Modelling motion:
from trace graphs to instantaneous change

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Modelling motion:
from trace graphs to instantaneous change

Het modelleren van beweging:
van discrete benadering naar momentane verandering

(met een samenvatting in het Nederlands)

PROEFSCHRIFT
TER VERKRIJGING VAN DE GRAAD VAN DOCTOR
AAN DE UNIVERSITEIT UTRECHT,
OP GEZAG VAN DE RECTOR MAGNIFICUS PROF. DR. W. H. GISPEN
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door
Lucas Michiel Doorman
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Promotoren:

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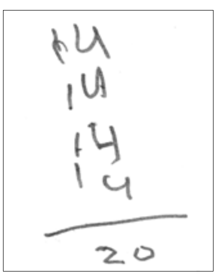
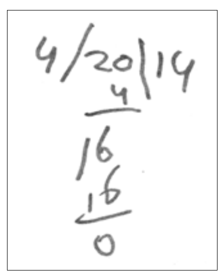
For Fleur and Sofie

Preface

In 1987, during a teacher training course which I attended before completing my mathematics study, George Schoemaker pointed out the possibility of doing compulsory social service at the Freudenthal Institute to me. I took the opportunity and have been at the institute since then. At first I was mainly occupied with software development. Since the 1990s this changed to an increasing role in curriculum development projects. The work at the institute modified my view on the learning of mathematics and on mathematics as a discipline, from an abstract language with hardly any connections to the world around us, to a language which emerges from organising phenomena and restructuring through reflection and generalisation.
The abstract nature of mathematics is experienced by many people because of the emphasis on algorithms in education: “dividing by a fraction is the same as multiplying by its inverse.” Teaching algorithms fosters beliefs such as mathematical rules having little to do with common sense, intuition, or the real world. I found another example of an algorithm on an old drawing of mine where adding 4 times 14 results in 20: adding the fours gives 16, and adding the ones makes 20. This seems correct, because 20 divided by 4 appears to result in 14. Similarly, you can add 7 times 13 and conclude that this has 28 as a result. Is this a standard algorithm? What does the algorithm express? Such issues are hardly addressed in education. Teaching mathematics from applications, as propagated by Hans Freudenthal, is an alternative for teaching algorithms.





From 1994 to 1998 I was involved in a curriculum project for upper secondary education. In this project Martin Kindt showed how history of mathematics and applications from physics could be used for the development of instructional materials for calculus. The experiences with students in this project indicated a parallel development of knowledge of mathematics and of applications where this mathematics originated from. It was shortly after that project that I got the opportunity to start a research study on an integrated approach to the learning of calculus and kinematics. The study created the possibility to investigate the students’ learning of motion and of the mathematics which models it. This thesis is the result of that research.

This thesis has one author, although the work is the result of many discussions with colleagues and relatives within and outside the institute. Unfortunately, it is not possible to thank all these persons, so let the following words of gratitude be dedicated to all of them.
First of all I am grateful to have had the opportunity to carry out the research at the Freudenthal Institute. An inspiring institute with a multi-disciplinary team of mathematicians, psychologists and pedagogues. I thank all my colleagues for their support during the research. Especially, I want to thank the following people: Paul Drijvers and Arthur Bakker who joined the research programme, discussed this study with me, and commented on prior versions of this thesis; Aad Goddijn for his time and outstanding answers on questions I posed him; Jackie Senior for correcting my written English, and Betty Heijman for processing the manuscript.
Secondly, I want to express my gratitude to the teachers and students who were involved in the experiments. This study could not have been accomplished without their willingness to invest time in an alternative approach of teaching and learning. Special thanks go to the teachers Albert Dorresteijn, Jeroen Zijlstra, Gerard Huls and Rob ten Broecke for their thinking along with the instructional design and their role in the teaching experiments.
Finally, I thank my supervisors Piet Lijnse and Koeno Gravemeijer. I learned a lot from their trying to convince each other during discussions with me. They were a great help, each in his own way, for focusing on the essential aspects of the study and for finishing this thesis.
This is the last sentence in the thesis where I use the personel pronoun ‘I’, which expresses how the exposed reasoning is the result of the contributions of many people.

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1 Introduction

In 1998 the Netherlands Organisation for Scientific Research (NWO) awarded funding to a research project on Calculus and Kinematics. The project is being carried out by the Centre for Science and Mathematics Education, which comprises the Freudenthal Institute, at Utrecht University. This thesis is one of the results of the project. In this chapter we describe the reasons for the study, its context and its aims, and we close with an overview of the contents of this thesis.

1.1 Reasons for this study

In the 1990s there was an increasing interest in the search for empirical evidence on the role of information technology in the learning of mathematics. 'Learning' here refers to a process that is based on students' contributions and constructions. Information technology (IT) can support constructive activities and can offer students the opportunity to experiment and to explore various situations. The question of how to employ IT in such a way that it supports students' learning of mathematics is the main theme of five projects within the research programme *IT as an aid to learning mathematics*. The programme was awarded an NWO grant (number 575-36-003). The research programme assumes that mathematics education requires both appropriate teaching sequences and a classroom culture conducive to learning. The programme focuses on the way in which IT can support both these elements.

One of the domains within the programme concerns the learning of calculus and kinematics. Much research points towards students' difficulties in learning calculus and kinematics. Students often have problems with understanding the basic concepts even after their courses have been completed. The introduction of graphing calculators and computer algebra has resulted in an increasing emphasis on these basic concepts at the cost of practising specific algorithms for differentiating and integrating functions. In addition, the use of computer tools – especially the distinction between building and exploring models – is often addressed in the context of the mathematics of change.

In this study *the basic principles of calculus* refer to what Freudenthal called differential and integral methods when he discussed an introduction to calculus:

> Calculus should be preceded didactically by something I propose to call Differential and Integral Methods. This topic deserves a place in an early stage of the learning process where algorithmisation has not yet been developed far enough as to allow teaching Calculus. It is an approach (in principle by graphic representations) initially merely qualitative and later on quantitatively refined (if possible). It aims at understanding and interpreting such ideas as the steepness of a graph and areas covered by the moving ordinate segment, maybe even curvature, in contexts where the drawing of the curve mathematises a given situation or occurrence in primordial reality.
> (Freudenthal 1991, p. 55)

Following Sawyer's book *What is calculus about?,* we took motion as the situation for developing these principles (Sawyer, 1961). Sawyer argued that calculus grows from questions about the idea of speed. It seems that the learning of calculus parallels an increasing understanding of speed and its relation with distance travelled.
Graphs are important in the teaching and learning of calculus and kinematics. In general, conventional mathematical symbols such as graphs, tables and algebraic notations play a central role in instructional sequences in mathematics education. Traditionally, math education has focused on the use of such ready-made symbols by presenting them to students, explaining what they mean and how they can be manipulated. In some topics it showed that this approach resulted in problems: the students did not 'see' the mathematical concepts through these symbolisations. We consider that this might also be the cause of the problems in learning with graphs about calculus and kinematics. Recent alternatives to the traditional approach make use of simulations and computer tools.
In mathematics and physics teaching we see an increasing use of simulations, which display scientific knowledge using models that are connected with idealised microworlds. These worlds behave according to the laws of the system that students are supposed to discover while *exploring* the simulation. Therefore, the simulation is designed to enable students to test and improve their hypothetical ideas about the scientific system. Such a use of simulations in education is referred to as *discovery learning* (de Jong & Joolingen, 1998a). The main activity for the students is to uncover and to try to reconstruct what was previously hidden from sight.
Guided reinvention can be characterised by the students performing a different modelling activity. The use of computers in this approach focuses on computer tools as an aid for modelling. These are tools which create opportunities for students to invent and express ideas, and to develop symbolisations from situation-specific to formal (Cobb, 2002; Gravemeijer, 2002a). The students can express and communicate their tentative ideas with the tools provided. The resulting modelling process with computer tools is also known as *expressive modelling* (Doerr, 1997).
For guided reinvention to work, it is necessary to know how students model new situations. Students are confronted with problem situations for which they do not have the appropriate models at their disposal; i.e. models which describe possible structures or patterns in the situations. Building upon the students' tentative expressions is one of the main objectives of guided reinvention. At this point, instructional designers should try to see whether ideas from theories on symbolising can be useful. We note that almost all human behaviour has a symbolising aspect in it. However, some general patterns can be identified that seem useful for education. We did not present students with mathematical and ready-made symbols, like graphs, in their conventional form. On the contrary, we will argue that there is a dialectic relation between students developing and using symbols on the one hand, and the development of their understanding of what is represented, on the other.

According to Meira (1995), the use of symbols influences actions, and the interpretation of symbols changes during the activities. We have adopted this dynamic approach, in which symbols and meaning co-evolve, as an alternative to an approach that focuses on appropriation of a ready-made symbol system.
Guided reinvention appears to be connected to Meira's argumentation for such a dialectic relation between symbolising and understanding. In line with Doerr and Meira, we investigated the possibilities of a modelling process from informal and intuitive notions to the basic principles of calculus and kinematics: an approach that progressively builds on students' symbolisations.
Finally, we note that the traditional border between the school topics of physics and mathematics is currently being reconsidered. Traditionally, the transfer of mathematical notions of change to physical situations appeared problematic. Our approach to the learning of calculus and kinematics seems a good way to investigate how these topics can be integrated, especially when their joint history is taken into account.
Problems with kinematics are often connected with an incomplete understanding of mathematical symbols. Conversely, the relation between speed and distance travelled is frequently used to explain calculations and interpretations of mathematical symbolisations of change, like the slope of a graph and the area underneath it. Consequently, it seems that the learning of calculus and kinematics are intertwined, and it is difficult, maybe even impossible, to say what must be taught first. We therefore investigated how, and to what extent, we were able to realise a learning process for modelling motion in which the teaching and learning of the basic principles of calculus and kinematics is integrated.

1.2 Context for this study

The paradigm for this study is a view on mathematics and physics as related disciplines. We believe that these disciplines do not primarily study sets of laws and algorithms which should be transferred to students, but we see them rather as disciplines that attempt to mathematise and physicalise our surrounding world, i.e. to describe phenomena in physical and mathematical terms in order to act and deal with them in a sensible way. It is precisely this activity that Freudenthal saw as fundamental for acquiring mathematical knowledge.
Freudenthal's point of departure is given in his critique of traditional mathematics education. He was fiercely opposed to what he called an anti-didactical inversion (Freudenthal, 1973), where the end results of the work of mathematicians were taken as starting points for mathematics education. Ernst Mach had already pointed this inversion out in the presentation of mathematical theorems in 1905:

> (...) more than in any other science, it is customary in the field of mathematics to erase any trace of its historical development. Yet even the completely evident knowledge of mathematical theorems does not come to light all of a sudden, but

> is introduced and prepared for by accidental remarks, conjectures, thought experiments and physical experiments.
> (Mach, 1980, p. 117; translated from Dutch)

By neglecting the history and presenting a final system for transferring mathematical ideas, students may have difficulties in understanding the relevance of, and the reasons for, its invention, and may not understand the connection with other topics. Hanson pointed to the importance of experiencing the very making of a scientific theory in order to be able to understand it:

> (...) a theory should not only be understood in terms of its formal generalisations, but also in terms of the interpretations of the formal statements (…) Interpretation is not something a physicist works into a ready-made deductive system: it is operative in the very making of the system.
> (Hanson cited in Ingerman, 2002, p. 45)

As an alternative for this inversion, Freudenthal advocated that mathematics education should take its starting point in *mathematics as an activity*, and not in the teaching of mathematics as a ready-made-system (Freudenthal, 1973, 1991). For him the core mathematical activity was mathematising, which means organising from a mathematical perspective. Freudenthal saw this activity as a way for students to reinvent mathematics.

Mathematising involves both mathematising everyday-life subject matter and mathematising the mathematical activity itself. In other words, mathematics involves organising phenomena into mathematical structures, studying these structures, and investigating the relations and transformations between structures and phenomena. The formal mathematical language originates from identifying patterns and relationships in phenomena, structuring and representing them, and progressively developing these representations in a process of reflection, restructuring and generalising.

Depending on one’s background and on the goal of the activity, the core of the mathematical activity varies from dealing with the elegance and efficiency of the pure mathematical structure, to its relation with or applicability in dealing with everyday-life phenomena.

Contrary to what seems to be implied by standard educational methods, the difference between a mathematical and a physical activity is not always clear. According to Klaassen (1995), physics deals with classifying objects and events, relating these objects and events to each other, and recognising patterns in these descriptions from a physical perspective. These recognitions may lead to the formulation of generalisations that make it possible to deal with, to describe and to predict physical phenomena. The physicist is not *primarily* interested in the mathematical structures that underlie these generalisations, but the physical language uses mathematics. However, at certain moments, classifying an activity as physical or as mathematical

is difficult. Both focus on organising phenomena. Bauersfeld described these processes of structuring the world around us as "throwing an organising net over it" (Bauersfeld, 1995, p. 278).
This view on the nature of mathematics and physics stresses the importance of the activities of students in education. Our emphasis is on the character of the learning process rather than on the inventions as such. The idea is to allow students to come to regard the knowledge they acquire as their own knowledge. Therefore, we looked for problems which students would recognise as relevant and real, and which would also evoke productive solution strategies. Then, we looked for a way to guide them to the intended goals that could progressively build on their ideas and strategies. These activities should help students acquire a scientific attitude, learn the nature of modelling, and develop new concepts from their experiences in everyday life.
At the Centre for Science and Mathematics Education, design heuristics are developed that can be used for setting up the teaching and learning processes aimed at. These heuristics can be used to realise a guided reinvention learning process starting from students' tentative symbolisations to the basic principles of calculus and kinematics. The design heuristics aim to realise a process in which students re-invent the intended concepts, while being guided by the teaching materials and the teacher. Hence, the *main question* for this research project was:

How, and to what extent, can the teaching and learning of the principles of calculus and kinematics be integrated into a guided reinvention course on modelling motion using computer tools?

Freudenthal has already pointed out the need for such a reinvention course and the difficulties that would be encountered:

> Reinvention is here [calculus] a bigger problem than in the domains I have dealt with so far. Reinventing something that since Archimedes has waited for about two millennia to be invented the first time is not that easy. It requires stronger but nevertheless more subtle guidance. It seems to me that we are just beginning to understand and tackle this problem. (Freudenthal 1991, p. 63)

1.3 Aims of this study

The aim of our study was to gain insight into the way students can develop scientific knowledge in a learning process characterised by guided reinvention. How can students be involved in a process of modelling motion in which they develop, use, and improve symbolisations together with learning the principles of calculus and kinematics? This should also give insight into the possibilities of design heuristics for realising these teaching and learning goals.
In addition, we aimed to gain insight into how computer tools can be used for supporting learning processes with a guided reinvention character. This should lead to

a better understanding of the way in which computer tools can help students develop, use and improve symbolisations.

Our final aim was to outline an exemplary, empirically tested, instructional sequence for the basic principles of calculus and kinematics. This sequence should contribute to the development of a local instruction theory for the teaching and learning of these topics.

1.4 Overview

This thesis starts with discussing literature on the teaching and learning of ideas related to modelling motion, and literature on symbolising processes which are related to (mathematics) education. These discussions help to specify our main research question and the methodology needed to answer this question.

In chapter 2 we give examples of problems in the teaching and understanding of the principles of calculus and kinematics. We claim that these didactical problems are more complex than appears from current school practice. A gap exists between daily life use and use in education of the notions related to motion and change. Recent educational approaches to these topics which try to overcome this gap are discussed. The main conclusions of this chapter refer to the interplay between using and interpreting graphical inscriptions on the one hand, and organising phenomena on the other.

