DIRK DE BOCK[1,2], WIM VAN DOOREN[1,3], DIRK JANSSENS[4] and LIEVEN VERSCHAFFEL[1]

IMPROPER USE OF LINEAR REASONING: AN IN-DEPTH STUDY OF THE NATURE AND THE IRRESISTIBILITY OF SECONDARY SCHOOL STUDENTS' ERRORS

ABSTRACT. Several recent ascertaining studies revealed a deep-rooted and almost irresistible tendency among 12–16-year old students to improperly apply the linear or proportional model in word problems involving lengths, areas and volumes of similar plane figures and solids. While these previous studies showed to what extent students' improper use of linear reasoning is affected by different characteristics of the task, it remained largely unclear what aspects of their knowledge base are responsible for the occurrence and strength of this phenomenon and how these aspects relate to other more general misconceptions and buggy rules identified in the literature. This paper reports an in-depth investigation by means of individual semi-standardised interviews aimed at analysing the thinking process underlying students' improper linear reasoning and how this process is affected by their mathematical conceptions, beliefs and habits. During these interviews, students' solution processes were revealed through a number of well-specified questions by the interviewer with respect to one single non-linear application problem, as well as through their reactions to subsequent kinds of cognitive conflict. The interviews provided a lot of information about the actual process of problem solving from students falling into the 'linearity trap' and the mechanism behind it. Although some students seem to really 'believe' that quantities are always linked proportionally, their improper use of linearity often results from superficial and intuitive reasoning, influenced by specific mathematical conceptions, habits and beliefs leading to a deficient modelling process.

KEY WORDS: illusion of linearity, length and area, misconception, ratio and proportion, similarity

1. Introduction

Linearity (or proportionality[1]) is, from a long way back, a key concept in mathematics and science education from elementary school to university. Both from a psychological and a mathematical point of view, the idea of linearity comes first. Rouche argues that because of their simplicity, linear functions immediately appear in human's mind ("C'est l'idée de proportionnalité qui vient d'abord à l'esprit, parce qu'il n'y a sans doute pas de fonctions plus simples que les linéaires", Rouche, 1989, p. 17). In its longitudinal development the linearity concept appears in many shapes:

Educational Studies in Mathematics **50:** 311–334, 2002.

from the old-fashioned 'rule of three' in primary school, to the idea of linear models and approximations in calculus and statistics at the secondary level, and to the abstraction in a vector space sense in university courses.

However, the reinforcement of linearity at numerous occasions in school mathematics, along with its intrinsic simplicity and self-evidence, may lead to a tendency in students – and even in adults – to see and apply the linear model 'everywhere'. As formulated by Freudenthal (1983, p. 267): "Linearity is such a suggestive property of relations that one readily yields to the seduction to deal with each numerical relation as though it were linear." The misuse of linearity in non-linear situations – sometimes referred to as the 'illusion of linearity' (or proportionality) – is a 'classical' misconception, probably one of the oldest in the literature of mathematical thought. The most famous and most often quoted example is the duplication of a square in Plato's dialogue *Meno*, in which a slave is asked to double the area of a given square and firstly proposes to double its side. The slave spontaneously applies the idea of linear proportionality (between length and area) and changes his mind only when Socrates helps him in diagnosing and correcting the error in his reasoning by confronting him with a drawing. Another well-known historical example is Chevalier de Méré's faulty linear approach to a dice game problem (see, e.g., Freudenthal, 1973). By experience, de Méré knew the advantage of betting on at least one six in 4 throws of a single die and he incorrectly deduced that it must be equally advantageous to bet on at least one double six in 24 throws of a pair of dice, because $\frac{4}{6} = \frac{24}{36}$. Later on, because he experienced that, notwithstanding his reasoning, bets on the latter event did not yield the financial gain he had hoped for, he brought this problem to the attention of Pascal and de Fermat which led these two mathematicians to work out the basics of probability theory.

As the concept of linearity itself, the misuse of linearity has many faces: it has been found at different age levels and in a variety of mathematical domains (see, e.g., De Bock et al., 1999). In elementary arithmetic, the phenomenon of improper proportional reasoning is often related to a 'lack of sense-making' in the mathematics classroom (Gagatsis, 1998; Greer, 1993; Nesher, 1996; Verschaffel et al., 1994, 2000; Wyndhamn and Säljö, 1997). When confronted with so-called 'pseudoproportionality problems' (such as, e.g. "It takes 15 minutes to dry 1 shirt outside on a clothesline. How long will it take to dry 3 shirts outside?"), many students give answers based on direct proportionality (i.e., tripling the drying time because the number of shirts is tripled). Also in cases where modelling with direct proportionality at best offers a very rough approximation (such as, e.g., in the runner problem: "John's best time to run 100 metres is 17 seconds. How long will it take him to run 1 kilometre?"), only very few students appear to show awareness that direct proportionality will give only an approximate answer. In secondary education, 'linearity errors' are often reported in the fields of algebra and (pre)calculus. Students tend to overgeneralise what has been experienced as 'true' for linear functions to non-linear functions (e.g. "the square root of a sum is the sum of the square roots" or "the logarithm of a multiple is the multiple of the logarithm"). This type of systematic errors has been discussed and illustrated by Berté (1987, 1993), Gagatsis and Kyriakides (2000) and Matz (1982). According to Matz (1992), these linearity errors result from students' overgeneralisation of the distributive law. The immense number of occasions wherein students add and use the distributive law in arithmetic and early algebra is very likely to reinforce students' acceptance of linearity. Students' excessive adherence to linearity has not only been observed in algebraic but also in graphical environments (for an overview, see Leinhardt et al., 1990). For example, Markovits et al. (1986) discovered that 14–15-year old students who were asked to generate examples of functions or to draw graphs of functions passing through given points stuck to linear functional thinking.

The best-known example of students' improper use of linearity, issued from the domain of geometry and measurement, is students' improper application of linearity in problems about the relationships between the lengths and the area and/or volume of similarly enlarged or reduced figures (De Bock et al., 1998). As the slave in Plato's *Meno*, students of different educational levels strongly tend to generalise changes in linear dimensions to changes in area and volume. In responding to questions about the effect of halving or doubling the sides of a figure to produce a similar figure, most students – and even prospective teachers – claim that the area and volume will be halved or doubled too (National Council of Teachers of Mathematics, 1989; Outhred and Mitchelmore, 2000; Simon and Blume, 1994; Tierney et al., 1990). Gaining insight in the quadratic, respectively, cubic growth rates of areas and volumes, appears to be a slow and difficult process, and, therefore, it deserves our close attention, both from a phenomenological and a didactical point of view. According to Freudenthal (1983, p. 401), "this principle deserves, as far as the moment of constitution and the stress are concerned, priority above algorithmic computations and applications of formulae because it deepens the insight and the rich context in the naive, scientific, and social reality where it operates."

