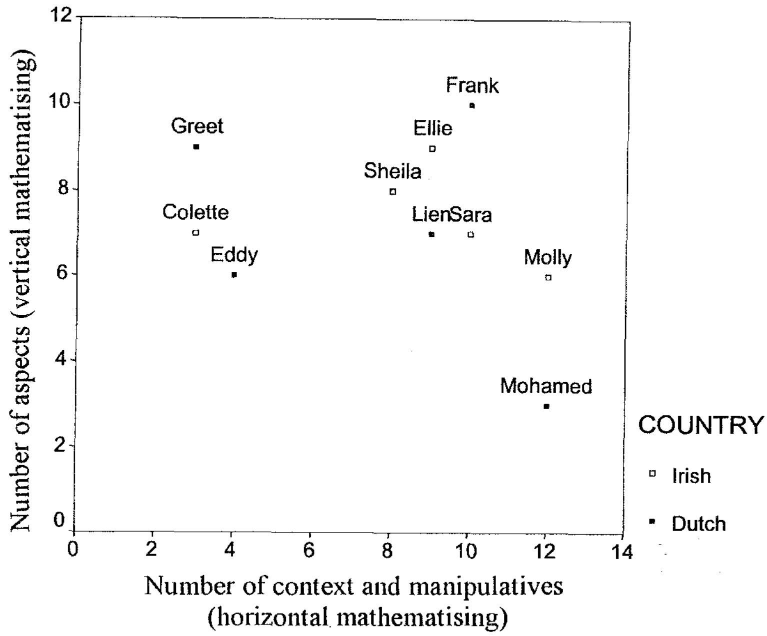

FIG. 2. The framework used in the study, showing individual PTs.

5.2 Framework for approaches to mathematics education

In Fig. 2, which illustrates the framework attempting to categorise the various approaches to teaching mathematics, each PT has been assigned a pair of co-ordinates: the number of aspects used by the PT is plotted against the total number of contexts and manipulatives that s/he used. Different symbols are used for the different nationalities. It is notable that the two national groups are differently distributed. The Dutch are widely dispersed, with three of them occupying 'extreme' positions. The Irish form more of a cluster (with one exception), and all lie within a comparatively narrow horizontal band.

The number of contexts and manipulatives (the horizontal co-ordinate in the graph) and the number of aspects (the vertical co-ordinate) are the operationalised measures of horizontal and vertical mathematising, respectively. Using this operationalisation, the closeness of a PT's point to the archetypal positions, as shown in Fig. 1 (Section 3), is taken as a measure of the PTs' tendency towards the four approaches to teaching mathematics. This interpretation was validated by analysing PTs' approaches as indicated by statements in the transcripts of the interviews, using protocol analysis. Results of the analysis for two PTs—Molly (with co-ordinates (12,6), suggesting a realistic tendency), and Greet (with co-ordinates (3,9) suggesting a structuralist tendency)—are given here as examples.

Pre-service teacher Molly (12,6) explained that she intended to start with a simplified definition "so they would understand it", and proceeded to count (unit) squares, making the students use pieces of squared paper as manipulatives. Next she had them measure and multiply lengths of sides of cut-out rectangles (hence, using the formula), counting squares as a means of checking; she then used formulae to find the area of other manipulatives, including cubes. At the end of the description she said: "So, I think that's really about it." She ended, noting:

> I'd get them to measure actual things in the classroom—like measure their desk, measure the table, a metre stick. Split them in groups and give each group something to measure. And then maybe if you have room in the classroom, measure the area of the floor, and so on.

So her approach was to introduce aspects using shapes and manipulatives as tools and sources. She used real-life contexts, but only as applications. She therefore missed one of the characteristics of the realistic approach: that of using real-life contexts as sources or tools. This is in accord with her position in Fig. 2, and supports the interpretation as "a tendency, but not a strong tendency, towards realistic mathematics education". It is likely that use of real-life contexts as sources or tools would have resulted in highlighting aspects she overlooked, such as the conservation aspect of area (for example, by using tiles to cover the ground).

Greet (3,9) worked very carefully with squares, rectangles, triangles, etc., and gave her students opportunities to master many aspects of area. However, she did this without using contexts or manipulatives, except, as she said, for less able students. In the interview, she talked about her—and university teachers'—worries regarding over-use of contexts in mathematics education:

> ... there are still many people with the idea that students are not well prepared with the mathematics education they get: in particular, not sufficiently prepared for study at university level in, say, the exact sciences. And they object to a move towards more contexts. Maybe more in later years with physics examples, or something, but no contexts on behalf of "just using contexts."