The interplay between graphs and organising phenomena is the starting point for the need to gain more insight into the development and use of graphical inscriptions as an aid to learning mathematical and physical concepts.

In chapter 3 the literature on this topic is discussed together with related theories on perception and interpretation. We reason that didactical problems in the teaching of calculus and kinematics might have their origin in overestimating the power of continuous graphs, and in insufficient attention being paid to students' symbolising activities. The consequences of theories on symbolising for education are analysed, and we hypothesise that a guided reinvention approach might overcome the didactical problems. At the end of chapter 3 the main question is split into *two research questions* for this project.

Chapter 4 describes the research methods. We focused on design research for developing an instruction theory for calculus and kinematics. This methodology contains a design phase in which an instructional sequence was designed for creating an educational setting in which our conjectures could be investigated. The sequence consisted of learning and teaching materials for ten lessons in grade 10 (16-year old students). These materials, together with the assumptions that underlie the hypothetical teaching and learning processes in relation to the use of these materials, have shaped a conjectured local instruction theory.

In chapter 5 we describe the emergence of the conjectured local instruction theory together with the instructional sequence. This design was inspired by the historical

development of calculus and kinematics. We paid attention to this history and to a pilot study that preceded the design of the sequence.
Chapter 6 describes our findings with the instructional sequence and the underlying assumptions in two teaching experiments. The first teaching experiment took place in two schools, and the second experiment in a third school.
In chapter 7 the conclusions with respect to the research questions are described. We have made recommendations for a local instruction theory for the teaching and learning of the basic principles of calculus and kinematics. In addition, we recommend the use of computer tools in trajectories that foster a guided reinvention learning process, and we draw attention to the use of theories on symbolising in mathematics and physics education, in general. Finally, we discuss the possibilities and constraints of integrating the learning of mathematical concepts with concepts from other disciplines.

2 The teaching and learning of calculus and kinematics

In this chapter we first discuss what the literature has to say on students' conceptual problems with calculus and kinematics, and possible relations with textbooks and current teaching practices (section 2.1). The main conclusions of this discussion deal with the gap between real life experiences with change and motion and students' ideas about them, and the teaching and learning of calculus and kinematics. Children interpret speed as an undifferentiated property, while we teach speed as a derived compound quantity that is related to two components which you can use for calculations. The frame of reference that is crucial for these calculations is influenced by our perception of motion. The causal and descriptive aspects of speed are intertwined in our everyday reasoning, and instantaneous change seems impossible. Getting students to interpret and use graphs appears to be insufficient for them to learn about these topics. Usually, we expect students to *see* the concepts in motion graphs, and then to be able to bridge the gap between the phenomena described and the concepts. However, these graphs are not as transparent as teachers sometimes think: the physical and mathematical knowledge remains isolated from students' everyday experience and reasoning.

Section 2.2 describes two recent alternative approaches using computer tools that try to bridge the gap between street wisdom and school knowledge. The first approach is referred to as discovery learning, and uses graphs in simulations that students can explore. The explanatory power of the graphical models in this computer environment is clear to the expert, but the student may relate only the visual similarities of graphs to changes in the situation without becoming aware of the underlying principles. The second approach is an inventive approach that tries to lead students into developing graphs guided by their interpretation and understanding of the context. This approach involves the students' reasoning and interpretations, but it is not clear to us how they will reach the taken-as-shared consensus on the meaning of calculus and kinematics.

We conclude in section 2.3 that if it is possible to allow students to invent distance-time and speed-time graphs by themselves and to let them experience a model-evolving process, the gap between formal mathematics and their own experience will not arise; the mathematical ways of symbolising such concepts will emerge naturally from the students' activities, and the accompanying formal mathematics will be seen as an extension of their own experience.

2.1 Conceptual problems in calculus and kinematics

It is difficult, and maybe even useless to talk about conceptual problems without relating these problems to a didactical implementation of the concepts, to the persons that perceive them, and to their previous education. Here we describe what the literature has to say on the problems related to teaching and learning the concepts of

kinematics and calculus. In some cases it proved difficult to reconstruct the classroom processes that caused these problems. Nonetheless, the problems are so widespread and persistent that they seem to be almost independent of the type education: they appear at all levels, from young children to university students studying physics and mathematics (McCloskey, 1983; McDermott et al., 1987; Orton, 1983a, b).
We focus here on how students develop the concepts for the basic principles of kinematics and mathematical aspects of modelling motion. These include the difference between average and instantaneous velocity, their relation with distance travelled, the use of graphs, graphical characteristics like slope and area, and the difference quotient. We also investigate the literature on the entanglement between the teaching and learning of the mathematics of change and the kinematical aspects of motion.

2.1.1 Intuitive notions of velocity

Velocity is used as an instantaneous property of an object ('at this moment I am driving at 90'), and as a relative property that describes a relation between objects. Such different ways of using and interpreting velocity are discussed here and related to the teaching of kinematics. With velocity we generally refer to the scalar quantity speed instead of a vectoral quantity.
Piaget studied young children's intuitions on time, velocity and distance travelled. One of his main conclusions was that velocity is a basic concept rooted in our experience (Piaget, 1970). He found that young children (7-8 year olds) see velocity not in terms of a relation between time and distance travelled, but rather between lengths of displacement and overtaking. They confused duration of motion and the length of a specific path travelled. Young children could not associate small displacements with large velocities, and in their explanations they used displacements independent of the duration of motion. They could compare velocities only in situations where they saw one object overtaking another, regardless of the objects starting position or time. This focus on displacement can also be recognised in the use of everyday language, where driving at 60 m.p.h. refers to a distance of 60 miles. The number of miles is treated as a measure for velocity. This may be why students said they could not answer the question: If a car drives at 50 m.p.h., how much time does it need to travel 1 mile? The students said that you cannot know the answer because the car's velocity can change continuously, the only thing you know is that it will take one hour to cover 50 miles. Velocity is conceived as the length of a displacement before it becomes a coordinated image of the rate of change of two quantities (Thompson, 1994a).
Piaget concluded that if velocity itself, as well as motion in general, is perceived in terms of relations of displacement, then the whole generic structure which follows will depend on these relationships. The children will comprehend velocity only in localised terms until they are able to grasp time operationally. Hutcheon specifies this aspect of grasping time in more detail and points out the idea of simultaneity:

> What is involved in the measure of time is simultaneity: the intuition that when two objects begin and stop moving at the same instant, the one which has gone the farthest has moved the fastest. The child who has acquired a grasp of simultaneity is able to coordinate these two velocities, and thereby compare them. (Hutcheon, 1996, p. 376)

From these studies we can conclude that children must perceive a measure of duration and be able to compare the time intervals of different events before they are able to interpret velocity as a compound quantity. As long as this notion is absent, children will not perceive velocity as a relationship between distance and time.

As an aside we should note that Piaget's study of young children was framed by scientific ideas of motion. In his experiments he investigated to what extent scientific notions of velocity exist in children's reasoning. His experiments were designed from this perspective. He was not investigating which notions were present or how they were used by the children in familiar situations. After all, children are able to move, to orientate, and to catch moving objects successfully. It is possible that from such a point of view you might derive different conclusions for the generic structure that has to follow.

Saltiel & Malgrange (1980) also investigated how students spontaneously reason about velocity (in approximately 700 first- and fourth-year university students). They observed how students used velocity in both a *descriptive* and a *causal* way. Descriptive aspects of velocity are related to perception: velocity is perceived as the property of an object, or as a relation between objects. Velocity as a property is called a true or real velocity that is relative to the ground. Motion and rest are in this interpretation fundamentally non-equivalent for students (a typical pre-Galilean view), e.g. 'object A has velocity v and object B stays in one position', is fundamentally different from 'object A has velocity v with respect to a moving object B'.

One problem with descriptive aspects of velocity is that velocity might be perceived relative to another object, but that it is interpreted relative to the ground. Often the perceived frame of reference is not the ground but another moving object. McCloskey (1983) gave a few examples of these illusions (fig. 2.1): a ball dropped by a running person is perceived as falling straight down (B), or even a bit backwards (A) and landing behind its point of release, while actually it follows a curve in the direction of the running person (C).

Our perception takes the running man as the frame of reference because he remains in view, and because we are able to identify with him. Tests on the computer have indeed pointed out that the moving carrier influences the interpretation of the motion of the ball. McCloskey designed an experiment with two situations. In the first situation, the moving carrier remained visible, while in the second situation the moving carrier disappeared from the screen after dropping the ball.

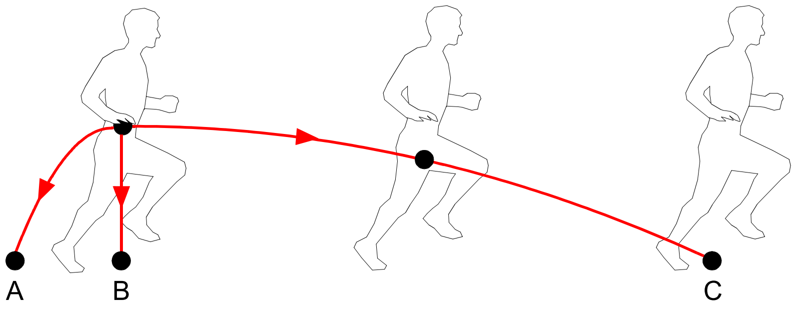


figure 2.1 What is the trajectory of a ball which is released by a running man?

They found that people were far more able to draw the real trajectory of the ball in the second situation. McCloskey concluded that everyday experiences do not guarantee that people will have reasonably accurate, physical, ideas about the motion of objects, or even have an informal grasp of the general principles that govern objects in motion (McCloskey, 1983).

As well as these descriptive aspects that are strongly related with perception, Saltiel & Malgrange (1980) noticed a causal aspect in students' explanations of perceived motion. In daily life, velocity is often associated with the driving forces that cause it. This aspect is also mingled with perception and comes to the fore in students' explanations about the trajectories of objects.

For instance, with the trajectory of the ball that is released by a running person, many students reasoned that this ball only follows a vertical path and immediately loses its horizontal velocity, because there is no longer a connection between the ball and the person. The only driving force is due to gravity, which is vertical. Another well-known example that illustrates this aspect is a ball that leaves a spiral-shaped pipe on the floor (fig. 2.2). The driving force on the ball in the pipe is circular and students therefore reason:

> The ball goes like this (curved path) because it still had some momentum when you were turning it in a circle, and it wants to go in a straight line (...) it goes in a curve until the momentum wears out (...) then it goes straight. (Halloun, 1985, p. 1060)

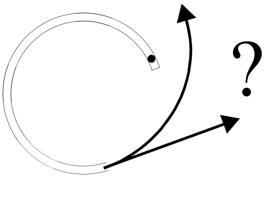


figure 2.2 How will the ball leave the curved pipe?

We conclude that our perception of motion is influenced by our ideas of a frame of reference, by a possible identification with a person or object in the situation, and by our supposed causalities in the situation.

People often relate the examples presented above with the history of kinematics, where scientists had similar thoughts and problems (e.g. Halloun, 1985; Eckstein & Kozhevnikov, 1997). A mixture of fragments of Aristotelian reasoning (the striving by the falling ball to return to its original or natural position), medieval reasoning (the ball that leaves the pipe has an ‘impetus’ that controls its motion), and Newtonian reasoning are found in students’ solutions of kinematical problems. However, you might say that there is a fundamental difference between scientists that try to frame motion from a theoretical perspective, and the situation-dependent reasoning described above (DiSessa et al., 1993). Everyday reasoning simply does not share the theoretical character of the reasoning of medieval scientists. In chapter 5 we will return to this historical relation, and to what we can learn from the historical development of these topics.

What can we learn from these studies about these naive intuitions of motion? The main reaction might be that students have intuitive ideas of motion that education has to correct, or for which education has to present a consistent alternative. However, the assumption that students have different ideas can also be the result of a misunderstanding between students and scientists. Scientists give very specific meanings to expressions like ‘to exert a force on’. It is therefore important to try to interpret students’ reasoning without taking into account the scientific conventions behind their expressions (Boyd & Rubin, 1996; Klaassen, 1995).

According to Klaassen, it is not surprising that students’ reasoning seems to conflict with scientific knowledge. Differences between these ways of reasoning do not give immediate information which can be used in teaching. Instead of correcting their ideas or presenting alternatives, teaching should start from the assumption that all our reasoning is consistent with our intuition built on our everyday experiences. This reasoning is based upon an underlying scheme of explanations that students and scientists have in common. From this starting point, teachers should try to evoke the scientific perspectives for modelling motion in such a way that students come to see the point of extending their knowledge in the intended direction, based upon this joint scheme of explanations.

To sum up, most differences between the scientific notion of velocity and students’ grasp of the concept of velocity are related to their experiences, perceptions, and situation-specific reasoning, in which a direct connection with time and distance travelled is not necessary. Lijnse described this distinction between daily life and theoretical perspectives for grasping a physical concept (Lijnse cited in Genderen, 1989 p. 86). He characterised differences between these street images and school images of physics (see table 2.1).

street image	school image
strongly rooted in everyday experiences	weakly rooted
vague concepts and unclear relations	well-defined concepts and relations
not logically consistent	logically consistent
individual, situational knowledge	public, generalised knowledge
instantaneously making sense of direct experience	permanent 'truth' of an idealised reality
intuitive frame of interpretation	reflective frame of interpretation

table 2.1 Differences between street and school images of physics

According to Lijnse, the street image of physical notions is strongly rooted in everyday experiences, and the transition to the school image is not a trivial process. Lijnse contrasted a not logically consistent street image with a logically consistent school image. However, we do not think that *logical* consistency is of great importance in street use, but consistency itself affects its use. You could say that consistency is related to situational use, while in the school image it is the theoretical, general usability, and the connection with related theories that requires logical consistency.

What causes these differences between the street and school images of science? In the next section we discuss the literature on research into textbooks on kinematics and calculus.

2.1.2 Systematics in textbooks

Until the 1980s, many teaching methods were qualified as taking a structuralistic approach (Dall'Alba et al., 1993; Tall, 1996; Treffers, 1987). The basic principle of this approach is the transmission of a logical structure within a scientific system. This system is a characteristic feature of adult science. The approach is appreciated because the theories themselves seem elegant and compact descriptions of what is to be learned. In chapter 1 we distinguished a structuralistic approach which starts with a ready-made system from an approach with gradually arising concepts. Teaching science according to its final structures might lead to teaching a mere system, rather than the science organised by it (Freudenthal, 1993). In this section we discuss problems which seem consequences of such a structuralistic approach to the teaching of calculus and kinematics.

Velocity is defined in textbooks as the compound quantity of distance travelled divided by the corresponding time interval. In this definition students hardly see the nature of this composition, the difference between instantaneous and average velocity, nor the meaning and goal of this difference (Halloun & Hestenes, 1985). The difference between average and instantaneous velocity is important for its physical interpretation and the difference comes to the fore in the use of time. Average velocity is related to distance travelled within a time interval, while instantaneous velocity is velocity *at a moment*, which can be approximated with a time interval that tends to zero.

Streefland (1981) pointed out a conceptual problem that is associated with the role of time in defining velocity. Velocity is an instantaneous property of motion. However, as soon as you want to give meaning to velocity, you use a time interval and lose the instant. This is a conceptual aspect that is not easy to reach from intuitive reasoning about velocity. Beth (1928) described this aspect in a paradoxical definition of instantaneous velocity:

> The velocity is what it would be if it remained what it was.
> (Beth, 1928, p. 54; translated from Dutch)

Regarding a frame of reference, Saltiel & Malgrange (1980) noticed that scientists are very careful about their frame of reference, while in daily life this care is not applied, as we noticed in the previous section. In teaching, this difference is hardly addressed. In a phenomenographic study, Walsh et al. (1993) found that physics students have difficulty in understanding relative speed, because they hardly think about a frame of reference – except for the ground – when they reason about speed. Some of their conceptions did not accord with scientific understanding, despite the fact that they had already passed various examinations in physics. Walsh et al. concluded that teachers often overestimate students' understanding of their lessons.