Recently, several studies have shown that – in the context of enlargements or reductions of plane figures and solids – students' improper proportional reasoning is a widespread and almost irresistible tendency among students (see, e.g., De Bock et al., 1998, 2002). In these studies, large groups of 12–16-year old students were administered (under different experimental conditions) a written test consisting of proportional and non-proportional word problems about lengths, areas and/or volumes of different types of regular and irregular figures. The next problem is an example of a non-proportional item about the enlargement of a square: "Farmer Carl needs approximately 8 hours to manure a square piece of land with a side of 200 m. How many hours would he need to manure a square piece of land with a side of 600 m?". The majority of the students in these studies failed on the non-proportional problems because of their alarmingly strong tendency to apply proportional reasoning 'everywhere'. Even with considerable support (such as drawings, metacognitive stimuli calling students' attention to the problematic character of the word problems, or embedding the problems in an authentic problem context), only very few students appeared to make the shift to the correct non-proportional reasoning.

Despite our rather extensive empirical data base about the phenomenon of improper proportional reasoning in this domain, the research method used so far, namely administering collective tests to large groups of students under different experimental conditions, did not yield adequate information on the *problem-solving processes* underlying improper proportional responses. Moreover, it remained largely unclear what aspects of students' knowledge base were responsible for the occurrence and strength of this phenomenon and how these aspects relate to other more general misconceptions and buggy rules identified in the literature. Therefore, we made a shift in our methodology by doing in-depth interviews with individual students who fall into the 'proportionality trap'. The interview study that will be reported here was preceded by a pilot study aimed at the development, try-out and definitive design of the different aspects of this interview technique (see De Bock et al., 2001).

2. Method

Twenty seventh graders (12–13-year olds) and twenty tenth graders (15–16-year olds) participated in the interview study. The participants came from a boarding-school located in a medium-sized Flemish town and were equally divided over most of the study streams of general secondary education. The number of boys and girls in the school as well as in our sample was more or less the same. All interviews were audio-taped and the interviewees had some pieces of paper, writing materials, a ruler and a scientific calculator at their disposal. The interviews consisted of five phases: after a short introduction, the interviewee was confronted with a non-proportional problem (Phase 1), followed by four subsequent forms of help to solve that problem (Phases 2, 3, 4, and 5). The interview stopped in the phase wherein the students discovered the non-linear nature of the problem and gave the correct response. If no form of help proved to be successful, the interview stopped at the end of Phase 5. The forms of help provided in Phases 2 to 5 aimed at eliciting a cognitive conflict in students who fell into the 'linearity trap'. Eliciting a cognitive conflict is a well-known method to create cognitive disequilibrium in a learner and to lead him or her to the discovery and development of new ideas (Forman and Cazden, 1985; Limón, 2001). In the present study, the cognitive conflict was evoked by presenting parts of the problem-solving protocol of a fictitious peer who proposes a (correct) non-linear solution to the problem. From Phase 2 to 5 this cognitive conflict in the interviewee was gradually increased by providing more and more evidence for the non-linear solution. As will be explained below, this procedure of stepwise bridging the distance between the linear and the non-linear solution originated partly from a rational analysis of the different obstacles students could encounter, and partly from specific hypotheses stemming from previous research findings (De Bock et al., 1998). This procedure was refined by the experiences in our pilot study (De Bock et al., 2001). For the sake of clearness, the main purpose of the cycle of cognitive conflicts used in the interviews was to unravel students' thinking processes by ascertaining how they reacted to certain kinds of help, and was thus not meant as a didactical trajectory. Preventing or remedying students for falling into the 'linearity trap' would require another approach (see, e.g., the pioneering work in this field by scholars of the Freudenthal Institute like 'Gulliver' in Treffers, 1987, or 'With the giant's regards' in Streefland, 1984), but the development and/or evaluation of such an instructional unit was not the focus of the present study.

Technically speaking, the interviews can be characterised as adaptive and semi-standardised: the interviewer-researcher followed a pre-determined scheme for the global development of the interview, asked specific standardised questions, but left enough room for spontaneous reactions of the interviewee and tried to respond to these reactions in a flexible way (Ginsburg et al., 1982).

We describe the five interview phases in more detail. A summary of each phase is given in Figure 1. The interview started with a standardised *Introduction* explaining the student that the interview was part of a research project on mathematical problem solving. Therefore, he would be confronted with one single problem. The interviewer stressed that, in every phase of the interview, the student was free to revise his answer whenever he felt this was necessary, since the interviewer was only interested in whether the student could give the correct answer *by the end* of the interview and in how this final answer was obtained. This aspect of the ‘experimental contract’ (Greer, 1997) was crucial, because we knew from our preliminary study (De Bock et al., 2001) as well as from other similar studies (e.g. Verschaffel et al., 1997) that some students tend to react in a self-defensive way and therefore try to withhold their original answer, even when they realise at a certain moment that this answer is untenable. So, we made every possible effort to create an experimental ‘climate’ in which a student would not resist changing his solution as soon as he thought or realised it was incorrect.

In *Phase 1* of the interview, the student received one non-proportional word problem about the enlargement of an irregular two-dimensional figure. To guarantee an appropriate and uniform interpretation, the problem was accompanied by a drawing of the original and enlarged figure. Previous research had shown that the vast majority of 12–16-year old students solve problems about the area of enlarged irregular figures in a proportional way, even when these problems are accompanied by ready-made drawings (De Bock et al., 1998). To prevent that students would get inside information about the problem and/or its correct answer from their classmates being interviewed earlier, we administered four isomorphic versions of the same problem, one of which is given in Figure 1. With respect to the choice of numbers in the problem, we avoided both too ‘easy’ numbers (which might make students suspicious about the nature and goal of the task, as documented in our pilot study, De Bock et al., 2001) and too ‘difficult’ numbers (which might necessitate students to pay too much attention to the purely computational aspects of the problem-solving process). The students were first asked to read the problem aloud and to ‘think aloud’ (Ginsburg et al., 1982) while solving it. At the moment when the student finished his first reading of the problem, a chronometer was started to measure the response time. When the thinking-aloud protocol did not yield sufficient information about the student’s thinking process, the student was asked to explain how his answer was found. Then, the student was asked to indicate how sure he was about the correctness of that answer, by choosing position on a five-point scale (from ‘certainly wrong’ to ‘certainly correct’). When a student did not indicate ‘certainly correct’, the interviewer asked why he was not absolutely sure, if there was anything that did arouse doubts and if he had considered alternatives. At the end of Phase 1, the student was asked to explain *why* he thought the problem had to be solved in that way. When a student could not justify his answer, the interviewer made use of a ‘teaser’ consisting of a nonsensical additive solution for the problem and asking the student why his solution was better than this one (referring to the problem given in Figure 1, the ‘teaser’ was: “The second

Father Christmas is 168 cm – 56 cm = 112 cm higher, thus Bart will need 112 ml more paint").