It thus appears that she sees a function for contexts only as applications. This supports the interpretation as "a tendency towards structuralist mathematics education".

Altogether, therefore, it can be concluded that the tendencies towards the approaches, as indicated by Fig. 2, are supported by these two examples. This is true also for the other results of protocol analysis, not presented here.

6. Discussion

The variables used in this study—the number of aspects, contexts, manipulatives and shapes—were sufficiently well defined and resulted in high reliability scores. This occurred in spite of language and communication problems (owing to the multilingual nature of the study), and of the fact that different student age groups and different education systems were involved. We conclude that the quality of data was reasonable.

The validity of the operationalisation of horizontal mathematisation as the number of contexts and manipulatives was affirmed by protocol analysis. However, some critical remarks can be made. In particular, they address the function of real-life contexts. As pointed out in Section 2, these can be used in all four approaches to mathematics education, but in different ways. The 'decoration' function of contexts described in Section 2 is typical of mechanist and structuralist approaches, and is purely motivational. Inclusion of such contexts may be inappropriate when using the number of contexts as a measure of horizontal mathematising.

In fact, it could be argued that only contexts and manipulatives used as tools or sources should be included in the analysis (and the same might be said for shapes). This would emphasise the difference between realistic and other approaches. However, the tool/source function relates contexts to the introduction of new aspects of a concept such as area, hence to vertical mathematising. Using many real-life contexts (as well as many manipulatives) related to horizontal mathematising is needed, from a realistic view, to supply students with a firm base on which to build the vertical structures. It can be argued that shapes can be used as a source or tool and as a means for horizontal mathematising as well. This is supported by one of the results of this study: for the PTs considered here, the number of shapes was related to the number of contexts and manipulatives.

No great differences were found between the Irish and Dutch with respect to either the number of aspects or the number of contexts and manipulatives. This is surprising, as the groups differ in age and mathematical maturity as well as with regard to educational level at which they plan to teach. However, patterns can be observed by looking at the *kind of aspects* used. (Differences in the kinds of contexts and manipulatives were not investigated in this study.) For both groups, the 'calculating aspects' (aspects d.)—in particular d.2 (counting unit squares) and d.4 (the formula), giving a numerical value for area—are well represented. This suggests that making calculations is seen by both groups as the core of the area topic. The high scores for the dimension aspect (a.1), the division and addition aspects (c), the unit aspect (d.1) and the length, width and configuration aspect (d.3) suggest that the PTs wanted to build up the concept from the basics. This holds in particular for the pre-service primary teachers (the Irish). The Dutch (secondary) PTs tended to overlook the basic aspects such as a.1, d.1 and d.3. Two Dutch PTs who were more inclined to go back to the basics thought of the aspects a.2 and/or b as well. Most Dutch PTs also included e.1. The differences in national patterns may stem variously from the Dutch PTs' greater mathematical knowledge and greater distance from their schooldays (perhaps leading them to include some aspects but to forget the need for others), and/or from the fact that they were targeting their lessons at rather more advanced students than were the Irish PTs (that is, at the upper rather than at the lower end of the specified age range). It might also suggest that the experience of the Irish students in their schooldays was more uniform than that of the Dutch.

Underlying the latter possibility is the assumption that PTs tend to 'recycle' the mathematical attitudes and educational conceptions of their former mathematics teachers (Galbraith in Haggarty, 1995, p. 7). The curricular backgrounds of the two groups of students were discussed in Section 4.1. In the light of that discussion, the categorisation of the PTs' approaches to teaching mathematics, as illustrated by Fig. 2, is illuminating. The position of the Irish students may reflect the structuralist or mechanist influences of their recent schooldays, together with the empiricist tendencies of their primary school curricula and their teacher education course. The Dutch tendency to occupy the more 'extreme' positions (nearest to the regions indicating archetypal views) may reflect exposure to widely differing curricular traditions in their schooldays. Also, or alternatively, it may be a result of more reflected and explicitly held opinions, stemming from their greater maturity in terms of both chronological age and mathematical experience.