Drake (1990) showed that in lessons on velocity there is the danger that the teacher makes implicit use of expert conventions. Teachers introduce velocity and acceleration as measures of instantaneous change of the distance travelled and velocity, respectively: $v = \mathrm{d}x/\mathrm{d}t$ and $a = \mathrm{d}v/\mathrm{d}t$. However, in the same lesson, they might also discuss difference equations on motion and add that position, velocity and acceleration can not change in zero time for the passage of time is crucial in the connections between all of these quantities. For students, time as a connecting quantity is often not a trivial consequence of the definitions. Formulas in textbooks suggest static relations, while the concepts are dynamic.

On top of this, the kinematic and dynamic aspects of velocity and acceleration (e.g. the trajectory of a thrown ball) are often treated in different chapters of textbooks. Consequently, these aspects are linked with different kinds of problems and situations. It appears that both aspects are mingled in students' intuitive reasoning. This mix is not explicitly unravelled by the textbooks (Dall'Alba et al., 1993). Dall'Alba et al. saw mainly operational definitions in textbooks, not conceptual explanations. The textbooks usually offer algorithms for solving quantitative problems, while exploration of the qualitative meaning is largely overlooked. It appears that students

can apply kinematical formulas in calculations, but in their reasoning about motion they stick to their intuitive notions. The introduction of quantitative methods in a formalised language is part of this problem.
Tall (1991) described a similar case in the teaching of calculus. He started with a mathematician's general approach by trying to simplify a complex mathematical topic, by breaking it up into smaller parts. These smaller parts can then be ordered in a sequence that is logical from a mathematical point of view.

> From the expert's viewpoint, the components may be seen as part of a whole. But the student may see the pieces as they are presented, in isolation, like separate pieces of a jigsaw puzzle for which no total picture is available.
> (Tall, 1991, p. 17)

It may be even worse, Tall continued, if the student does not even realise that there is a big picture. The student may imagine every piece in isolation, which will severely hinder synthesis. The result may be that the student constructs an image of each individual piece, without ever succeeding in bringing all the pieces together in one whole. As an example, Tall described an average textbook sequence for differentiation. To be able to understand the derivative $f'(x)$, one has to have a concept of the limit, because one has to take the limit of the difference quotient $(f(x+h) - f(x))/h$, where h tends to zero. Thus the concept of a limit has to precede the derivative. Furthermore, one might decide that it is easier to take the limit in the case where x is fixed. The next step would then be to let x vary, to introduce the idea of a derivative. For the student, however, the introduction of the limit concept suddenly appears for no reason, with all the cognitive problems this may bring. The next big problem is in the shift from a limit with a fixed x to a varying x, since taking a limit at one point is substantially different from perceiving $f'(x)$ as a function of which the values describe the gradient of a graph of $f(x)$.
When we return to Lijnse's distinction between the street image and school image of science in the previous section (p. 14), we conclude that in school we teach velocity with a clear frame of reference, while in a street image this frame will depend on the perception of the situation. In a street image there exists an unclear distinction between the descriptive and causal use of speed, while in school we teach two connected and consistent theories (kinematics and dynamics). Velocity is used as a property in a street image, while in a school image it is taught as a compound quality, a relation between distance travelled and time. A street image deals with real motion, while the school image deals with theoretical motion in an idealised world.
Mathematical descriptions of calculating change are taught according to a logical consistency that is unclear to students. It is therefore questionable whether teaching these topics according to a scientific system actually helps students to learn and understand the underlying principles.

In the following two sections we discuss the literature on the role of formal language and the use of graphs in the teaching of these basic principles. We hope that this will give the reader more insight into the didactical problems as well as clues for an approach to overcome these problems.

2.1.3 Formal language

Researchers have paid much attention to conceptual problems with formalisations in calculus and kinematics (e.g. Barnes, 1995; McDermott et al., 1987; Dall'Alba et al., 1993). There are a few recurring themes in these studies, which do not appear to have been solved yet by educational research, such as the relation between algebraic and graphical aspects, and the limit concept. This section gives an overview of this kind of problem with the formalisation of the relation between velocity and distance travelled and the mathematics of change. The role of graphs in this formalisation process is discussed in the next section, because of its special intermediary role in both the learning of kinematics and of calculus.

It is often pointed out in physics and mathematics that there is a risk in making a too quick formalisation (Dall'Alba et al., 1993; Roschelle, 1998). Textbooks do not make clear the relationship between students' ideas and the formulas relating to a physical concept, but rather focus on the algorithmic aspects: "Textbooks focus on knowledge in the form of equations" (Roschelle, 1998). As a result, students have to 'guess' what knowledge is aimed at and what the relation is with their ideas and experiences (if they try to establish such a relation). Velocity is often introduced in a uniform motion as a compound quantity with the formulas $v = \Delta s/\Delta t$ or $v = (s(\mathit{final}) - s(\mathit{initial}))/t$. Both formulas have their connotations and use for scientists. However, for students it is not directly clear why a function-notation like $s(\mathit{initial})$ is necessary. It becomes even more obscure when the instantaneous value of a changing velocity v is defined as the limit of $\Delta s/\Delta t$. In student activities, practising working with these formulas takes a central position. The choice of the formulas is often triggered by the values of the quantities that are given and the missing value that is asked for, or by expressions like 'uniform' or 'uniform accelerated'.

Freudenthal (1973) pointed out implicit conventions behind the formal language in kinematics and calculus. In mathematics, characters like f, g, h, ..., are used as function symbols, while a, b, c, x, y, ..., represent quantities. These characters are used in specific situations with specific operations, that differ from their use in physics. In physics the characters s, v, and a have both roles, representing both quantities and functions. The physicist sees the situation through the formula, and knows what interpretation makes sense.

Another issue of confusion is in the formalisation of *average* velocity. The quotient $\Delta s/\Delta t$ is related to calculating average velocity, while students in mathematics classes learn to calculate an average by adding values of one quantity and dividing by the number of these values. The resulting average value has the same dimension as the added values. From this point of view it is not strange that students might think that the division $\Delta v/\Delta t$ also has average velocity as a result. Freudenthal (1983) added to this that the quotient $\Delta s/\Delta t$ refers to an external proportion, a relation between two different quantities, which is conceptually much more difficult to understand as internal proportions. In the next section on graphs we will return to this issue.

Orton (1983b) and Thompson (1994b) also stressed the importance of the notions of ratio and proportionality. They concluded from studies on students' understanding of differentiation that it is necessary to intertwine the learning of rate of change together with developing an understanding of ratio, proportion, and of graphical representation. This seems necessary to overcome problems with the interpretation and use of variables in difference quotients.

The mathematical formalisation of a difference quotient introduces an extra problem. The quotient is often introduced in the physical context of dividing change of position by change of time. After this introduction, a quick step ahead is made from $\Delta y/\Delta x$ to its representation with function symbols: $(f(x+h) - f(x))/h$. Students can see and understand the meaning of the intervals Δy and Δx in a given situation as the increase of given quantities. The values of these intervals can be determined in arbitrary order, while in the formula $(f(x+h) - f(x)) / h$ the x has disappeared from the denominator in the quotient, and the order for calculation is fixed, in the sense that you have to determine an x and an h before you can calculate function values. Moreover, the intervals Δy and Δx refer to change or increase, and it is not directly clear why the subtraction in the nominator and the variable in the denominator also refer to such increments.

In addition to these problems with the difference quotient, Pence (1995) pointed out a necessary understanding of the variable concept. He noted that many students starting a calculus course did not realise that $2x$ is twice as far from 0 as x. They were not able to locate $2x$ on a number line when x was already positioned. The students did understand that $2x$ represented a multiplication, but were not able to interpret $2x$ as representing a quantity twice as large as x. Freudenthal (1984) pointed out the problem that variables are often taught and understood as placeholders or letters to be manipulated, so that the kinematic understanding that the letters refer to something which varies is lost. White & Mitchelmore (1996) found that many student errors in applying calculus were caused by a weak concept of variable:

> Students frequently treat variables as symbols to be manipulated, rather than as quantities to be related. Three examples of such a 'manipulation focus' have been identified: failure to distinguish a general relationship from a specific value; searching for symbols to which to apply known procedures regardless of what the symbols refer to; and remembering procedures solely in terms of the symbols used when they were first learned.
> (White & Mitchelmore, 1996, p. 91)

The last problem in the formalisation of the basic principles of calculus and kinematics which we mention here concerns the formalisation of the limit concept. This limit concept seems fundamental for differential and integral calculus and for the formalisation of instantaneous change. However, the many ways to approximate limits, and the different notations, make this concept hard to teach (Cornu, 1991; Williams, 1991; Orton 1983a). Monaghan (1991) pointed to differences between students' and mathematicians' interpretation of words like *tends to*, *approaches*, *converges* and *limit*. Tall (1986), Rosenquist et al. (1987) and Cornu (1991) pleaded for an implicit use of the limit concept in an introductory course on the basic principles of calculus.

Many researchers have already pointed out the problems of a too quick formalisation in mathematics and physics:

> Rather than being defined, these terms are introduced by exposure to examples of their use, examples provided by someone who already belongs to the speech community in which they are current. (Kuhn, 2000, p. 11)

Kindt (1995) and Barnes (1995) stressed problems related to of this focus on 'exposure of use' in the teaching of calculus. Kindt pointed out the tension between algebraic manipulations and graphically supported concepts. The manipulations are relatively simple in school problems, while the understanding of concepts is difficult and is often dealt with rather too quickly. Barnes stated that students reacted by neglecting these conceptual aspects and focusing on the operation of algebraic manipulations at a rote level. From these problems in the formalisation process, we can conclude that a didactical implementation is difficult to realise. Such a didactical implementation often proceeds too quickly, or too far (e.g. the limit concept), the relation with intuition is not paid much attention, and symbols are introduced with implicit conventions that are clear for the experts but not for the students. As a result, students try to focus on the algebraic manipulations.

These conclusions are not revolutionary. However, it is surprising that the problems are still unsolved, and that there is still not much exchange of knowledge between physics and mathematics educational research about these problems. Graphs should have an intermediate role in the didactical implementation of the formalisation, and in the correspondence between these formalisations from a physical and a mathematical perspective. In the next section we investigate why this has not happened.

2.1.4 The role of graphs

Graphs play a didactical role in the teaching of calculus and kinematics. They should clarify the concepts described, and connect these concepts with the phenomena they are referring to. However, graphs are apparently not a solution to the evident conceptual problems, which is why we discuss the role of graphs more thoroughly in this section. We hope to provide insight into specific problems in the didactical use of

graphs for these topics, as well as considering whether, and how, these problems can be solved.
In an extensive descriptive study, McDermott et al. (1987) identified a number of difficulties that students have in making connections between kinematical concepts, their graphical representations, and the motion of real objects. They separated difficulties with connecting graphs to the real world, the actual motion it is referring to, from difficulties with connecting graphs to physical (and in our case mathematical) concepts. We first focus on the connection with the real world, the understanding of graphs as models of motion, and secondly on the use of graphs for conceptual reasoning.

Graphs as models of motion

A frequently recurring issue in using graphs to describe motion is their interpretation as representing the actual trajectory of motion (an iconic representation) (McDermott et al., 1987; Clement, 1985; Leinhardt et al., 1990; Dekker, 1991; Monk, 2003). Clement discerned two problems. The first was that students connected the global shape of the graph with visual characteristics of the situation (e.g. a bump in a distance-time graph is associated with a hill in the trajectory). The second problem was that students tended to associate local characteristics of the situation with corresponding characteristics of the graph.

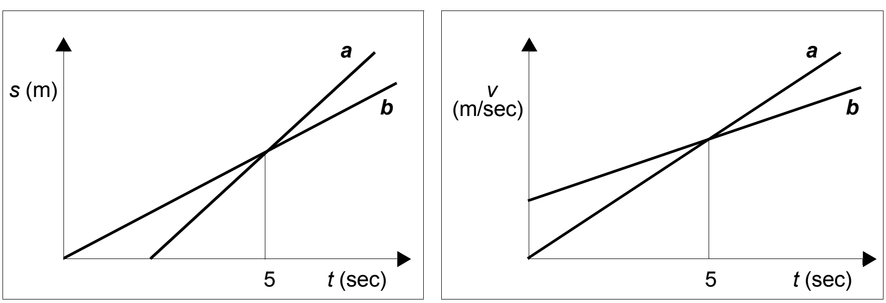


figure 2.3 Point of intersection in a distance travelled graph and in a velocity graph

Figure 2.3 shows two examples of this second problem, in which students connect points of intersection with events in the real world. In the graph on the left, students associated the same location with the same velocity. When the students were shown this graph and asked “what can you say about the velocities of ***a*** and ***b*** at t = 5 sec.?”, they answered that the velocities were equal because ***a*** and ***b*** had travelled the same distance (Halloun, 1985). At that moment, both graph lines have the same height, which suggests equality. Another well-known example is the right-hand graph, from which students concluded that ***a*** catches up with ***b*** after 5 seconds. Leinhardt et al.

(1990) discussed such slope/height confusions extensively. Goddijn (1978), Berg (1994) and Dekker (1991) discussed *iconic interpretations* of graphs (the graph as a literal picture of the situation). We use illustrations in graphs, like cars that drive along a curving velocity-time graph or the parachute jumper in (fig. 2.4) that refer to the actual situations and, as a consequence, suggest similarities with the pictured situation. Goddijn saw the problem not only in the shapes of graphs and these illustrations, but also in the language we use when talking about graphs with words like ‘crossing’, ‘slope’, and ‘rising’.

Dekker (1991) also discussed the interpretation of the word ‘distance’ in distance-time graphs. Some students described the motion of a walking person with a distance-time graph with descending parts. Fellow students commented that a descending graph was impossible, because “when you walk, you don’t go backwards”. These students interpreted ‘distance’ as the distance travelled, while the students who drew the graph used it as the distance to a certain point. You can walk farther away from and return closer to that point. Brungardt & Zollman (1995) found students were hindered in their understanding of kinematical concepts by sometimes using the words ‘up’ and ‘down’ for the value of the represented quantity, and at other times for the direction of the motion or the graph.

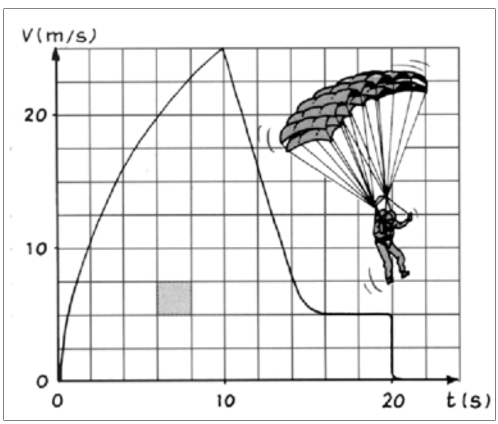


figure 2.4 Velocity-time graph of a Parachute jumper (Biezeveld & Mathot, 1998)

While working on questions about a parachute jump described by a graph (fig. 2.4), a student responded:

> After 10 seconds he is at his highest point ...
> The distance fallen is 50 squares, which is 250 cm^2.