In *Phase 2*, we tried to raise a first, weak form of cognitive conflict in the students who had solved the problem incorrectly by means of proportional reasoning. This conflict aimed at testing the hypothesis that these students were simply inattentive in the first phase and to enable them to discover as yet the correct solution. It was realised by confronting the student with a manipulated frequency table presenting an overview of the answers given by a group of fictitious peers (see also Figure 1). The frequency table contained two major answer categories. For the Father Christmas problem in Figure 1, for instance, it indicated that 43% of the peers answered 18 ml (which is the incorrect, linear answer given by the student himself), but another 43% answered 54 ml (which is the correct, non-linear answer). The remaining 14% in this frequency table gave other answers or could not give any answer at all. At first, no questions were asked and we looked if the student spontaneously searched for the origin of the equally popular alternative. If the frequency table did not elicit a spontaneous reaction, the interviewer asked if the student had any idea where the alternative non-linear answer did come from and if this alternative did raise some 'seeds of doubt' about his initial answer. Finally, the student was asked which answer he preferred: the initial answer or the alternative that emerged in the peer group. After the student made his decision, the interviewer once more asked for a justification.

For the students who did not abandon their initial linear answer at the end of the second phase, a stronger conflict was elicited in *Phase 3*. In this phase, we gave the argumentation of a fictitious peer from the 43% who answered the problem correctly. For the example listed in Figure 1, the following argumentation was given: "One student told me that if the Father Christmas becomes three times as high while keeping the same shape, not only his height is multiplied by 3, also the *width* has to be multiplied by 3, so that you have to multiply the amount of paint by 9"). Moreover, the calculation '9 × 6 ml' was written down next to the answer '54 ml' in the frequency table. So, although the arithmetic operation underlying the non-linear answer was uncovered, no reference was made yet to the concept of area. If the student did not react spontaneously, the interviewer asked if the argumentation of the peer did (not) raise doubts about his initial answer. Finally, the student was invited once again to indicate his preference between the linear and the non-linear answer. Students who did not exchange their original linear answer for the correct non-linear one, went to Phase 4.

Phase 1
Solving a word problem

Bart is publicity painter. In the last few days, he had to paint Christmas decorations on several store windows. Yesterday, he made a drawing of a 56 cm high Father Christmas on the door of a bakery. He needed 6 ml paint. Now he is asked to make an enlarged version of the same drawing on a supermarket window. This copy should be 168 cm high. How much paint will Bart approximately need to do this?

Bakery's door Supermarket window

Phase 2
Weak form of cognitive conflict

"Last week, we gave this problem to pupils in another school …"

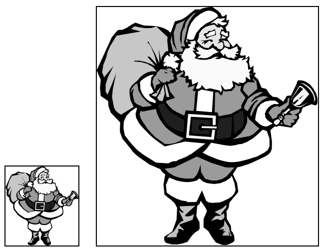
Answer	*Number of pupils*
18 ml	43 %
54 ml	43 %
Other answers	14 %

Phase 3
Argumentation of fictitious peer

"One student told me that if the Father Christmas becomes three times as high while keeping the same shape, not only its height is multiplied by 3, but also the width has to be multiplied by 3, so that you have to multiply the amount of paint by 9.
9×6 ml = 54 ml"

Phase 4
Solution process of fictitious peer

"One student drew rectangles around the Fathers Christmas …"



Bakery's door Supermarket window

Phase 5
Explicit link with area measurement

- "Can you calculate the area of the two rectangles?"
- "How much larger is the area of the large rectangle compared to the small one?"
- "How much larger is the area of the large Father Christmas compared to the small one?"
- "How much more paint do you need to paint the large Father Christmas?"

Figure 1. Summary of the interventions in each phase of the interview.

In *Phase 4* an even stronger cognitive conflict was installed by demonstrating (visually) the reasoning behind the argumentation of the fictitious peer. Therefore, the interviewer showed again the two Fathers Christmas of Figure 1, but now inscribed in exactly fitting rectangles (see Figure 1). He explained that the peer who multiplied by 9 drew the rectangles around the Fathers Christmas and then saw that it enlarges 3 times in both dimensions. So, the amount of paint needed for the big Father Christmas is $3 \times 3 = 9$

times the amount needed for the small one. This intervention was inspired by previous studies in which students more easily discovered the two-dimensional impact of an enlargement in regular than in irregular figures (De Bock et al., 1998). Once more, after leaving room for spontaneous reactions, the interviewer asked if the solution strategy of the peer did (not) raise doubts about his initial answer. Finally, the interviewee was invited to indicate his preferential answer. If this preferential answer was kept to be the incorrect linear one, the student arrived in the last interview phase.

The strongest cognitive conflict was installed in *Phase 5* wherein for the first time we created an explicit link with the concept of area of regular and irregular figures. The student was consecutively asked (1) to calculate the area of the rectangles wherein the two Fathers Christmas were inscribed, (2) to compare the area of these two rectangles, (3) to compare the area of the two inscribed Fathers Christmas, and (4) to compare the amounts of paint needed for painting them (for the exact phrasings, see Figure 1). If this fourth scaffold proved to be ineffective, the interviewer asked to compare the amounts of paint needed for painting the two rectangles. Because every step in Phase 5 could be helpful, the interviewer left room for spontaneous reactions after each step. At the end, the interviewee stated his definite preference and whatever this answer was, the interview stopped here.

3. Results

Table I presents the number of students who chose the correct answer in each phase. A detailed overview of the answers of the different students in each phase of the interview is given in the Appendix. As expected, the tendency to give a linear answer was strongly present in both age groups. Initially, only two students, both 15–16-year olds, solved the problem correctly. The other thirty-eight students gave an erroneous, linear answer in Phase 1 of the interview. The subsequent cognitive conflicts of Phases 2 to 5 proved to be effective for, respectively, one, seven, five, and three 12–13-year olds and for, respectively, one, seven, four, and two 15–16-year olds. By the end of the interview, four students of both age groups had not exchanged their wrong, linear answer for the correct, non-linear one. We now look at each phase in more detail, particularly from a qualitative point of view. The codes between squared brackets are individual student codes and refer to the overview in the Appendix. Differences between the two age groups will only be described if relevant.

TABLE I

Absolute and cumulative number of students who chose the correct answer in each phase

Age group	*N*	Phase 1	Phase 2	Phase 3	Phase 4	Phase 5
12–13-year ($N = 20$)	Absolute	0	1	7	5	3
	Cumulative	0	1	8	13	16
15–16-year ($N = 20$)	Absolute	2	1	7	4	2
	Cumulative	2	3	10	14	16
Total ($N = 40$)	Absolute	2	2	14	9	5
	Cumulative	2	4	18	27	32

3.1. *Phase 1: Solving the word problem*

The mean response time after the first reading of the word problem was 98.18 seconds, with a standard deviation of 73.20 seconds. Without the response times of the two 15–16-year olds who gave the correct non-linear answer in Phase 1 (125 and 130 seconds), the mean and standard deviation were still, respectively 96.56 and 74.91 seconds. This mean is rather high, considering that most students used very simple and straightforward proportional calculations. It is, however, considerably raised by the response times of a few students, which also explains the high standard deviation. These students were having difficulties with manipulating the available calculator or had to re-read the problem because they had not written down nor memorised the relevant numerical data. None of the forty students made a drawing or any other kind of external representation involving more than writing down the three given numbers.