With respect to one of the main concerns of this article, Fig. 2 indicates a tendency towards the realistic position amongst several of the PTs. However, in some cases the tendency may be quite weak, since high scores on both axes do not guarantee the *linking* of horizontal and vertical mathematising that is a feature of realistic mathematics. Some PTs show structuralist tendencies; there are few PTs close to the empiricist or mechanist positions.

7. Conclusions and Recommendations

It was pointed out in Section 1 that, in examining the ideas that PTs bring to their teacher education courses, the authors have aimed to identify frameworks that might be robust across cultures and education systems, with a view to applying these frameworks within teacher education courses. Different frameworks may serve different purposes (see, for example, Pirie & Schwarzenberger, 1988). The Oslo Mathematics Group has already developed a framework with regard to PTs' knowledge (Berenson *et al.*, 1997) and is working on one for pedagogical content knowledge (Oldham *et al.*, 1998). This paper continues the group's endeavour with reference to a framework for describing the curricular approaches—realistic, structuralist, empiricist and mechanist—that are used by the PTs.

The earlier parts of the paper have concentrated on development of the framework. In Section 2, the four curricular approaches were outlined, in particular with regard to the way in which they can be characterised by their use of vertical and/or horizontal mathematising. Section 3 then operationalised the ideas in the context of the concept of area, setting up a framework with two dimensions—the number of aspects of area, and the number of contexts and manipulatives, used in the lesson plan on area—in order to describe the PTs' approaches. In the later sections of the paper, the framework is tested by applying it to two small groups of PTs who might be expected, from their backgrounds, to be influenced by different approaches to mathematics. While the results from such a small study must be treated with caution, especially as there is no guarantee that the national groups are representative of their countries, two findings are worthy of note:

- the two groups of PTs are characterised in distinctive ways broadly consistent with their curricular backgrounds;
- moreover, the characterisation of individuals is validated by protocol analysis of the transcripts of their lesson plans.

In the course of setting up the framework, the aspects of area, and also the contexts, manipulatives and shapes, used by the PTs in their lesson plans were examined. Several patterns of interest, in particular the PTs' tendency to concentrate on the 'calculating' aspects (assigning a numerical value to area), were noted. More generally, the value of making a didactical phenomenology of the area concept was manifest.

It remains to consider recommendations for further research and applications to teacher education courses. As pointed out above, numbers in the present study are small. An obvious extension to the work is analysis of the remaining data (from Sweden and the USA) available to the Oslo Mathematics Group, to see if the framework is robust across other education systems. Further, an examination of the different functions of contexts—source/tool, application and decoration—is needed, as well as an analysis of the role of shapes and manipulatives. An investigation of the kinds of contexts and manipulatives used, such as that carried out for aspects, might give a clearer picture of the similarities and differences with respect to primary and secondary PTs. Moreover, the entire study might be replicated, *mutatis mutandis*, for different topics. It would also be useful to extend the study to PTs at the end of their courses and to in-service teachers.

As regards application in teacher education courses, the methodology used in this paper—transcribing discussions of lesson plans and mapping the results on to a framework—could be used with pre-service or in-service teachers as one means of helping them to identify and develop their philosophies of mathematics education. The framework may be particularly useful in helping to find an answer to a typical teacher educator's question: "are there indicators in a (say) PT's lesson plan (plus interview), with respect to horizontal and vertical mathematising, that can be used as starting points to educate this particular PT in the direction of being a realist?" Alternatively or in addition, excerpts from the transcripts on which this study is based could be used as exemplar material to promote discussion and reflection (as has been done with earlier work of the Oslo Mathematics Group (Oldham *et al.*, 1998)).

A major concern for pre-service teacher education is that of 'recycling' attitudes and approaches, as mentioned in Section 6, for it appears very resistant to change. Perhaps a start can be made in breaking this 'recycling' by such activities. Thus, challenges formulated by Haggarty can be faced:

> to help each [PT] to become aware of her or his own thinking and agenda so that she or he cannot only question their existing beliefs but also so that each can provide an agenda for their own learning; and creating experiences which help all [PTs] to recognise the importance of issues which might not be part of this agenda but which are considered essential by the tutor and mentors. (Haggarty, 1995, p. 108)

In this way, PTs may be enabled to implement the spirit of new or reformed curricula, in particular those based on a 'realistic' philosophy, and a growing number of students may experience realistic mathematics education in their schools.

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