It appears that not much attention is paid to the question *why* in teaching. Dekker (1991) observed, for instance, that after students saw a relation between straight distance travelled-time (*s-t)* graphs and constant velocity, the teacher confirmed this relation, but did not ask why this relation exists. Ainly (2000) stated that using graphs has the risk of a ‘false construction of metaphoric resonance’ between visual characteristics in the situation and in the graph. Graphs can be misleadingly suggestive. The danger of using graphs is that understanding stays rooted at a visual level. These visual resemblances might lead to the problems outlined above.

Graphs as models for conceptual reasoning

Graphs as models of motion are often the starting point for formalisation in kinematics and calculus. Reasoning about distance travelled with velocity-time graphs provides meaning to the area under the graph line. The gradient of distance travelled-time graphs supports the understanding of instantaneous velocity. Both issues are discussed below.

A *v-t* graph is used to explain how area is related to distance travelled as well as the relation with average velocity. An object that travels with a changing velocity covers the same distance as an object that travels with a constant velocity v_{av} (fig. 2.5).

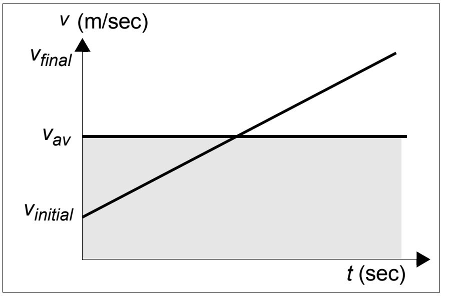


figure 2.5 Area and distance travelled in a velocity graph

The line of reasoning is that the distances travelled are the same, because the area under both graphs is the same. Moreover, velocity v_{av} is the average of the velocities of the other object. However, it is crucial here that v_{av} is not the average of $v_{initial}$ and v_{final}, but that v_{av} really is the average of *all* velocities of the other object. It is rather difficult to connect the area-calculation of an average quantity with the usual calculation of averages: $(n_1 + n_2 +... + n_k)/k$, where n_i represent k values of a quantity. Students are not used to connecting such computations with thinking about different quantities, and external proportions. It becomes even more difficult for them to distinguish the different calculations after they learn to use intervals to calculate an average velocity: $\Delta s/\Delta t$. The *v-t* graph and this division might suggest that an average velocity can be found by $\Delta v/\Delta t$.

In addition, the constantly increasing graph also implies: $v_{av} = (v_{initial} + v_{final})/2$;

which is difficult to relate with the area-reasoning that deals with infinitely many different velocities at infinitely different instants of time. However, this last quotient only works in the case of a uniformly accelerated motion.
Notice that we, as experts, use specific reasoning with specific graphs. We do not ask ourselves what the meaning is of the area below an *s-t* graph, or the meaning of average distance travelled. Such questions are hardly ever posed in textbooks.
The question about the meaning of area and average values becomes even more difficult in the *v-s* graph below. It may seem cumbersome to put distance travelled on the horizontal axis. However, it can represent the road, the very place where you measure the velocity which you put on the vertical axis. It was found that many students initially use the horizontal axis for position when they have to draw a graph of motion (Boyd & Rubin, 1996). This can also be influenced by graphs in sports: 'it takes *y* seconds to arrive at position *x*'. In sports graphs the position often appears as an independent variable displayed along the horizontal axis (e.g. see figure 2.6).

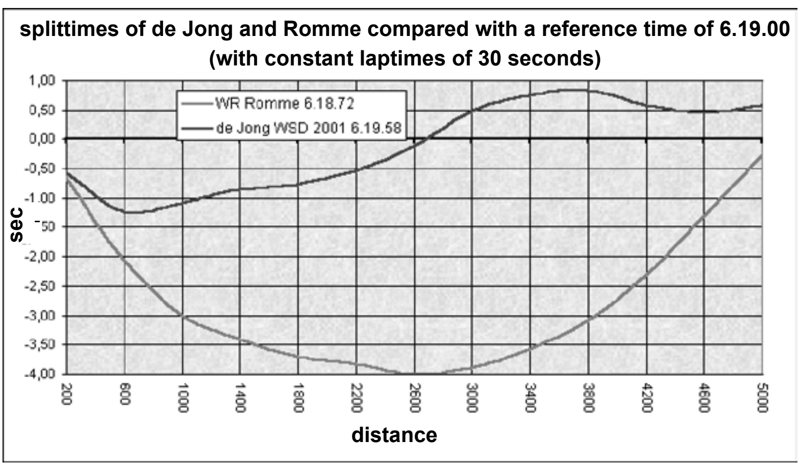


figure 2.6 The course taken by two speed skaters in a 5 km race

Streefland (1981) analysed problems with interpreting area related with kinematical concepts. He noticed that students must be able to disconnect the direct interpretation of area. Area has a mediating role. The goal is to determine the distance travelled, a one-dimensional quantity that is 'accidentally' represented by the area under the graph.
Similar problems come to the fore in reasoning about slope. In the previous section, we discussed resemblances between slopes in a distance travelled graph and hills in the actual motion, and the slope-height confusion. This is especially difficult for students reasoning about increasing or decreasing slope. A decreasing positive slope

can be associated with speeding-up (McDermott et al., 1987), and an increasing negative slope with a 'decreasing gradient' (Orton, 1983b).
Orton (1983b) also pointed out that students found it difficult to understand the principle of a tangent as a limit of chords (or secants). He found that students had difficulties in understanding that the rate of change is based on proportionality, i.e. the rate of change of a straight line is always the same, no matter how large the intervals for calculating this rate.
Students also had problems with the difference between rates of change of straight lines and of curves. The *average* rate of change over an interval is calculated in the same way for both graphs, but the *instantaneous* rate of change is not. For curves you have to take a limit, draw a tangent or derive a function, while for straight lines all instantaneous rates of change are identical to the average rate of change.
The problems in working with slopes become even more difficult when different kinds of notations are used. The left-hand graph below is copied from a mathematics textbook, while the right-hand one is from a physics textbook. In mathematics the focus is on how to approximate change with formulas, while in physics it is focused on sketching a tangent to approximate instantaneous change (fig. 2.7).

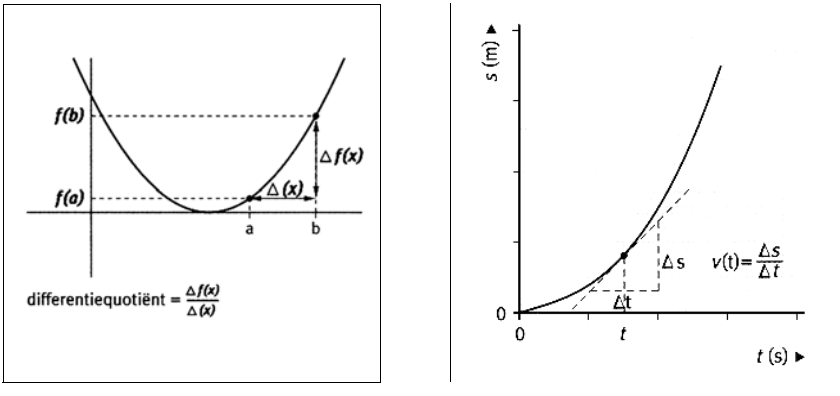


figure 2.7 On the left: mathematics textbook (Staal et al., 1998, p. 189)
On the right: physics textbook (Kortland et al., 1998, p. 197)

This example illustrates the different goals of physics and mathematics education in the use of notations and words, such as slope, rate of change, gradient, chords, secants and tangents. The physics textbook aims at working with global graphs (resulting from a number of measurements) and sketching tangents, while the illustration from the mathematics book aims at differentiating functions.
Nemirovsky & Rubin (1992) pointed out a recurring heuristic of students to assume resemblances between graphs of a function and graphs of its derivative. Students tried – even at the end of a teaching programme – to match global features of the two graphs, like increasing, decreasing and sign, instead of their actual relationship.

The last problem we will consider here concerns differences in graphical and phenomenological patterns. Standing still and moving with a constant velocity are similar from a graphical point of view. Both graphs have a horizontal velocity-time graph. But from a phenomenological perspective both situations are different. Using this example Boyd & Rubin (1996) showed that there is a difference between scientific discourse with graphs and everyday experience.
We select graphs with horizontal time axes, and use these graphs because they illustrate the relation between distance travelled and velocity, and the fundamental quantity acceleration. We think that graphs are helpful for understanding this, but we forget that these graphs are selected very carefully, and this aspect often remains implicit.

> In fact, in most math classrooms, the conventional format for distance/time graphs is taught as if it is the only way to express the relationship between these two variables. (Boyd & Rubin, 1996, p. 77)

2.1.5 Conclusions

Students' conceptual problems which we have sketched in this section concern the concept of velocity, instantaneous change, and the concept of limit. In addition, we noted differences in notations between physics and mathematics education, quantitative methods, and the use of graphs. Intuitively, velocity is understood as a property of a moving object and causal and descriptive aspects of velocity are mingled in everyday reasoning. Velocity is taught as a compound quantity, with respect to a frame of reference. However, students find it difficult to understand velocity as an external proportion and to calculate an average with such a proportion.
In the use of graphs we have pointed out resemblances between the global shapes of graphs and the characteristics of the phenomena they describe, and the graphing conventions needed to understand the concepts that can be derived from them. These conventions differ between mathematics and physics, and remain implicit; with the risk that students' understanding stays rooted in visual characteristics. Arguments that recur in discussions about problems in teaching calculus and kinematics are:

- almost no connections with day-to-day reasoning,
- use of implicit conventions and connotations that students do not recognise,
- differences between notations and goals in maths and physics classes, and
- a transfer of quantitative methods, building on graphs that are not as transparent as intended.

The gap between a street image and a school image of science, or, in other words, between intuition about motion and the formal theory of change is an example of what Kaput calls "the island problem in mathematics education" (Kaput, 1994b). This gap between the island of mathematical representations of motion and the mainland of everyday experience is not bridged by the use of graphs.

Graphs are meant to connect motion phenomena and quantitative methods. However, on this boundary between scientific representations and everyday experience, conceptual problems become visible: graphs are used to refer to *the expert's* frame of reference; apparently, they do not function as a tool for students to picture and to order *the students'* ideas and knowledge.
To overcome this gap we discuss recent alternative approaches to the learning of calculus and kinematics. These approaches take modelling motion as their central idea and use computer tools for supporting the students' modelling activities. Physical and mathematical perspectives are integrated in these approaches.

2.2 Approaches to the learning of calculus and kinematics

Analyses of students' conceptual problems are often accompanied by recommendations for education. These recommendations point to new approaches for research and lead to new developments in the teaching of calculus and kinematics. In this section we discuss some recent approaches in teaching that enable students to use computer tools for modelling activities.
The discussion of the approaches is split into two parts: discovery-based and invention-based approaches. With discovery-based approaches we refer to trying to make formal concepts accessible to students by presenting visualisations of concepts in specially designed environments. Students can use these environments for real experiments or exploring simulations. While doing so, they should be able to discover the related concepts. In contrast, invention-based approaches present a learning trajectory by way of model-building, to encourage students to construct the formal concepts by working from intuition to intended model.
This separation of approaches can be illuminated with a distinction between explorative and expressive modelling made by Doerr (1995, 1997). She characterised two kinds of student activities in computer-based learning environments. *Explorative modelling* can be associated with students' discoveries while they explore relations between representations. Literally, students discover a meaning by exploration, a meaning that was previously covered up by the instructional designer. An explorative modelling approach models a domain using representations in the experimental setting or in a computer program. The model that is enclosed in the environment determines the boundaries of the domain and describes relationships between variables. The variables are presented to the student in a simulation or with representations like tables, graphs, or value entries. By changing the values or properties of a variable they can observe the effect on the other values. This connection between representations helps students discover the connection between formal concepts and perceived experiences during exploration (Hulshof, 2001).
Expressive modelling stresses the importance of students' constructions and the progressive development of these constructions from intuitive to formal.

In an expressive modelling approach, the students' initial inventions are the starting point for a model building trajectory from intuitive interpretations towards formal concepts.

Thus, there is a difference between these approaches in the character of the mathematical model and the character of the students' modelling activities. In the explorative approach, the mathematical model is the final, formal model that has to be discovered by the students. For instance, this model determines the motion of the objects in a simulation and its graphical representations, or it determines how real motion can be represented with distance, velocity and acceleration graphs in the graphical software connected to a motion detector.

In the expressive approach, the mathematical models are temporary, tentative models in a trajectory that might vary from informal to formal models. The program offers students didactical models for describing relations within the described situation and can be used to solve the problems presented.

Gravemeijer et al. extended Doerr's distinction between expressive and explorative modelling (Gravemeijer et al., 2000). Explorative modelling is associated with discovery-oriented approaches and characterised by the implementation of an expert model that the students have to discover. Expressive modelling is associated with learning trajectories from students' inventions to the final, intended models.

The previous section 2.1 ended with Kaput's 'Island Problem': the gap between the island of formal concepts and the mainland of human experience. In short, you could say that the *discovery principle* is an attempt to make the island accessible with environments (e.g. realistic simulations) in which students can connect formal concepts with their everyday experiences (on which their ideas are based). The *invention principle* tries to prevent the gap from appearing by progressively building on students' intuitive reasoning.

In describing and discussing the approaches below, we encountered a problem in collecting information of both the instructional sequences *and* empirical findings. It was therefore often difficult to clarify a didactical implementation of an approach and to relate it to research results.

2.2.1 Discovery learning approaches

The approaches to calculus and kinematics discussed in this section share a common aspect: the intended target concepts are made accessible by linked representations: formal representations, like graphs, are connected to realistic simulations or experiments. These linkages are implemented in an environment for experiments or in a computer simulation. The connections illustrate the relationship between the representations and the phenomena. As a result, students can explore the educational setting, pose questions, test (intuitive) ideas and, in due course, discover these relationships and the underlying concepts. Firstly, we discuss discovery approaches using simulations, and secondly, using motion detectors connected to graphing software.

With computer simulations

Simulations are less realistic than real experiments or video recordings, but they have an extended functionality. Simulations enable students to not merely model phenomena, but also to generate phenomena using mathematical notations and controls (Nemirovsky et al., 1998). Kaput felt that today's students can have access to calculus thanks to technological possibilities like realistic simulations (Kaput, 1994b). He emphasised the relationship between mathematical symbol systems like graphs and everyday reality, and thought simulations could be a solution for the 'island gap'. To bridge the gap, Kaput looked for situations in which students could most exploit their own authentic experiences for investigating and coming to grips with formal representations and the related concepts. He tried to create such a situation with software developed in the Simcalc project (www.simcalc.com). The power of the device lies in the internal link between the various display systems. In this way, the everyday experience of motion can be linked to formal graph representations. This link offers students the opportunity to test the ideas they develop about the graphic representations.