Except the two 15–16-year olds who gave the correct non-linear answer in this phase, all students calculated the amount of paint needed in a proportional way. Only one of these 38 students expressed some doubts about this linear approach ("You need more data, for instance the width" [10–19]), but this did not affect his solution. Strategies for finding the linear solution were, in order of importance, the internal ratio strategy (i.e. using the ratio between the heights of the Fathers Christmas: 168 ml/56 ml = 3, thus 3×6 ml = 18 ml), the 'rule of three' (i.e. via the amount of paint needed to paint 1 cm of the Father Christmas: 6 ml of paint for 56 cm, thus 0.107 ml for 1 cm, thus 0.107×168 ml = 18 ml for 168 cm) and the external ratio strategy (i.e. using the ratio between the amount of paint and the height in one Father Christmas: 6 ml/56 cm = 0.107 ml/cm, thus 0.107 ml/cm $\times$ 168 cm = 18 ml).

Most students were very sure or quite sure they gave a correct answer. On the five-point scale, 20 students indicated to be 'certainly correct', 16 students to be ‘probably correct’ and the remaining 4 students to ‘have no idea’. Remarkably, the two 15–16-year olds who gave the correct non-linear answer in Phase 1, indicated to be ‘probably correct’ [10–17] or to ‘have no idea’ [10–20]. The choices of all individual students on this scale are given in the Appendix. Typical justifications of the students who indicated ‘certainly correct’ (while actually having answered incorrectly) were: “It’s an easy problem. I just used the three numbers and the formula, so it must be correct” [10–01], “It’s logical, the Father Christmas becomes three times bigger” [10–02]. Students who expressed doubts about the correctness of their answer gave rather superficial and general reasons for their doubts that did not address the correctness of the applied model (e.g., “I’m not completely sure because I haven’t carefully read the problem” [7–12], “Maybe I made a computational error. That can always happen” [10–18], “That was the first thing that came to mind, but maybe I didn’t use the correct procedure” [7–18], “You are never absolutely sure” [7–06], “Mathematics is not my cup of tea, so I am not sure that my answer will be correct” [10–15]). The great self-confidence observed in most students seems to indicate that for them, the linear model was self-evident. Moreover, students’ reasons for being uncertain about the correctness of their given answers show their habits and beliefs when approaching word problems: the chance for success is mainly due to general mathematical ability, problems are read superficially, while possible mistakes are attributed purely to technical calculation errors.

Despite the fact that students were quite sure about the correctness of their answers, they had great difficulties explaining *why* their methods were correct. Initially, most students were unable to give any explanation at all. After insisting on a justification, students (a) referred to the fact that their solution is the most logical one, (b) explained that the Father Christmas is higher, so you need more paint and because it is three times as high, you need three times as much paint, (c) referred to the fact that the problem is about ratio or proportion. These superficial answers seem to indicate that students typically use the linear model in a spontaneous and thoughtless way and do not check whether this model is applicable in a given situation. Students do not seem to have clear arguments justifying its use, nor do they realise that there are competing models. Even the few students who realised that the enlargement acts in two dimensions did not necessarily give up the linear model. On the contrary, it was among these latter students that we observed the purest and most general expressions of the linear misconception (“It’s 3 times bigger, not only the height but also the width. You can see it on the drawing. The whole thing is enlarged by factor 3, so you will need 3 times as much paint” [10–08], “I knew it was enlarged, but not how much, so I calculated 168: 56 and then I knew the multiplier" [10–05], "Because the picture becomes larger, you need more paint, so you have to multiply by three" [7–19], "It has the same shape, but it is enlarged, so you have to multiply the amount of paint by the same number" [10–13]). Arguably, these students identified 'increase' with 'proportional increase', without making a clear mental representation of the problem situation.

3.2. *Phase 2: Reactions to the weak form of cognitive conflict*

After being confronted with the manipulated frequency table with the correct answer and the erroneous linear answer, only two of the thirty-eight students involved in Phase 2 began to think about a mathematical model that also takes into account the width of the enlarged Father Christmas. Both students ([7–10] and [10–14]) decided in favour of the correct, non-linear solution, although one of them argued that the width surely enlarges, but that it is impossible to know exactly how much ([10–14]). Being asked why they had given a wrong linear answer during Phase 1, both students admitted they had not thought about that (non-linear) solution and had not paid attention to the given drawings.

For the majority of the thirty-six students who stuck to their original linear answer, the confrontation with the frequency table with the answers of fictitious peers really induced a cognitive conflict too: they started wondering where that other frequently-chosen answer could come from. Remarkably, thirteen students discovered that this answer was obtained by a multiplication by 3^2 (or 9), but most of them immediately rejected this method as erroneous ("They multiplied by three two times! You see, they made a mistake. My answer is correct" [7–01]). The twenty-three other students who did not find the origin of the alternative answer, searched for the rationale behind it in a more superficial way. For instance, they tried out 'randomly' some combinations with the basic arithmetical operations (+, –, ×, :) on the given numbers, regardless of their contextual meaning (e.g. trying to arrive at the alternative solution by subtracting the heights of the Fathers Christmas). The reasons of the thirty-six students for persisting in their linear solution were typically very general and extrinsic (e.g., "In general, I'm good in mathematics" [10–03], "You better always stay with your first solution" [10–16]). Besides, they often indicated that the linear answer was self-evident (e.g., "I would think my solution is much more logical" [10–04], "It's evident, you cannot do it otherwise" [10–12]), while the non-linear answer was qualified as 'counterintuitive' or 'illogical' (e.g., "In my opinion, this is a strange reasoning" [10–15], "That's too far-fetched" [7–13], "54 ml is quite a lot" [10–04]).

3.3. *Phase 3: Reactions to the argumentation of a fictitious peer*

In the third phase, another fourteen students of the remaining thirty-six students (equally divided among both age groups) changed their incorrect answer into the correct one. Apparently, the argumentation of the fictitious peer who answered the problem correctly provided them the insight that to maintain the same shape, a figure has to be enlarged in all dimensions, having a quadratic effect on the area of the figure. The fourteen students who changed their answer were asked to explain why they originally gave the wrong linear solution. A first category of explanations referred to the fact that they did not approach the problem in a thoughtful way, but instead immediately and routinely (student [10–11] called it 'instinctively') started to reason proportionally ("I started to solve the problem that way, and, so to speak, I closed myself for the other reasoning. No other reasoning could come up in my mind any more, also because this is the easiest way" [10–18]). Second, some students argued that they did not notice the width, because they were fixating on the problem statement which only referred to the height ("I only paid attention to the text . . . a little bit to the drawings, but above all to the text . . . and in the text only the height is mentioned" [7–12], "In the text, only the height is given. If both the height and the width were mentioned, I probably would have used another formula" [10–18]). Third, there were students who realised that the enlargement also had an effect on the width, but *deliberately* did not take it into account *because* the width was not explicitly mentioned in the problem statement ("I thought that the width was relevant too, but because in the text no reference was made to the width, I decided to work with the height only" [7–06]).