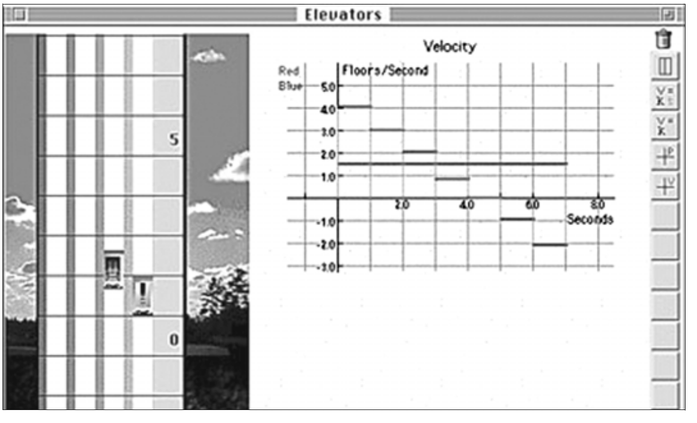


figure 2.8 Elevators in Simcalc

Figure 2.8 shows a screen dump of Simcalc. Changing the velocity graph affects the motion of the elevators. According to the graph, the elevator rises in the first second with a velocity of 4 floors per second. Every second thereafter, the velocity decreases. Students can predict at which floor the elevator will finally stop and then they can view an animation of the elevator's motion. Another elevator can be added to investigate what constant velocity graph is needed to let the second elevator move the same distance as the first. The task (teaching material retrieved from http://www.simcalc.umassd.edu in October 2003) focuses the students on understanding

the relation between an average velocity and a varying velocity. This graphical approach illustrates how the variation across the average balances out: the total area above the average velocity equals the total area below the average.
The designers stated that this approach makes it possible to build on students' intuitions about areas and averages. Their goal was to engage students in modelling situations that enable them to approach the idea of average velocity in relation to a same distance constraint and a same time constraint. In the teaching material, the relation between area, average velocity, and distance travelled is summarised in the graph of (fig. 2.9).

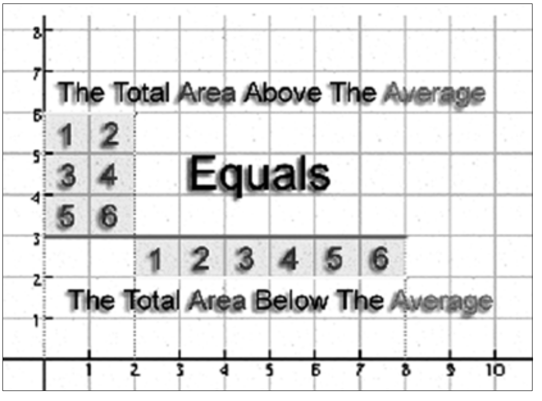


figure 2.9 Relation between area and average velocity

The effect of Simcalc has been investigated in formative studies that conclude that the software motivates students, and helps them to reason intelligently about slopes and areas of graphs (Kaput & Schorr, in press). Moreover, the dynamic images make a lasting impression, to which students can refer in describing their reasoning during lessons held after they have used the software. However, it is not clear from these studies whether the students are able to use the related concepts in new situations. The reasoning presented by the students was related to questions that varied only in complexity of shape from the Simcalc graphs and situations.
A similar idea can be found in the Trips software. The Trips software uses an animation of two running children whose motion is linked to a speed indicator and a graph (fig. 2.10). These links should enable students to correctly interpret the relation between motion and distance travelled graphs. On the same web site, the ease of using the software is emphasised:

> This computer simulation uses a familiar context that students understand from daily life, and the technology allows them to analyse the relationships in this context deeply because of the ease of manipulating the environment and observing the changes that occur. (retrieved from: http://standards.nctm.org/document/eexamples/chap5/5.2/index.htm in September 2002)

However, the problem may not be how easy it is to manipulate the software, but how easy it is to interpret the graphical relationships. The question is whether students understand these relationships? It could be that being able to connect graphical properties, such as slope, to what is happening in the animations may not be based on understanding, and therefore may not prevent reasoning like in the examples of conceptual problems in interpreting graphs (see section 2.1.4).

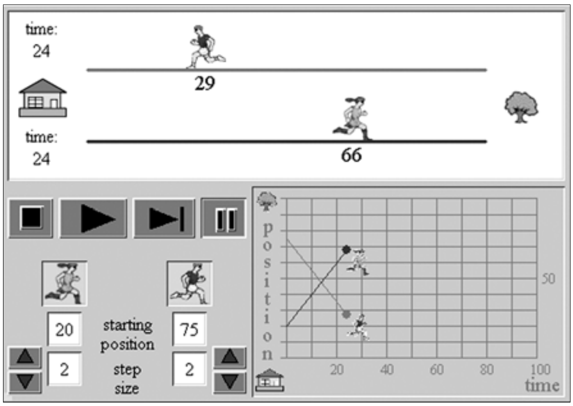


figure 2.10 Trips

In addition, we must note that instruction is not based on the presentation of and activities with only one tool. Noble et al. discussed the importance of using different environments (Noble et al., 2001). They investigated the relation between the mathematical tools in these environments and the students' reasoning and strategies. In an instructional unit, the Trips software was one of these environments. Within each environment, the students developed mathematical reasoning, but it is not clear from their report to what extent the students understood *why* and *how* their reasoning was related. The researchers seemed to think this was a follow-up of the students' work in the different environments:

> Of course it is sometimes important for a teacher and her or his students to step back from a diverse set of activities and ask what they have in common and to reflect on the general mathematical principles that describe the activities. However, we argue that these general principles become meaningful and relevant only to the extent that they are rooted in an ongoing background of experiences. (Noble et al., 2001, p. 106)

De Jong et al. (1998b) noted that much research is needed to understand how the implementation of such tools is related to a didactical embedding in education. We reflect on these approaches at the end of this section, but first we discuss discovery-oriented approaches with motion detectors connected to graphing software.

With motion detectors

Simulations refer to reality, but are not directly connected with it. Motion detectors enable a direct connection to be made between actual motion and graphing software or graphing calculators. In the case of a connection with a computer, such an educational environment is called a 'computer-based laboratory' (CBL). This technology offers real-time data acquisition, displays in various representations, and tool analysis.

CBL allows students to perform and to repeat experiments. They can choose the representations for the data display and which tools they use for their analyses. Theoretical concepts can emerge from experiments, experiences, observations and modelling activities when the appropriate educational material is used (Schecker, 1998). For example, body motion that is traced by detectors and displayed in graphs on a computer screen allows students to see how their motion influences the construction of the graph. Students can experiment with their ideas about the relation between real motion and those graphs. According to McDermott, this resulted in a better understanding than traditional methods (in Tiberghien et al., 1998).

The difference between using software with simulations like Simcalc and using motion detectors is that in Simcalc the link works in two directions: it is also possible to create simulations by adjusting the graphs. Nonetheless, motion detectors are connected to real motion, while simulations remain artificial microworlds.

Thornton and Sokoloff (1990) described an CBL-based curriculum, which starts by looking at velocity graphs in which students walk quickly and slowly, towards and away from a motion detector. In these tasks, students learn to relate velocity graphs to various kinds of motion. The tasks are supposed to clarify sign conventions for velocity and the relationship between the course of an actual velocity and the vertical distance of the graph to the horizontal time axis. After the activities students were asked to move in such a way that they matched a graph shown on the screen. Figure 2.11 shows a student's third attempt to match a given velocity graph.

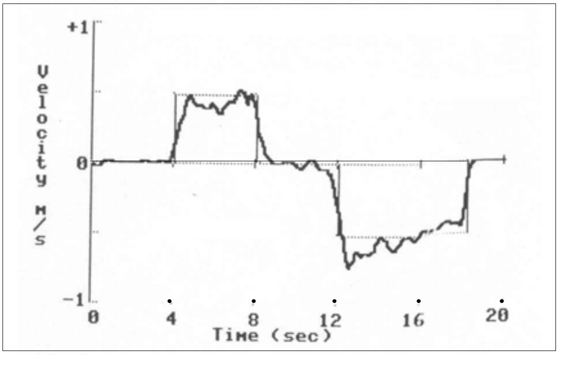
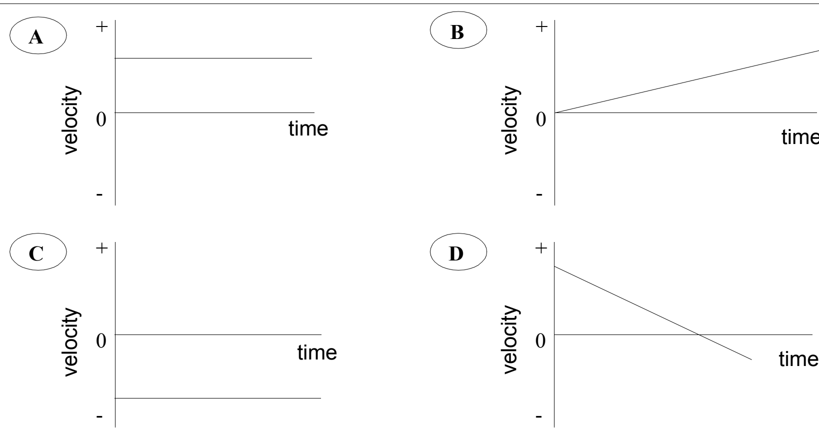


figure 2.11 Student's attempt to match her body motion to a velocity graph

In the subsequent tasks the students were asked to predict piecewise constant and linear distance, velocity or acceleration graphs, when one of the other graphs was given. They could check their predictions by creating the given graph with their body motion, and comparing the other graphs in the CBL to their prediction.
The situations in the tasks became more complicated and increasingly made use of changes in sign of velocity and acceleration. During these activities students could also use a cart moving along an inclined ramp. At the end of the CBL-based curriculum there is a focus on quantitative aspects of modelling motion. Students had to compare values of acceleration and velocity by using slopes in distance and velocity graphs. This curriculum can be incorporated into a traditional introductory physics course.
The educational results of using CBL have been investigated in summative studies, for which Thornton & Sokoloff (1998) designed the 'Force and Motion Conceptual Evaluation multiple choice test'. They used the test to compare traditional teaching methods with CBL-based methods. These studies strongly indicate that students from CBL-based methods scored better. Figure 2.12 shows a part of a test item. The item was introduced with a story about a car driving along a line. Students had to connect possible car motions with the graphs in a multiple choice test.



1. Which velocity graph shows the car moving away from the origin at a steady (constant) velocity?
2. Which velocity graph shows the object reversing direction?

figure 2.12 Part of an item from the Force and Motion Conceptual Evaluation multiple choice test

Many of the test items appear to be similar to the carefully designed motion activities in the CBL curriculum: motion with constant or linear distance, velocity, and acceleration graphs that students have to relate to each other. It might be the case that students mainly focused on using iconic characteristics for the relations between the motions and the graphs.

Hand-held technology, like the graphing calculator TI-83 and Texas Instruments' Calculator-Based Ranger (CBR), opens up even more possibilities. Body motion, like walking and waving hands in or outside classrooms can be recorded by a CBR. The calculator shows graphs or tables of these recordings. It is assumed that body motion, language and the use of technology will mediate and support students' transition from perceived motion to symbolic representations. Students connected live experiences through gestures and words with the data representation (Arzarello & Robutti, 2001). They had to walk from a sensor to a red line, stop there and return to the sensor (see figure 2.13 for the task and the resulting graph on a calculator).

The CBR will record your position with respect to time and will collect the data in a graph and in a table. The data are expressed in seconds (s) and in meters (m) respectively. Each 1/10 s a couple of data (time and position) are collected.
a. Describe the kind of motion you made in the corridor.
b. Using the graph and the table, describe how space changes with respect to time (increase, decrease, . . .).
c. Analyse the graph. Is it like a line? Is it like a curve? Does that curve increase? Does that curve decrease?
Consider the ratio: $m = \frac{s_2 - s_1}{t_2 - t_1}$ and use it to describe mathematically the graph of your motion (t_1 and t_2 are two subsequent time data and s_1 and s_2 are two subsequent position data).

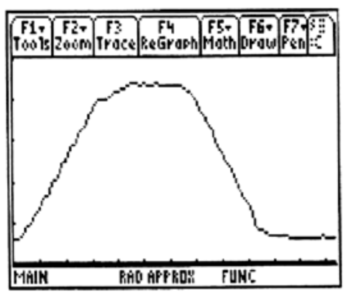


figure 2.13 Graphing calculator and motion detector

When the students tried to interpret the linear part of their distance graph, Arzarello et al. (2001) revealed how their reasoning changed. Their first reactions were: 'steps in the same time interval', 'the same pace', 'walking the same distance'. Then Fabio, one of the students said: 'to always keep at the same speed', a remark eventually reformulated as constant speed. This is an example of language development since graphical and conceptual ideas were mingled in the beginning but each became clearer. Brungardt et al. (1995) found that students had difficulty in connecting graphs to motion because they "tended to consider every little bump on a graph as significant, ignoring the fact that irregularities are often due to the many error sources inherent to recording motion." Brungardt et al. think this problem is reinforced by text books which often have smooth kinematical curves that make realistic curves harder to interpret. Beichner (1990) and Mokros & Tinker (1987) pointed out that the kinesthetic feedback and real-time graph construction of CBL materials contributed to its success in kinematic studies.

A recurring issue seen in these studies is that students are better motivated and discuss the situations and relations with the graphs more vividly than in traditional approaches that do not use such technologies (Brungardt et al., 1995; Nemirovsky, 1994). Hand-held technologies make these curricula more accessible in classroom situations. However, the practical problems of implementing such activities are not yet fully solved. The instructional setting demands tremendous effort and is essential for the students' activities, like hypothesis generation, reflection, and generalisation (Beichner, 1996).

Similar comments to those made for the simulation approaches can be given for this approach. Even when the instructional setting is optimised, understanding the relation between graphs and motion might be mainly based on visual relationships, without understanding of the underlying concepts. It is argued that establishing a dynamic connection between a phenomenon and the construction of graphs does not guarantee that students will interpret all the graphs' features correctly. There is a danger that interpreting and using graphs does not necessarily clarify the conventions for students, for example, how carefully chosen quantities for the horizontal and vertical axes can give more meaning to graphical features like slope and area (Brungardt et al., 1995; Kanselaar et al. 1999).

Testa et al. (2002) observed this problem in the teaching of kinematics through reading and interpreting continuous real-time graphs. They compared novice and advanced students' reasoning about real-time generated kinematics graphs. They found that the students' previous graphing experience influenced their reasoning. Novice students did not have the cognitive 'lenses' of the expert, which made it difficult for them to understand what they were supposed to do in the experimental setting. Testa et al. concluded that to overcome kinematic-related and iconic difficulties, an effort to address both these difficulties simultaneously is needed. This effort should address a shift from observing and exploring phenomena and searching patterns and regularities to be modelled from a certain perspective, to theoretical extrapolations in ideal cases.

Nemirovsky (1994) described a qualitative study with one student, Laura, working in an CBL environment. He described how Laura was grasping the meaning and representation of negative velocity. Her learning was triggered by a graphical feature that did not correspond with her expectations. As a consequence, she tried to interpret the graph in another way to become familiar with a new field of knowledge. She had to refine her idea in order to develop a new approach to graphing motion in which the computer representation made sense. She used hand gestures and tested ideas during her attempts. It was not the CBL curriculum, but her own actions and ideas that led to her development. According to Nemirovsky, the learning environment offered Laura opportunities for symbol use that encouraged her to revise what she knew and expected, to re-conceptualise the situation symbolised. However, the reasoning underpinning her revision seemed to remain implicit.

Reflection on discovery approaches

An important aspect in the discovery approaches described in this section is the computer-based link between formal knowledge of graphs and everyday reasoning. The software and the link with motion detectors seem attractive for students and are easy to use. The question is whether students can understand the graphical relationships? The ease of seeing patterns in various situations might hold the danger that connecting graphical properties such as slope or area to what is happening with motion, is not based on a true understanding of the underlying concepts, but on guess and check strategies. In these cases their understanding is jusitfied because they did not encounter a contradiction yet.

The mechanism that creates velocity graphs in the simulation-based software and in the micro-computer-based laboratories remains invisible to students. In addition, the computer tools take a ready-made symbol system of two-dimensional continuous graphs as a point of departure. This system is consistent with an expert concept of a mathematical system and the physical relations between velocity and distance travelled, which can be quite distinct from everyday experiences.

We do not want to argue that these approaches are non-productive or incorrect. They are used by a large number of researchers and need further investigation, especially with respect to the didactical possibilities of new technologies. Such research should focus on the students' learning, the role of the teacher, and the way in which using these technologies supports the learning and teaching of kinematics.

However, the teaching trajectories based on an extensive use of explorative tools do not seem to focus on creating a trajectory from everyday reasoning and informal symbolism to the formal concepts and symbols in the software. The implementation of explorative tools aims at visualising theoretical concepts, and connecting these concepts with real-life phenomena and reasoning, in order to bridge the gap between formal knowledge and everyday reasoning. Questions like how to understand velocity as a compound quantity, and how to measure velocity, do not seem to be addressed. Moreover, in the teaching process, the importance of the choice for a horizontal time axis might well be overlooked. Finally, the microworlds suggest a reality that cannot be found in real life: smooth graphs cannot be realised in real-life situations. The microworlds are idealised fiction, and the step from real situations to fiction is great.

These discovery approaches have similarities to the long didactical tradition in which instructional designers have tried to make abstract mathematics accessible for students by presenting concrete representations of mathematical structures. The designer takes the high level of abstract mathematics as a starting point and tries to bridge the gap with the students' level by using so-called structuring materials. Students are supposed to discover the mathematical relationships that were previously hidden (Freudenthal, 1991). We could consider that computer tools replace these materials in the approaches described; instead of exploring structuring materials, the students explore environments and graphical relations. These relations are based on a formal model that describes graphical relations and the system is consistent with an expert image of the theoretical system. However, it is questionable whether students come to understand that this consistency is the result of a long process (Gilbert & Boulter, 1998). According to Gilbert & Boulter, these computer environments show how a consensus model works, but education within these environments runs the risk of skipping the process of reaching a consensus. Students might not gain a correct understanding of velocity, the goal of the activities, and the constraints of the knowledge they acquire. Friel et al. commented on graphs structuring situations (e.g. motion):

> Can one interpret data accurately without having a significant level of understanding of the context? How do the characteristics of the information (e.g., similarity or difference in magnitudes of data values and frequencies) affect the interpretation? (Friel et al., 2001, p. 152)

The teacher can discuss and problematise such issues as how to measure velocity and the horizontal time axis. However, it is essential in these discovery approaches that students uncover the expert's model.

An alternative is an approach that tries to build on reasoning and informal symbols invented by the students while modelling motion. Such approaches provide insight into how students perceive motion phenomena and what they judge as relevant, which could then be used for developing the formal concepts. These invention approaches stress the importance of trying to get the students to experience the learning process as if they could have invented the new concepts and symbolisations themselves. In this sense, they differ fundamentally from the discovery approaches.

2.2.2 Invention approaches

The starting point for invention approaches lies in trying to create situations in which students invent symbolisations by themselves. These approaches try to build progressively on these inventions towards the formal concepts and symbol systems. In this section we discuss two invention approaches.

DiSessa et al. (1991) and Sherin (2000) described an approach that focuses on students' graphing inventions for modelling motion. They looked at lessons in which the students created graphs to describe and represent motion of a car which slows down and then speeds up (fig. 2.14).The students had been programming simulations of real-life motion with a Logo-like turtle that left a trail of dots across the screen. Next, the students were asked to come up with a paper-and-pencil way to represent the motion history of one of the simulations they had worked on. The students' solutions, which were to some extent inspired by the dot-tracking of the computer simulation, formed the starting point for a series of discussions and activities in which a graph-type representation of the motion history emerged. Sherin (2000) analysed the students' solutions and their reasoning, focusing on the developing generic symbolising competences of the students.

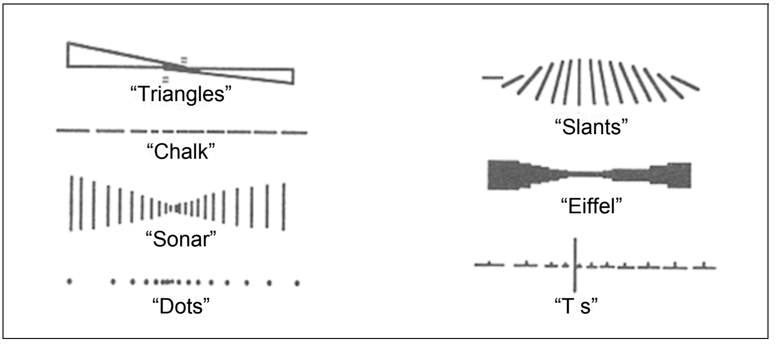


figure 2.14 Creating graphs to model motion (from DiSessa et al., 1991)

He contrasted this with their capabilities related to specific representational forms as seen in the discovery approaches. Instead of bridging a gap between everyday reasoning and formal knowledge, Sherin argued that such generic competences are useful and meaningful resources for the students. This graph drawing approach makes it possible to discuss with the students why a particular representation takes the form it does, what situation it is appropriate for, and relate it to the adequacy of alternative representations. Moreover, giving students the opportunity to express their creativity is important for developing a scientific attitude, and for developing the skills needed for an intelligent use of computers with a wide variety of representational facilities. Sherin also stated that such an invention approach places strong demands on the teacher:

> The teacher must be prepared to respond flexibly to a wide range of student inventions and ideas. It requires, at a minimum, that the teacher knows some of what to expect as the class proceeds: what kind of inventions students might offer, which inventions offer productive avenues to follow, and how to guide students in these productive directions. (Sherin, 2000, p. 438)

Sherin's analysis did not give answers on *how* to guide the students, but does make an important contribution to the problem of how to move from students' reasoning to formal knowledge.

A similar approach focusing on students' construction of graphs and based on interactive video instead of computer simulations, was investigated by Boyd & Rubin (1996). They found that students did not automatically construct a two-dimensional graph with a horizontal time axis and a vertical distance axis. This surprised the authors, given the students' prior education, which included a lot of two-dimensional graphing activities. In earlier studies with more explicit questions in this direction, it appeared that many students constructed graphs with a horizontal position axis and a vertical time axis. This choice was probably inspired by the graph's resemblance to a horizontal road, but we found students made the same choice in situations where they had to describe data on free fall.

Boyd & Rubin used a video of motion in their research. Video has the disadvantages of providing no physical experience (unlike motion detectors) and the students cannot influence the motion by adjusting the graphs, as in Simcalc. However, the advantage of video is that it can be played frame by frame. The starting point in Boyd & Rubin's research was that students were asked about the motion shown in the video, and to create graphical models of the motion to solve the questions:

> (...) students need to use their prior knowledge and mathematical sense-making skills to create their own representations. (Boyd & Rubin, 1996, p. 62)

The authors chose a setting in which students had to draw graphs by themselves after working with software that enabled them to create trace graphs of Quicktime movies. They could drag the ball on a snapshot of the movie in order to trace the motion. This has similarities with educational environments in which video is integrated with mathematical software (e.g. Measurement in Motion). However, software like VideoGraph is essentially different from such a setting, because it immediately generates conventional graphs like two-dimensional position, velocity and acceleration graphs, and also tables after tracing a point in every frame.

The authors observed a 6th grade student (age 12) in more detail: after working with the video and the software, her first problem was to create a graph that she could use to explain the motion of a ball that was slowing down in the video to a third person. She constructed the following one-dimensional trace-graph (fig. 2.15).

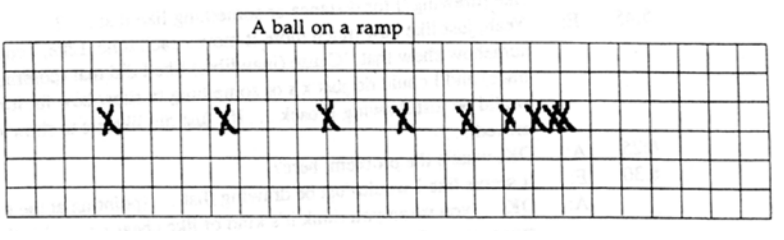


figure 2.15 First trace graph (from Boyd & Rubin, 1996)

Boyd & Rubin concluded that the interactive video focused her attention on change of position in equal time intervals or video frames. Although, it could be that she did not realise that the equal time intervals were a consequence of her focusing on displacements between frames.

In her reasoning she mainly used the distances between the crosses on her graph. These intervals appeared to be a basic structure element of both the motion and its representation. In drawing her graph she used her previous knowledge of graphs, e.g. a conventional horizontal axis (although the motion was vertical). What was striking was that she did not feel the need for a second axis to indicate how time related to distance (see section on conceptual problems). When the observer asked her to show *how* the ball slowed down and to display time, she did not draw a two-dimensional graph, but adjusted her first graph (fig. 2.16).

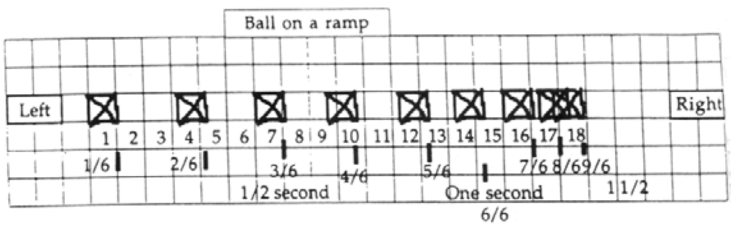


figure 2.16 Adjusted graph (from Boyd & Rubin, 1996)

With her adjusted graph she developed language for reasoning about the motion and her representations. She talked about speed as: ‘how long it takes to cover a certain distance,’ a characteristic kind of reasoning in sports. This differs from the conventional formulation: “how much distance is covered in a certain time interval”. Her formulation is remarkable given that the constant time intervals between the video frames might have focused her attention on time as an independent variable.
During the subsequent discussion her reasoning gave clues to the way she constructed the graph. When the observer asked: “can you tell what speed is?”, she reformulated the question with time needed for covering the total distance, instead of referring to how *far* the ball goes in a certain amount of time. When the observer asked her to draw a graph from which you could read the change of speed during the motion, she drew a two-dimensional graph (fig. 2.17).

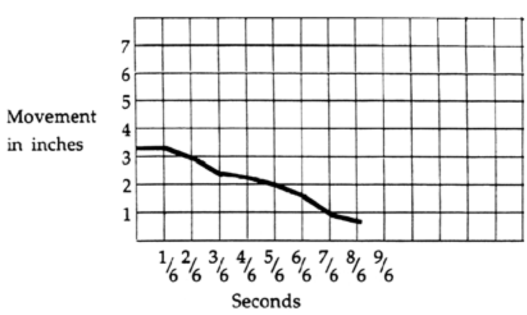


figure 2.17 Two-dimensional graph of movements (from Boyd & Rubin, 1996)

These activities changed her reasoning with displacements to reasoning with displacements *and* time intervals, during which an interpretation of speed as a compound quantity emerged. Boyd & Rubin argued that the need to represent a different quantity made her move away from her trace graph. Actually, speed is represented in this graph as movement in a time-interval. After the observer asked about the precise value of the speed at any moment, she reformulated her definition of speed to miles per hour.

Boyd & Rubin concluded that these activities are important, because they reveal ways to learn about conventional graphs from intuitive interpretations of and reasoning about motion. Moreover, the relationship between an actual motion, a graph of total distances, and a graph of movements in subsequent time intervals, is central to the learning of calculus. However, it is not clear to what extent the student had, so far, really experienced this relationship.

Reflections on invention approaches

These invention approaches give insight into students' reasoning that might form possible starting points for a learning trajectory from everyday reasoning to formal concepts. If it is possible to design such a trajectory, the gap between reasoning and formal knowledge will not arise.

It is remarkable that the students (especially in Boyd & Rubin's research) drew more primitive graphs than expected given their previous training. The graphs appear to be sufficient for expressing the students' thinking about the problems they were presented with data of subsequent positions. Apparently, it is not obvious to the students that they could calculate velocities from the data and draw velocity graphs.

Such analyses are valuable for gaining insight into students' intuitive reasoning and ideas with respect to the problems presented. Their descriptions provide a paradigm that can aid further research and design. Their development of making sense, of language and gestures is related to their development of the graphical models of motion.

A difficulty with invention approaches concerns the strong demands made on the teacher. Students cannot be expected to arrive at all the target knowledge by themselves. Sherin (2000) pointed out that the teacher must be able to value the students' contributions along the learning trajectory. In classroom situations, this means the teacher must know what kind of strategies students may come up with, and ways to build productively on such strategies. The teacher must be able to think flexibly within a certain range along the intended trajectory, in order to value the students' contributions and to provide them with content-related motives to aid their progress. Consequently, realising invention approaches in a classroom situation demands a lot from both the teaching materials and from the teacher.

2.3 Conclusions and discussion

We have discussed students' conceptual problems in calculus and kinematics, and described the gap between their everyday reasoning about velocity and the formal knowledge that is the starting point for teaching these subjects. Intuitively, velocity is understood to be a property of a moving object, but velocity is taught as a compound quantity with accompanying calculations. Graphs play an important role in teaching because they help to illustrate concepts, but it appears that students often focus on similarities between the global shapes in graphs and the characteristics of the phenomena described. Graphing conventions remain implicit, although they need to be dealt with explicitely for understanding the concepts that can be derived from them. Moreover, the conventions differ in mathematics and in physics. Students' understanding stays rooted in visual characteristics and in guess-and-check strategies.

Next we described two different approaches – discovery and invention – that aim to reduce the gap between students' everyday reasoning and formal knowledge by modelling motion with computer tools. In discovery approaches, these tools are designed to constitute productive learning environments. The tools allow students to discover the meaning and use of graphical models of motion, like time-distance graphs, by linking them to referents (in animations or with real motions). However, it is unclear whether students' understanding of velocity is sufficient for correctly interpreting the graphs.

The focus in the alternative invention approaches is on the contributions of individual students. Although the reasoning of these students change, it is not clear from these studies how taken-as-shared meanings about calculus and kinematics will evolve in classroom situations or to what extent the teacher can guide and influence the invention process. Nevertheless, we argue that, if it were possible to have the students invent distance-time and speed-time graphs by themselves, the gap between formal mathematics and authentic experience would not arise, because mathematical ways of symbolising would emerge naturally in the students' activities, and the accompanying formal mathematics would be experienced as an extension of their own authentic experience.

We have discussed discovery and invention in their pure forms, but in educational practice they can be mixed or be close to each other. Nonetheless, it seems there is a fundamental difference between starting from each of the two underlying principles. This difference between these ideas was also addressed by Clement in his article on model-based learning as a key research area for the teaching and learning of science. In this article he distinguishes between learning with models that are accurate and as rich as possible *versus* learning via a model evolution process by building on intermediate models that are only partially correct (Clement, 2000).

Somehow we have to help students learn to see conventional symbols as symbols of their own developing scientific activity. In the next chapter we will discuss theories on symbolising in order to understand how we symbolise and how this is related to concept development. These theories support an approach to teaching and learning calculus and kinematics by progressively building on students' inventions guided by instructional materials and a teacher. This approach was investigated in our research project.

3 Theoretical framework

This chapter describes the didactical and theoretical framework for this research and the underlying reasoning and design heuristics. The use and the interpretation of graphs in teaching is situated in a semiotic perspective. This perspective is based upon a selection of semiotic literature which appears relevant to the didactical and conceptual problems presented in preceding chapter 2. We cannot present an elaborate semiotic study here, but will outline some recent findings that may explain the possible causes of these problems. These explanations support an invention oriented approach to the teaching and learning of calculus and kinematics.