The justifications of the twenty-two students who decided to stick to their original answer were diverse, but can be grouped into three different categories.

A first group of students gave the non-linear reasoning a serious thought and were torn between the two alternative solutions, but they finally opted for the familiar linear model because they insufficiently understood the mathematical principles relevant to this problem. Some realised that the enlargement had an impact on the Fathers Christmas' width too, but were unsure about how much this width increased ("Height and width are not that much related to each other" [10–04], "The width changes too, but you cannot know how much" [10–16]). Others struggled with the quadratic impact of a linear enlargement on a figure's area (or on the amount of paint, the indirect measure that is linearly related to this area) ("6 ml is for the *whole* Father Christmas, not only for the height. And 18 ml is for the *whole* large Father Christmas, for the height as well as for the width" [7–08]).

For a second group of students the argumentation of the fictitious peer was an immediate cause to formulate their mistaken linear belief more clearly and more convincingly than ever before ("if you need 6 ml for this area and this area fits three times in that area, you need three times more paint" [10–04]). This reasoning is similar to that of those students in the first group who struggled with the quadratic impact of a linear enlargement on a figure's area in the sense that they both improperly assumed a linear relationship between two quantities. However, this second group differs in the sense that its linear reasoning seemed not to be inspired by any specific mental representation of the problem situation, but rather by an application of linearity 'everywhere'.

A third group of students justified their answer by referring to the implicit rules for solving school mathematics word problems (Verschaffel et al., 2000; Wyndhamn and Säljö, 1997). These students demonstrated a simplistic view on school word problem solving, assuming that all word problems can be solved by using simple mathematical calculations on the numbers given in the problem, and that real-world knowledge and context-based considerations should not be involved in the solution process. Sometimes, this type of arguments also occurred in students belonging to the first two categories. Examples of justifications in this group are: "I think that the pupils who gave this answer make it too complex for a word problem" [7–13], "In your calculation you can only involve numbers that are given" [10–04], "The word problem says nothing about the width at all, so it must be wrong" [10–15]. Typically, these students rejected conflicting evidence arising from the given drawings ("In the drawing it is wider, but not in the word problem. The word problem is about the height only" [7–09]).

3.4. *Phase 4: Reactions to the solution process of a fictitious peer*

After the confrontation with the fictitious peer's solution process, namely actually drawing rectangles around the Fathers Christmas' irregular shapes, in Phase 4, another nine of the remaining twenty-two students (five 12–13- and four 15–16-year olds) exchanged their initial linear answer for the correct non-linear one. It appeared that for these students, the circumscribed rectangles seemed to function as a real 'Gestaltwechsel' (Wertheimer, 1945), since they immediately and conclusively made a shift in their answers: "Oh yes, now I see it. Indeed, it is nine times larger because the small rectangle also fits nine times in the large one. With the help of these rectangles I understand it. I am sure now, it should be 54 ml" [7–04]. The uncertainty about *how much* the width increased, which was noted down several times at the previous phases, disappeared completely in these students. Before concluding the interview, these students were also asked why they originally gave a wrong linear answer and why they stuck so long to it. Their reactions were very similar to those given on this question by the students who found the correct response in Phase 3.

In the thirteen students who stayed with their original linear answer during Phase 4, no reflections were made about a mathematical model that takes into account the increased width of Father Christmas. Most students justified their answers by expressing their beliefs about how to solve mathematical word problems ("Don't look too far for the solution of a school word problem" [10–07], "It's possible, but the width is not mentioned in the problem statement" [7–09]) and about the role of and the relationship between textual and graphical information in a word problem ("It is nine times for the drawings, but three times for the word problem" [10–08], "Drawings are less accurate" [10–04], "You never should ground a solution in mathematics on a drawing. You have to ground it on formulas" [10–15]) (cf. also Phase 3). Some students could not give any justification at all or just repeated their mistaken linear belief, sounding an attempt to convince the interviewer ("The little Father Christmas fits three times into the large one. It's the same for the rectangles. Three times more area, thus three times more paint" [7–20]).

Remarkably, the confrontation with the rectangles brought two of the thirteen students who stayed with their original linear answer ([7–03] and [7–13]) to express for the first time a strange misconception: the increase of area is different for the enlarged rectangles than for the irregular figures inscribed in these rectangles (in this case, the area scale factor was supposed to be *nine* for the rectangles, but *three* for the Fathers Christmas). Student [7–03] formulated it this way: "What they do with the rectangle is correct: it enlarges in two directions. But within the rectangles is an irregular figure. And that's completely different. See here and here" (points at the white parts in the rectangles).

3.5. *Phase 5: Reactions to the explicit link with area measurement*

In this final phase, yet five other students of the remaining thirteen students (three 12–13- and two 15–16-year olds) left the wrong linear answer and exchanged it for the correct non-linear answer. All five students correctly calculated the area of both rectangles and, to their astonishment, found that the area of the larger rectangle was indeed nine times the area of the smaller one. Three of them felt no difficulties to generalise this last finding to the 'irregular' Fathers Christmas within these rectangles and to the amounts of paint needed to paint them. However, two students ([7–15] and [10–03]) only succeeded in drawing this conclusion after the interviewer provided them with still an extra scaffold (e.g. after the interviewer had asked them to compare the amounts of paint needed to paint the two *rectangles*). Moreover, an ultimate choice for the correct answer did not necessarily remove all doubts (e.g., "OK, it is nine times more paint. But I still don't see why my original calculation was not correct" [10–03]).

Even after four types of increasing cognitive conflicts, eight students maintained their linear answer until the very end of the interview. While arguing their choice, these students even stronger stuck up for their beliefs about how to solve word problems and about the role of drawings in word problem solving ("For the drawings, it is nine times, but this is not that relevant. In a word problem you are expected to work with the data that are provided in the text. Using drawings and measuring is less accurate" [10–04]) (see also Phases 3 and 4). Some of these persisting students also seemed to be worried about the role of 'amount of paint' as an indirect measure for 'area' in this context ("It's about 'consumption of paint' and you don't have to solve it via area" [7–02], "For the area, it is nine times, but for the amount of paint, I'm not so sure. Millilitre is not referring to area, but rather to volume" [10–08]).

4. Discussion

The interviews confirmed the existence of a very strong and deep-rooted tendency among 12–13- and 15–16-year olds to stick to the linear model when doing non-linear word problems about the enlargement of two-dimensional figures, even when confronted with very strong contradictory evidence for the tenability of that model in the given context. Indeed, after the first confrontation with the word problem, almost all students improperly applied the linear model and, in each of the subsequent phases of the interview, only a limited number of students – often hesitatingly – left that model. After four types of increasing cognitive conflicts, providing increasingly strong evidence for a non-linear approach, still one fifth of the students stuck to the linear model.

More importantly, the interview study provided a lot of information about the reasoning and problem-solving processes of students falling into the 'proportionality trap' and the mechanisms behind it. The results enabled us to identify the role of different aspects of students' knowledge base that were responsible for their inappropriate proportional responses. These aspects can be grouped in four distinct categories, which we will explain in detail now. However, it is important to mention here that students cannot always be exclusively or straightforwardly put into one of these categories. Most often, the reactions of a student during the five phases of the interview involved a complex interplay of elements originating from different categories and some of these elements were more prominently present in particular interview phases.

A first category of explanations refers to the *intuitiveness of linear relationships*. According to Fischbein (1987), intuitive cognitions have an obvious, self-evident, and coercive character, receive great confidence and persist despite formal learning. These characteristics seem to apply to the incorrect reasonings of the students in our study too, particularly in the first phase of the interview: the use of linear relationships was perceived as correct without a need for any further justification, students were overconfident in it, and were reluctant to question the correctness of their linear approach when confronted with conflicting evidence. Proportions appeared to be deeply rooted in students' intuitive knowledge and were used in a spontaneous or even unconscious way, which made the linear approach quite natural, unquestionable and to a certain extent inaccessible for introspection or reflection. While thinking aloud, most students immediately used proportions, they were convinced about the appropriateness of the proportional model and of the correctness of their answer, but it was virtually impossible for them to justify what they did. Later on during the interview, some students qualified the non-proportional solution as 'counterintuitive' or 'illogical' (see, e.g., students' reasons for persisting in their linear solution in Phase 2). There also seems to be a parallel between students' problem-solving process – especially in the very first encounter with the problem – and the 'intuitive rules' described and studied by Tirosh and Stavy (1999a, 1999b). These authors claim that there are some common, intuitive rules that come in action when students solve problems in mathematics and science. Two such rules are manifested in comparison tasks: 'More *A*–more *B*' and 'Same *A*–same *B*'. In the problem we presented to our students, it is quite natural (and correct) to apply the 'More *A*–more *B*' rule (the *more* height, the *more* area/paint). However, the 'Same *A*–same *B*' reasoning might occur too (figures share the *same* shape, so everything enlarges by the *same* factor), leading to an incorrect '*k* times *A*–*k* times *B*' judgement (three times more height, so three times more area/paint). Expressions in line with these schemes occurred several times during the interview and were sometimes phrased literally.

Besides the intuitiveness of linear relationships, students' improper use of linear reasoning seems – at least in some cases – to be the result of a conscious and deliberate application of linear functions in situations wherein they are not applicable. This kind of misconception can be legitimately called the '*illusion of linearity*': students really believe that the linear model is applicable in a given situation. This second category differs from the first in the sense that students no longer implicitly or automatically rely on proportions, but rather do this in an explicit and deliberate way. For some students, this deliberate application of linearity was related to the specific context of enlarging geometrical figures. In their conviction, the same scale factor (namely factor 3) applies for both the lengths and the area of a geometrical figure (e.g. "The height and the width of the figure are tripled, so the area is tripled too" [7–05]). However, the illusion of linearity also appeared at a more general level. Some interviewed students tended to quantify every relationship between variables (whatever they are) into a proportional relationship between these variables. Support for this assertion can be found in students' reactions on the question why the problem had to be solved that way (e.g. "I knew it was enlarged, but not how much, so I calculated 168: 56 and then I knew the multiplier" [10–05], "Because the picture becomes larger, you need more paint, so you have to multiply by three" [7–19]). Apparently, these students identified 'increase' by 'proportional increase', always and everywhere. Models that contradicted this conviction, such as the non-proportional solution given by the fictitious peer, were immediately rejected.

Third, we found that many students (the younger as well as the older ones) suffer from *shortcomings in their geometrical knowledge*, especially about the effect of a similarity on the lengths and area of a figure. Utterances of this fuzzy geometrical knowledge are the confusion of area and volume, not recognising indirect measures for area (such as the amount of paint), or the convictions that (1) when a figure is enlarged but maintains its shape, the height and width not necessarily increase by the same factor, (2) enlargements have a different effect on the area of a regular figure than on the area of an irregular one, and (3) only regular figures have an area. Although students learned these basic concepts and principles of geometry and measurement at the elementary school level, they seemed to have a bad or weak understanding of them, or at least they were not able to apply them correctly.

Fourth, the described findings seem also closely related to students' *inadequate habits and beliefs* about solving word problems, which is supported by a vast amount of research (see, e.g., Verschaffel et al., 2000; Wyndhamn and Säljö, 1997). The intuitive reasoning in the first phase of the interview and the moderate impact of the conflicts in the subsequent phases could occur only because the students approached the word problem in a superficial way, mainly looking at the numbers (and key words) in the problem statement without making a clear mental representation of the problem, assuming that word problems have little or nothing to do with reality, and that everything you need to solve the problem is always given in the problem statement. A mechanism that may underlie some of these habits and beliefs is the so-called *didactical contract* (Brousseau, 1984) which can be described as a system of implicit norms, rules and expectations being in force between a student and his teacher in school settings. Related to the problem-solving activity, students did not spontaneously use and even distrusted heuristic methods that might have facilitated problem solving (e.g., students rejected conflicting evidence arising from the given drawings, they assumed that formulas are a more valuable and trustworthy problem-solving tool than drawings), possibly because they more or less implicitly received this message from their teachers. They even didn't acknowledge it as a necessary and valuable part of the problem-solving process. These observations are in sharp contrast with the vast amount of research showing how and why drawings may help people solve (mathematical) problems, such as, e.g., Larkin and Simon's (1987) famous article *Why a diagram is (sometimes) worth ten thousand words*. In addition to students' poor mastery and depreciation of heuristic methods, they were also affected with deficient metacognitive knowledge (e.g., some of them were convinced that a first and quickly found idea is always the best and therefore it is always better to stay with your first solution, and that evaluation and reflection are not an essential part of mathematical problem solving).