In section 3.1 we consider the relation between perception and interpretation. It is likely that students do not perceive the same information in graphs as professional mathematicians do. Students should have an appropriate preparation and a content-specific motivation to see important and useful characteristics in graphs and for preventing dominating visual resemblances. Knowledge of a phenomenon seems to be a condition for understanding mathematical descriptions of that phenomenon. In brief, we use graphs to organise motion and to teach students the relation between velocity and distance travelled, whereas motion has to be organised in order to correctly interpret the graphs. This problem is known as the learning paradox and is discussed in section 3.2.

The learning paradox provokes the question: how can we learn students new representations? How do humans develop symbolisations and how is this related to knowledge development? We discuss research on symbolisation in section 3.3, and discuss its consequences for the use of symbols in teaching and learning in section 3.4. We conclude that for an alternative approach to the teaching and learning of calculus and kinematics we must look for a dynamic process in which symbols and meanings develop together. We conjecture that such a process will prevent the problems observed with these topics. The characteristics of such an approach and the design heuristics for achieving it are described in section 3.5. Whether this alternative approach will yield the intended results is addressed as the research questions in section 3.6.

3.1 Notes on perception and interpretation

In his contribution to a research companion to the NCTM standards, Monk (2003) described a shift in mathematics education, from teaching graphs as standard objects involving skills and procedures towards the graphing activities and use of representational forms, like graphs, for giving meaning and forming communicating processes. Monk referred to an increased awareness of the visual aspects in mathematics, and to an increase in knowledge about how we see things, and how this is tied to what we know, think, say and do. This influences our thinking about graphing in mathematics education:

> This new interest in visuality in mathematics and how changes in the way we see things can be fostered, has promoted new points of view from which to address the problem of the persistent errors students make in graphing. (Monk, 2003, p. 260)

The close reciprocal influences between knowledge, looking and perception are nowadays generally accepted. What we look at and what we see in a situation is largely determined by what we know about the situation and by our purpose. Research in neurophysiological and psychological areas investigates how we mentally represent and recognise objects and patterns in various situations. This appears to be an ongoing process that develops in connection with experiences within a social environment (e.g. Wallis & Bülthoff, 1999; Jarvilehto, 1999; Goldstone, 1998).

In the next two sections we focus on two aspects of that process. First we discuss the relation between concept development and perceiving and organising phenomena. Second, we discuss the relation between perception and the interaction with an environment.

Organising phenomena

Brouwer (2003) investigated how we interpret poetic texts where the interpretation involves subjective representations of meaning. At first glance, this seems rather different from learning to interpret and use graphs in physics and mathematics. However, it appeared that the underlying ideas of understanding and concept formation are useful for understanding the conceptual problems described in chapter 2.

Interpretation is tightly interwoven with knowing. Brouwer refers to the philosopher Bartsch and describes a process of concept development based on recognising similarities and capabilities of association. Concepts are formed through the organisation of experiences (Bartsch, 1998). The development depends on both the individual's processing of the perceptual information, and the individual's social environment. To this second aspect, Bartsch adds that learning involves a process of expressing ideas and correction and approval within a community. Common ground is created by the shared experiential grounding for the conceptual system, the causal relations between concepts and perceptually processed properties of reality, and by social interaction within a community. Cunningham (1992) combined these individual and social aspects of knowledge formation within a culture:

> The world, as we know it, is culturally coded. What we experience as reality is really prior cultural and personal codings, prior structures invented (not discovered) both collectively by our culture and individually by us. (Cunningham, 1992, p. 170)

The organisation of experiences is structured in so-called similarity classes. Something is understood as a collection of situations gathered under similarities, which can vary from perceptual to structural similarities. The formation of a concept thus consists of an evolving structure of collections of experienced situations, while what is perceived as a similarity depends on the knowledge of the observer. According to Bartsch (1998), this organising process presupposes the availability of pre-cognitive perspectives and dispositions that take care of selecting relevant similarities and cancelling out irrelevant ones. The notion of perspective can be understood as a guided path of associations, guided by the individual's intentions or by the collective intentions of a community (Searle, 1995). Brouwer added to this with respect to learning language:

> Generally, when learning language, we copy behaviour by tentatively applying expressions to new situations, all the while meeting approval or objection. (...) not the meaning but the symbol itself is perceptually processed.
> (Brouwer, 2003, p. 190)

In the following sections we will return to the role of symbols in learning. In this section we focus on the dialectic process of organising experiences as a fundamental drive for perception and learning. Organising seems the core of mathematical and physical practices. In the beginning of the 20th century, the mathematician L.E.J. Brouwer described such a process when referring to mathematical activity:

> People succeed in detecting a regularity in a restricted area of phenomena independent of other phenomena, that can therefore remain completely latent during an intellectual consideration.
> To preserve the certainty of a perceived pattern as long as possible, people try to isolate systems, i.e. ignore the aspects interfering with the pattern; in this way man creates many more regularities in nature than originally occurred; he desires these patterns, because they strengthen him in the struggle for life, by allowing him to make predictions and to take measures.
> (Brouwer, 1907, p. 82; translated from Dutch)

From these findings we may conclude that students' intentions, as provided within an educational setting, are important for concept formation. The studies described above present a dynamic and situational view on learning governed by our experience. Traditionally, concepts are thought of as standing for something in an ideal sense, such as the mental object 'tree' that refers directly to the physical object of a tree, independent of the situation where it is used. Instead, nowadays concepts are thought of as *dynamic structures* by which mental representations are associated. In this dynamic view, concepts emerge as cognitive representations in relation to other cognitive representations. A tree becomes meaningful when it can be connected to a mental representation according to certain similarities of the object and situational relations. Trees are perceived as trees when these representations can be triggered.

Perceiving a tree is processing something as being similar to previously perceived objects according to similarities that are individually and socially determined (e.g. trunk, leaves, needles, forest). Representations can be perceived or retrieved in such a process of conceptualisation. All are triggered through focusing in perception, which is through attention. Conceptual associations connect mental representations to one another in a conceptual network. This seems to be related to Skemp's (1979) notion of cognitive nets in the learning of mathematics. The constructed net of connections determines mathematical actions in a given situation. This net shapes our existing knowledge and frames our aims, expectations, and perception associated with a concept. It acts as a tool for learning by making understanding possible and assimilating something to an appropriate position in this net (Kemme, 1990).

Finally, we would like to mention Bartsch's claims on stability with respect to her dynamic model of concept formation. Concepts resulting from associative structuring are not necessarily stable, since they may result from coincidentally activated mental representations, yielding associations that could be momentary or situational, and which might never be reinforced. However, understanding aims at keeping stable structures intact. When an utterance or a visual characteristic cannot be processed coherently, we have three strategies available before destabilising a stable concept: (*i*) discard the situation as nonsensical, (*ii*) discard it as false, or (*iii*) create a new 'quasi-concept' for this particular situation, by combining *some* similarities.

When recalling the conceptual problems on graphs in chapter 2, we might say that students' focus on similarities between graphs and situations being modelled – iconic interpretations – is probably the result of insufficient intentions or preparations. With an insufficient preparation, we refer to a conceptual network that does not afford the intended extension of this network. Students probably use this third strategy to process the graphical symbol and to keep their conceptual network stable. It is not the use of slope and area that they focus attention on, because these relations are not developed yet. Their prior work with graphs (rising is associated with something going up) makes them focus on iconic similarities between the graphs and the problem situations.

The same can be said about their notion of velocity as an undifferentiated property of moving objects. In a school situation they are presented with algorithms for calculating average and instantaneous velocities that are hardly connected to their notion of velocity. These differences might result in the dichotomy between street and school use of kinematical notions, and of notions of the mathematics of change. Consequently, it is not amazing that when we present students with a series of tasks, their learning is triggered by the variation in these tasks. When they do not understand the principles that govern the rules they have to apply, they can only focus on how to perform the algorithms, without understanding why, and without connecting them with related notions. Whether the discovery and invention-oriented approaches will solve this problem is discussed in section 3.4.

Interacting with an environment and tool use

Past experience and intentions within a (social) context play an important role in knowledge formation. Recent research has shown that past experience even influences our processing of stimuli. Jarvilehto is a behavioural scientist who investigated this connection between our senses and knowledge (Jarvilehto, 1999). He specified the relation between knowledge formation and perception from a neurological perspective:

> Every organism 'assumes' something about its environment in the sense that it has a structure into which only certain parts of the environment may be fitted. This idea was expressed several hundred years ago by Spinoza (1677), who stated that perception is a truer reflection of the structure of our body than any outer object as such. (Jarvilehto, 1999, p. 98)

Evidence for this idea comes from neurological studies. Recent studies have shown that neural responses did not simply follow the given stimuli. A dynamic organism-environment system is central to his study: receptors of stimuli are not simple transmitters of information, but give the possibility of direct contact to the environment, which is necessary for successful behaviour. This possibility is given by the structure of our present knowledge of the situation. Knowledge frames our intentions and expectations about a certain situation within an environment.

> The stimulus is a part of the process of reorganisation of the structure of the organism-environment system, which forms the basis of new knowledge. (Jarvilehto, 1999, p. 97)

Gibson (1979) emphasised the role of the environment in perception and interpretation. An individual does not simply assume something about the environment. It is the desired interaction with this environment that plays an essential role in his perception. Gibson called the perceived possibilities for interaction the *affordances* of the environment. An affordance is a resource or support that the environment offers. An affordance is perceived when it fits a purpose or activity and when you have the capacity to detect it. Examples of affordances include objects that provide actions (e.g. doorknobs, ripe blackberries). Before you are able to perceive such an affordance, you must have learned to detect the information and the intention to utilise the possibilities that are afforded to you.

Pea (1993) uses the notion of affordance-fit for tool-use related to learning. The knowledge carried out in tools may be exploited in activity by a new learner through a variety of paths: through observed use by others and attempts to imitate it, through (playful) discovery of its affordances in solitary activity, and through guided participation in its use by more knowledgeable others. As an illuminating case, Pea used the example of a forest ranger who each year measures the diameters of trees in a forest to estimate the amount of lumber. With a conventional tape measure he has to remember circumference as a measurable property of the tree, related to the diameter of the tree by a formula. This procedure for determining the diameter is prone to error and effort. As a result a new tape measure might be invented with the numbers on the new tape scaled so that the algorithm is built into the tool. When the new tape measure is wrapped around the tree, he can read off the diameter directly. The affordances of the tool parallels the achievements of the forest ranger.

Cobb (1999) underlines this interpretation of affordances as a person's achievements. An affordance is not an objective property, perceiving it is a personal achievement. We should add that the forest ranger's colleagues can use the new tape measure without understanding the algorithm that is built into the tool. Nevertheless, for using the tool in various situations, it is necessary to understand that the scale is related to the shape of the tree. It is not evident from the tool whether it can also be used for measuring the diameters of square rafters or lamp posts.

Kuhn (1970) described this subjectivity with respect to scientists working within a paradigm. The use and meaning of concepts and tools change within different paradigms after a 'scientific revolution'. Scientists practice their trades with different interpretations of the same words (like force, mass, etc.) and see different things when using the same tools. Kuhn wrote of such paradigms as incommensurable.

Instead of thinking of incommensurability as a relation between concepts, Klaassen & Lijnse (1996) emphasized changes of discourses in which vocabulary and interpretation have different meanings and conventions. Scientists see and interpret phenomena in a particular way, taking a certain perspective with a certain goal according to a paradigm. These perspectives and goals differ within different paradigms. In education differences in the way in which teachers and students see and interpret the same situations can be seen. We should be aware of these differences and of the relation between intention and perception in order to understand how students experience affordances of tools within a certain situation.

Conclusions

Our senses are not objective windows for the transmission of knowledge. Prior knowledge and our intentions for the process of organising and reorganising phenomena determine how we receive and process stimuli and recognise affordances in presented tools. Therefore, prior knowledge and intentions constitute learning, and concept development is a dynamic, context-dependent process. In this process, classroom communication plays an important role in creating common ground.

As a result of the subjectivity of perception and of this dialectic relation between knowing and perceiving, one could say that by extending our knowledge, we widen the range of possibilities of perceiving and interpreting and have more options in how we act and interact with our environment.

In education, we have to prepare students in such a way that they perceive the presented situations and tools as intended, relevant and useful. This is less simple than we are inclined to think.

3.2 External and internal representations

External representations are supposed to make certain notions accessible for students and are often used in education. Such representations reflect structures or characteristics of expert knowledge. The idea is that students adopt this knowledge by working with the representations. In mathematics education, this idea is also referred to as a representational view of mathematics education (Cobb et al., 1992). In this section we consider this representational view and discuss Cobb et al.'s comments on it. This discussion leads to an understanding of ways to prepare students for working with external representations in a meaningful way.

The representational view

Continuous distance-time and velocity-time graphs are used in kinematics education as 'instructional representations'. We – as mathematics teachers – see the relevant characteristics for the relation between velocity and distance travelled in these graphs. The graphs are object-like and transparent entities for us as expert users, and are immediately connected to our understanding of the notion of slope, and to difference quotients as a rate of change. A resemblance to the actual trajectory of the motion does not come into our mind. Cobb et al. (1992) and Gravemeijer (2002a) argue that this is because we 'automatically' see the mathematical and kinematical relations, and constantly experience that we can easily reason and communicate with these objects. For students, however, it appears that this transparency is not self-evident.

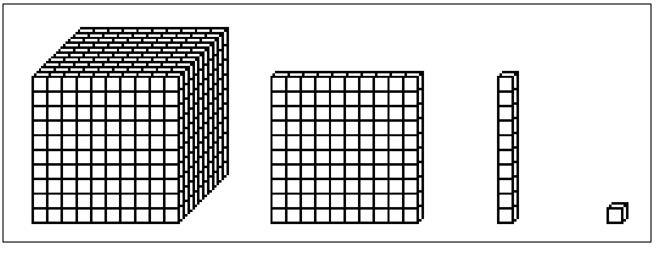


figure 3.1 Dienes' blocks representing the decimal place-value notation of 1111

As an example we will discuss the use of Dienes' blocks in arithmetic education. The blocks were thought to be transparent for students, in the sense that students were expected to see the benefit of tens and ones in the material (fig. 3.1).
The Dienes' blocks were designed to function as an instructional representation of

the place-value notation. The prescribed actions with the blocks could be mapped with steps of written algorithms for adding and subtracting numbers. Students used the blocks for operations on numbers, and while doing so they were supposed to gain understanding about the written algorithm. However, it was found that students who already knew how to perform the written algorithm could see the connection between the Dienes' blocks, the numbers and the algorithm, whereas students who did not understand the place-value algorithm, also had no clue how to solve the problems with the blocks (e.g. Holt, 1982).

The fact that the Dienes' blocks represent a place-value system is not discovered simply by looking at and manipulating the blocks. This is what makes mathematics so hard for students: visible objects refer to invisible mathematical concepts or relations that are still to be constructed by the students (Bakker, 2004). From the experts' point of view, the external representations make their knowledge accessible and communicable. However, the example of the Dienes' blocks shows that this might not be true from the students' perspective.

It is possible that there is the same problem, the same difference between expert knowledge and students' view, with the use of graphs in calculus and kinematics. Graphs may be thought of as transparent objects; in the sense that students are expected to see the benefit of, for example, the horizontal time-axis in distance travelled graphs. On the basis of this transparency, students have to understand tangents, average and instantaneous velocity, and difference quotients. Thus, the problem might be that students cannot see the mathematical and physical relationships in the graphs because these relationships are more sophisticated than their current understanding.