All four elements mentioned before (intuitive reasoning, the linearity illusion, shortcomings in geometrical knowledge, inadequate habits and beliefs) seem to be a fertile soil for a superficial or deficient mathematical modelling process. As several authors have stressed, mature mathematical modelling involves a complex, cyclical process consisting of a number of subsequent steps: understanding the situation described; selecting the elements and relations in this situation that are relevant; building a mathematical model and working through it; interpreting the outcome of the computational work in terms of the practical situation; and evaluating the results and the applied model itself (Burkhardt, 1994; Greer, 1997; Verschaffel et al., 2000). In the modelling process observed in many of our students, some of these steps were completely bypassed. Little effort was invested in understanding the problem situation and in making a clear mental representation of the relevant elements and relations. The mathematical model then mainly occurred on the basis of 'reflex-like recognising', and is almost immediately translated in calculations. These calculations received the most time and attention in the problem-solving process. The superficial modelling was moreover affected by inadequate habits and beliefs (e.g. solving word problems is just doing the correct operations with the given numbers, drawings are less trustworthy than formulas). The superficial character of students' modelling process also appeared in the last phases of the modelling cycle: no critical evaluation of the model itself, nor of the results obtained by applying this model was undertaken. With the exception of a quick control on calculation errors, students did not spontaneously verify their answer by means of their common-sense knowledge or the given drawings, and in no phase of the modelling cycle (Burkhardt, 1994), students spontaneously compared the applied linear model with alternative models and even when confronted with these alternatives, they did not seriously take them into consideration. Taking into account these final considerations, students' improper use of linear reasoning can also be seen as a symptom of an immature and even distorted disposition towards mathematical modelling.

The interview study enabled us to unravel the complex interplay of different elements in students' knowledge base that was at the origin of their adherence to a proportional solution. A major open question relates to the characteristics a powerful learning environment should have to be successful in defeating students' deep-rooted tendency towards (improper) proportional reasoning. In an ongoing follow-up design experiment with 13-14-year old students, Van Dooren et al. (2002) are designing, implementing and evaluating an ecologically valid learning environment for providing answers to that question. The environment is based on (1) the results of this study (such as, e.g., focussing on the underlying concepts and relationships of problem situations, working on shortcomings in specific geometrical knowledge, acting on students' conceptions, habits and beliefs), (2) principles of realistic mathematics education (such as, e.g., starting from meaningful contexts for concept development, connecting different mathematical strands, building on students' own productions and informal knowledge, integrating authentic assessment procedures) (de Lange, 1987; Gravemeijer, 1994; Treffers, 1987), and (3) more general research-based design principles for substantial learning environments aimed at enhancing higher-order thinking skills (such as the use of instructional techniques like coaching, scaffolding, articulation and reflection) (Collins et al., 1989). The preliminary results of this experiment are both promising and disappointing. They are promising in the sense that the students at the end of the teaching experiment no longer automatically applied the linear model. The instruction at least installed some kind of critical mindset towards linearity. However, some undesirable side effects emerged, such as, e.g., students' overgeneralisation of non-linear reasoning to linear problems and their use of superficial task characteristics to distinguish between linear and non-linear problems about the effect of an enlargement or reduction on area and volume.

5. Appendix

Overview of the answers of the different students in each phase of the interview

Grade-Student	Phase 1	Certainty	Phase 2	Phase 3	Phase 4	Phase 5
7–01	L	4	L	NL		
7–02	L	5	L	L	L	L
7–03	L	4	L	L	L	L
7–04	L	4	L	L	NL	
7–05	L	5	L	L	NL	
7–06	L	4	L	NL		
7–07	L	4	L	NL		
7–08	L	5	L	L	NL	
7–09	L	5	L	L	L	L
7–10	L	5	NL			
7–11	L	4	L	NL		
7–12	L	4	L	NL		
7–13	L	5	L	L	L	L
7–14	L	3	L	L	NL	
7–15	L	5	L	L	L	NL
7–16	L	4	L	NL		
7–17	L	5	L	L	NL	
7–18	L	4	L	NL		
7–19	L	4	L	L	L	
7–20	L	5	L	L	L	NL
10–01	L	4	L	L	NL	
10–02	L	5	L	L	NL	
10–03	L	5	L	L	L	NL
10–04	L	5	L	L	L	L
10–05	L	5	L	L	L	NL
10–06	L	5	L	NL		
10–07	L	5	L	L	L	L
10–08	L	5	L	L	L	L
10–09	L	4	L	NL		
10–10	L	4	L	NL		
10–11	L	5	L	NL		
10–12	L	4	L	NL		
10–13	L	5	L	NL		
10–14	L	5	NL			
10–15	L	3	L	L	L	L
10–16	L	5	L	L	NL	
10–17	NL	4				
10–18	L	4	L	NL		
10–19	L	3	L	L	NL	
10–20	NL	3				

Note. L and NL indicate that a student chose, respectively, for the linear or the non-linear answer in that phase of the interview. The column 'certainty' contains the scores indicating how sure a student was about the correctness of his answer (chosen on a scale from 1 to 5, from 'certainly wrong' to 'certainly correct')

6. Acknowlegdements

This study is part of the research project OT/00/10 "The illusion of linearity in secondary school students: Analysis and improvement", funded by the Research Council of the University of Leuven, Belgium.

Notes

1. In this paper the terms *linearity* and *proportionality* are used as synonyms. In mathematics nowadays, *linearity* refers to relations of the form $f(x) = cx$, i.e. functions for which the properties $f(a + b) = f(a) + f(b)$ and $f(ka) = k\,f(a)$ hold, graphically represented by a straight line through the origin. Although the concept encompasses both properties, the term 'linear reasoning' as currently used in mathematics education especially refers to the multiplicative aspect of the concept which is also the case for this paper.

References

Berté, A. (Réd.): 1987, *Enseignement des mathématiques utilisant la 'réalité'*, Tome 1, IREM, Bordeaux.

Berté, A.: 1993, *Mathématique dynamique*, Nathan, Paris.

Brousseau, G.: 1984, 'The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics', in: H.G. Steiner (ed.), *Theory of Mathematics Education*, Occasional Paper 54, Institut für Didaktik der Mathematik, Bielefeld, Germany, pp. 110–119.

Burkhardt, H.: 1994, 'Mathematical applications in school curriculum', in T. Husen and T.N. Postlethwaite (eds.), *The International Encyclopedia of Education*, Pergamon Press, Oxford/New York, pp. 3631–3634.

Collins, A., Brown, J.S. and Newman, S.E.: 1989, 'Cognitive apprenticeship: Teaching the crafts of reading, writing, and mathematics', in L.B. Resnick (ed.), *Knowing, Learning, and Instruction. Essays in Honor of Robert Glaser*, Lawrence Erlbaum Associates, Hillsdale, NJ, pp. 453–494.

De Bock, D., Van Dooren, W., Verschaffel, L. and Janssens, D.: 2001, 'Secondary school pupils' improper proportional reasoning: An in-depth study of the nature and persistence of pupils' errors', *Proceedings of the 25th International Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, Utrecht, The Netherlands, pp. 313–320.

De Bock, D., Verschaffel, L. and Janssens, D.: 1998, 'The predominance of the linear model in secondary school students' solutions of word problems involving length and area of similar plane figures', *Educational Studies in Mathematics* 35, 65–83.

De Bock, D., Verschaffel, L. and Janssens, D.: 1999, 'Some reflections on the illusion of linearity', *Proceedings of the 3rd European Summer University on History and Epistemology in Mathematical Education*, Vol. 1, Leuven/Louvain-la-Neuve, Belgium, pp. 153–167.

De Bock, D., Verschaffel, L. and Janssens, D.: 2002, 'The effects of different problem presentations and formulations on the illusion of linearity in secondary school students', *Mathematical Thinking and Learning* 4(1), 65–89.

de Lange, J.: 1987, *Mathematics, insight and meaning*, OW&OC, Utrecht.

Fischbein, E.: 1987, *Intuition in Science and Mathematics*, D. Reidel, Dordrecht.

Forman, E.A. and Cazden, C.B.: 1985, 'Exploring Vygotskian perspectives in education. The cognitive value of peer interaction', in J.V. Wertsch (ed.), *Culture, Communication, and Cognition: Vygotskian Perspectives*, Cambridge University Press, New York, pp. 323–347.

Freudenthal, H.: 1973, *Mathematics as an Educational Task*, D. Reidel, Dordrecht.

Freudenthal, H.: 1983, *Didactical Phenomenology of Mathematical Structures*, D. Reidel, Dordrecht.

Gagatsis, A.: 1998, 'Solving methods in problems of proportion by Greek students in secondary education, ages 13–16', *Scientia Paedagogica Experimentalis* 35(1), 241–262.

Gagatsis, A. and Kyriakides, L.: 2000, 'Teachers' attitudes towards their pupils' mathematical errors', *Educational Research and Evaluation* 6(1), 24–58.

Ginsburg, H.P., Kossan, N.E., Schwartz, R. and Swanson, D.: 1982, 'Protocol methods in research on mathematical thinking', in H.P. Ginsburg (ed.), *The Development of Mathematical Thinking*, Academic Press, New York, pp. 7–47.

Gravemeijer, K.: 1994, *Developing Realistic Mathematics Education*, Freudenthal Institute, Utrecht.

Greer, B.: 1993, 'The mathematical modelling perspective on wor(l)d problems', *Journal of Mathematical Behaviour* 12, 239–250.

Greer, B.: 1997, 'Modelling reality in mathematics classroom: The case of word problems', *Learning and Instruction* 7(4), 293–307.

Larkin, J.H. and Simon, H.A.: 1987, 'Why a diagram is (sometimes) worth ten thousand words', *Cognitive Science* 12, 101–138.

Leinhardt, G., Zaslavsky, O. and Stein, M.K.: 1990, 'Functions, graphs, and graphing: Tasks, learning, and teaching', *Review of Educational Research* 60(1), 1–64.

Limón, M.: 2001, 'On the cognitive conflict as an instructional strategy for conceptual change: A critical appraisal', *Learning and Instruction* 11(4-5), 357–380.

Markovits, Z., Eylon, B.-S. and Bruckheimer, M.: 1986, 'Functions today and yesterday', *For the Learning of Mathematics* 6(2), 18–24, 28.

Matz, M.: 1982, 'Towards a process model for high school algebra errors', in D. Sleeman and J.S. Brown (eds.), *Intelligent Tutoring Systems*, Academic Press, London, pp. 25–50.

National Council of Teachers of Mathematics: 1994, *Curriculum and Evaluation Standards for School Mathematics*, Author, Reston, VA.

Nesher, P.: 1996, 'School stereotype word problems and the open nature of applications', *Selected Lectures from the 8th International Congress on Mathematical Education*, Sevilla, Spain, pp. 335–343.

Outhred, L.N. and Mitchelmore, M.C.: 2000, 'Young children's intuitive understanding of rectangular area measurement', *Journal for Research in Mathematics Education* 31(2), 144–167.

Rouche, N.: 1989, ‘Prouver: Amener à l’évidence ou contrôler des implications?’, *Actes du 7ème Colloque inter-IREM Epistémologie et Histoire des Mathématiques*, Besançon, France, pp. 8–38.
Simon, M.A. and Blume, G.W.: 1994, ‘Building and understanding multiplicative relationships: A study of prospective elementary teachers’, *Journal for Research in Mathematics Education* 25(5), 472–494.
Streefland, L.: 1984, ‘Search for the roots of ratio: Some thoughts on the long term learning process (Towards ... a theory). Part I: Reflections on a teaching experiment’, *Educational Studies in Mathematics* 15, 327–348.
Tierney, C., Boyd, C. and Davis, G.: 1990, ‘Prospective primary teachers’ conceptions of area’, *Proceedings of the 14th International Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, Oaxtepex, Mexico, pp. 307–314.
Tirosh, D. and Stavy, R.: 1999a, ‘Intuitive rules: A way to explain and predict students’ reasoning’, *Educational Studies in Mathematics* 38, 51–66.
Tirosh, D. and Stavy, R.: 1999b, ‘Intuitive rules and comparison tasks’, *Mathematical Thinking and Learning* 1(3), 179–194.
Treffers, A.: 1987, *Three dimensions. A model of Goal and Theory Description in Mathematics Instruction. The Wiskobas Project*, D. Reidel, Dordrecht.
Van Dooren, W., De Bock, D., Hessels, A., Janssens, D. and Verschaffel, L.: 2002, ‘Remedying secondary school students’ illusion of linearity: A developmental research’, *Proceedings of the 26th International Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, Norwich, UK, p. 370.
Verschaffel, L., De Corte, E. and Lasure, S.: 1994, ‘Realistic considerations in mathematical modelling of school arithmetic word problems’, *Learning and Instruction* 4, 273–294.
Verschaffel, L., De Corte, E. and Vierstraete, H.: 1997, ‘Teaching realistic mathematical modelling in the elementary school: A teaching experiment with fifth graders’, *Journal for Research in Mathematics Education* 28(5), 577–601.
Verschaffel, L., Greer, B. and De Corte, E.: 2000, *Making Sense of Word Problems*, Swets and Zeitlinger, Lisse, The Netherlands.
Wertheimer, M.: 1945, *Productive Thinking*, Harper and Brothers, New York.
Wyndhamn, J. and Säljö, R.: 1997, ‘Word problems and mathematical reasoning: A study of children’s mastery of reference and meaning in textual realities’, *Learning and Instruction* 7(4), 361–382.

[1]*Center for Instructional Psychology and Technology (CIP&T),*
University of Leuven,
Vesaliusstraat 2,
B-3000 Leuven, Belgium
E-mail: dirk.debock@avl.kuleuven.ac.be
[2]*Europese Hogeschool Brussel, (EHSAL)*
[3]*Research fellow of the of the Fund for Scientific Research,*
Flanders (Belgium)
(F.WO.– Vlaanderen)
[4]*Department of Mathematics,*
University of Leuven