When we take into account both the traditional role of graphs in the teaching of calculus and kinematics, and the dialectic relation between knowledge and perception, we could claim that students hardly exploit the specific characteristics that are put into the graphs. With respect to this, Roth & Bowden (2001) pointed at the knowledge students should have of the situation and the pattern or regularity that is represented:

> (…) competent readings are related to understanding of both the phenomena signified and the structure of the signifying domain, familiarity with the conventions relating the two domains, and familiarity with translating between the two domains. Graphs are not significant (signifying!) signs on their own. (Roth & Bowden, 2001, p. 189)

Students will probably use regularities in the situation that fit their thinking for interpreting graphs. In calculus, the structure of the signifying domain concerns a measure for the slope of a graph to quantify change. Velocity, as a compound quantity, is used for teaching a way to measure the slope of a graph. The result of this seems to be that students should have a thorough understanding of velocity for interpreting

graphs and for developing the notion of a measure for the slope of graphs.
Continuous velocity-time and distance-time graphs are models of motion, which are probably not as transparent to students as we would like to think, because they have not yet organised motion sufficiently. For correct interpretations of these graphs, students should bear in mind that time and distance are primitive quantities, and that pieces of a graph can be associated to horizontal and vertical intervals which refer to increases of corresponding quantities. Moreover, areas and slopes in the graphs can be calculated with the lengths of these intervals, and these calculations can have a meaning in the context of motion. Without this understanding, velocity and distance-travelled graphs are not self-evident in supporting the concept-development of slope, the difference quotient, and of instantaneous change.
A teacher may spell out the relations between the graphs and the calculations, and students may be able to check and copy them. However, in such cases their understanding is not built upon their day-to-day reasoning about motion and might remain restricted to classroom solution procedures.

Signifiers and the signifying domain

Lemke (in press) gave an example of the complexity of graphing conventions in scientific publications. He claimed that such pictures are not iconic representations but refer to related verbal language and mathematical concepts. One of the pictures he discussed illustrates the change of temperature through a fluid (fig. 3.2).

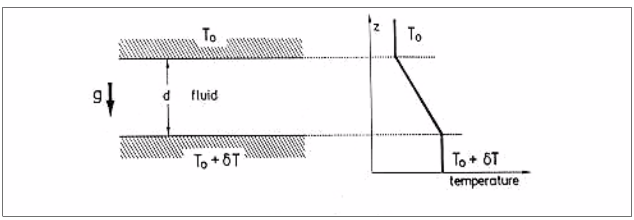


figure 3.2 Change of temperature through fluid
(from Berge et al., 1984, referred to by Lemke, in press)

The graph on the right shows a relation between position and temperature as if it were a shape in space. However, this signifier brings a continuous variation in all sorts of phenomena to the mind of the scientifically educated readers of this publication. Continuous variation is directly translated into verbal language like “the temperature rises linearly”, or “temperature is proportional to the depth in the fluid”. With this example Lemke pointed to the dependency between understanding, visual representations and related verbal language.
The author Robert M. Pirsig discusses the relation between recognising structures and understanding a manual in his novel ‘Zen and the art of motor cycle mainte-

nance' (1974). He describes this relation in the context of motorcycle maintenance. For understanding the manual of a motorcycle, it is necessary to understand *the structure of the signifying domain* and to recognise the type of razor used by the author of the manual to take the motorcycle to bits. He illustrates how beginners do not recognise either the razor nor the underlying systematics. They cannot connect it with their knowledge of the machine and, consequently, they have and develop inert knowledge. This can also be seen with first-time buyers of a computer or a video recorder who try to understand the manual. An adaptation to knowledge for beginners can be seen in flora determining cards for children, which use completely different systematics than a professional flora determination book.
These examples show why descriptions of the structure of a new domain might be a problematic starting point for learning. We therefore need another approach for developing these notions from students' knowledge.
Cobb et al. (1992) concluded that – in the representational view – meaning is analysed in terms of fixed mappings between arbitrary symbols and objects or events in the real world. This view results in using these mappings instead of using students' purposeful and socially situated mathematical activities as sources of meaning. Consequently, the representational view constitutes a duality between external and internal representations by trying to project expert mathematical interpretations into the students' environment as mind-independent external representations. In the previous section we noted a dialectic relation between knowledge and perception, thus this duality cannot be maintained.

> In other words, the assumption that students will inevitably construct the correct internal representation from the materials presented implies that their learning is triggered by the mathematical relationships they are to construct before they have constructed them. (Cobb, Yackel & Wood, 1992, p. 5)

Moreover, the mappings with external representations will presumably not bridge the gap, because they are perceived and interpreted differently by students. How can you see something in an external representation that does not (yet) belong to your knowledge repertoire?

The learning paradox

The consequence of the relation between knowledge and perception is that students must understand specific aspects of motion before they can see how these are displayed in graphs. They need to have identified and structured the signified domain before they can understand and use the signifier. Instead, we use graphs to isolate time and displacement, and to explicate the relation between these variables. Can we use graphs when students do not have sufficient understanding of the referring domain or of the purpose of displaying specific variables? More generally, can contexts, didactical models, or tools have a function in education when students do not see what they are supposed to discover? This problem is referred to as a *learning paradox* (Bereiter, 1985).

It can be argued that it is the teacher's responsibility to guide the learning process in such a way that students come to understand these relationships. But the gap between the students' understanding of the representations and the related concepts might not leave the teacher many options other than to spell out the correspondence between the graphs and the underlying physical and mathematical concepts. This results in presenting the rules of the algorithm (Cobb et al., 1992) or in patterns in the teacher-student interaction that disguise real understanding (van den Boer, 2003). The educational questions remain: What is the relation between knowledge and symbol development? How is knowledge development possible at all? Theories on symbolising should give us some clues on symbol development and the relation with construction of meaning.

3.3 Symbolising

In this section we discuss developments in the area of semiotics and symbolising. The semiotic perspective should help us to understand how the construction of meaning can be related to solving problem situations. The symbolising we are interested in concerns the development of representations from a mathematical or physical perspective, an activity with a tentative and heuristic character. We should realise that symbolising in general is a much more encompassing notion. Almost all our actions have a symbolising character.

Current ideas about the relation between a symbol, an object and a user are similar to recent understanding of perception and interpretation as described in section 3.1. In the past, it was thought that a symbol had a static meaning, independent of the user or a specific activity. As a consequence, one could spell out its meaning to students. Nowadays, a more *dynamic meaning* is assigned to symbols. The meaning of a symbol is evolutionary in a dialectic relation with the knowledge of the user and dependent on a context and the activity (van Oers, 1998). Symbols derive a meaning when someone thinks about them and works with them in a specific activity. Consequently, the construction of meaning is a dynamic and constructive process.

One of the pioneers in this field is C.S. Peirce (described in Edwards, 1972). Peirce argued that a symbol is not a static relation between an inscription, sound, or utterance and what it signifies. He claims that 'someone takes x to be a symbol of y' rather than 'x is a symbol of y.' A symbol x standing for y for a person P in a context C implies that when P becomes aware of x in context C, it calls y to mind. What is required for the interpretation of a symbol is regular practice in using the 'symbol' in a certain way.

A symbol has meaning by convention and possibly by character. Similarity between the symbol and the signified (a possible iconic character of symbols) plays a role in

connection with the intentions of the producer and the consumer of the symbol. Moreover, symbols (as concepts) develop in a dialectic growth process. Symbols organise reality, and as such, they change and shape the perceived reality.
Roth (2002) analysed such an organising process when observing students who were investigating a balance beam. Piaget investigated the same situation and concluded that students reasoned in an increasingly sophisticated way with weight and distance. However, Roth found that the students' identification of these variables took some time. Initially, the students focused on the position of the weights along the beam, followed by paying attention to change of location, size of the change, and finally to relative distance.

> Only when the relative distance was indexed to the fulcrum did these students perceptually attend to distance as conceived in the developmental literature. That is, the children did not perceive and act in a world as Piaget and others understood; rather, the perception of this world in these terms was the outcome of interactions of a behavioural environment perceived in very different ways. (Roth, 2002, p. 31)

Apparently, these interactions resulted in a progressive growth process from what appeared to be the case on the balance beam and what the students subsequently learnt.
Roth & McGinn (1998) presented another illustration of such a growth process in teaching. They studied 8th grade students doing an eco-zone inquiry near their school grounds. The students' activities started with digging in soil and finding different properties at different depths. Next, influenced by the teacher's remarks and the resource materials, they measured the depths of soil layers and calculated the relative composition of the samples for comparing their data with others. They collected these numbers in drawings, and then represented the properties in graphical drawings as dots on charts. Not the eco-zone, but their practices, the changing perspectives and the discussions became the context for the graphing. Numbers, drawings and charts were objects of discussions, served their reasoning, and influenced the way they proceeded. Consequently, in this dialectic growth process of symbolising practices, structuring the eco-zone became part of their activities.
The final example in this section concerns the modelling of a rolling ball, and also illustrates the relation between reasoning and external representations. Roth & Tobin (1997) describe how scientists construct *inscriptions* – e.g. drawings, graphs, tables – through series of other inscriptions. They used this notion for analysing translations between inscriptions in a physics lecture. Representations like numbers, tables, and graphs came to represent a rolling ball in one physics lesson. The translations between different representations that built on each other in the students' activities formed cascades of inscriptions. The analysis of the learning process from this perspective appeared to be a useful tool in understanding how the phenomena

became represented in a variety of mathematical forms. It appeared that some of the students' problems were related to implicit assumptions or conventions in the translations which the teacher made from one inscription to another (e.g. from numbers in tables to graphs without asking what should be placed along the axes).
We conclude that the invention and use of inscriptions by students cannot be separated from their understanding of the signifying domain – the context – and the practices in which they are produced. The terms 'context' and 'practice' are in this interpretation closely related to the way they are used in activity (van Oers, 1998). As a result of these relationships, students need to appropriate the use of inscriptions by participating in symbolising practices (Gravemeijer, 2002a).
In the next section we discuss the consequences of these ideas on symbolising, perception and knowledge development for the discovery and invention-oriented approaches in chapter 2. This discussion should guide us to an alternative approach for the teaching and learning of calculus and kinematics.

3.4 Reflection on discovery and invention-oriented approaches

This section starts with a reflection on discovery approaches (see section 2.2.1) from a symbolising perspective. We consider that a central aspect in these approaches is that the symbols – continuous velocity-time and distance-time graphs – are a starting point. Secondly, we focus on the invention approaches, and consider that, from a symbolising perspective, a growth process in symbolising and concept development can be recognised in these approaches. The question that arises from this discussion is how to guide such a process of teaching and learning in order to reach the intended goals.
Nemirovsky et al. (1995) investigated a discovery-oriented approach and focused on the development of language, gestures and the students' interpretations of graphs. They created an educational environment with moving vehicles and a direct link between the real motion and graphing software. During the activities, Nemirovsky et al. observed a *fusion* between students' reasoning about the shape of the graphical symbols in the graphing software and their reasoning about the corresponding motion:

> Merging qualities of the symbols with qualities of the signified events or situations, that is talking, gesturing, and envisioning in ways that do not distinguish between symbols and referents. (Nemirovsky et al., 1995, p. 38)

Nemirovsky & Monk (2000) took fusion as a useful process for construction of meaning by students. This is an associative process of talking, acting, and gesturing. The associations are determined by the learning situation and the students' prior knowledge. During the activities the students' associated specific graphical characteristics with specific characteristics of motion (e.g. pointing at a rising graph: 'here velocity rises').

In discovery-oriented approaches, the starting points are the activities with presented graphs. This seems to be an alternative for teaching a symbol system. The focus is on designing activities where symbol use results in understanding (e.g. Nemirovsky, 1994). However, in the approaches of Nemirovsky and Kaput (section 2.2.1), students primarily learn the language of graphs. What remains is the teaching of this language without attention for a conceptual development of underlying notions like velocity and instantaneous change.

The notion of fusion does not describe a growth process of concepts *and* symbols. In contrast with associations between observable events – from the students' perspective – construction of meaning can also be built from mathematical and physical aspects of modelling motion. From a symbolising perspective, we may conclude that establishing associations between graphical characteristics and an algorithm does not necessarily mean: understanding this relation in a mathematical and physical sense. In addition, we could not find a description of how the students' hypothetical ideas of the relations would build upon *their* kinematical and mathematical notions of graphs and the signifying domain including velocity. The discovery approaches remain a risk that students will construct associations through a process of trial and error, without being able to use or deduce a reasoning for these associations.

Consequently, from a semiotic perspective, it is not clear how discovery approaches can overcome the conceptual problems as sketched in the previous sections. The graphical symbols presented do not guarantee a reference to concepts that students know, but refer to concepts that we assume students will construct.

Instead of learning to work with presented symbols we advocate a growth process in symbolising activities. Such a growth process can be recognised in Boyd & Rubin's invention-approach (see section 2.2.2), although their example does not show how this process can be realised in a classroom context. However, it gives some clues for the way to design such a growth process supported by a series of inscriptions that might parallel concept development.

An illuminating example of such a growth process, we refer to the work of Meira (1995). He studied students while they constructed and used (graphical) inscriptions without a guiding teacher. His study showed how students' inscriptions organised their activity and sense-making in situations involving mechanisms that produced linear relationships. Moreover, students appeared to conjecture ideas that were based on the inscriptions they created. According to this dynamic point of view, it is in the process of symbolising that inscriptions were drawn and developed their meaning. In this process, notational systems shaped the very activities from which they emerged while, at the same time, the activities shaped the meanings that emerged. Meira (1995) proposed an activity-oriented view with a dialectic relation between notations-in-use and mathematical sense-making. Such an approach is recently advocated by many educational researchers (Gravemeijer, Lehrer, Van Oers & Verschaffel, 2002).

We argue that, if it were possible to get the students to invent distance-time and velocity-time graphs by themselves in a process of organising motion (and problematising velocity), the dichotomy between formal mathematics and authentic experience would not arise, since the mathematical ways of symbolising would emerge in a natural way in the students' activities. The acquired knowledge would be experienced as an extension of their own authentic experience. Such a process could start by picturing motion with trace graphs as in the study by Boyd & Rubin (1996). The pictured displacements between the successive positions could gradually acquire a more graphical than contextual meaning. This could be reflected by the fact that discrete graphs of displacements would start to signify constant or changing velocity (as a property) and the relation with the total distance travelled. The discrete graphs together with the notions signified constitute an initial structuring of motion.

Average displacement – calculated as the sum of the lengths divided by the number of displacements – and instantaneous change can be problematised in order to develop the notion of velocity as a compound quantity, together with advanced graphing. It is this emergence of meaning and symbolisations on the basis of modelling motion, which constitutes a dialectic growth process: both developing velocity as a quantity related to displacements and time intervals, and graphing practices from trace graphs to two-dimensional graphs of motion.

The challenge is to arrange classroom practices in which the tentative inscriptions and ideas of the students involve the intended characteristics of the signified target situation (in our case motion), and are the subject of discussion. Inscriptions and ideas which develop in these practices from the relevant prior knowledge into the notions aimed at.

In such learning processes, students do not have to subtract from what they already believe, but will mainly have to build on, and extend, their reality from a mathematical and physical perspective. According to this view, learning is a process that involves changes of intention and meaning, and developing an experiential base, with the aim of organising and explaining natural phenomena (Klaassen & Lijnse, 1996).

Concluding remarks

We are now in a position of having a better understanding of how to resolve the learning paradox. Knowing, symbolising and perceiving are tightly related. We can deduce that the variable *velocity* as a compound quantity (distance travelled divided by travelling time) was invented to organise motion. This notion of velocity frames the perception of motion and of motion graphs.

In their education, it is possible that students do not feel the need to mentally construct this variable to organise their experiences on motion, because they already have a vague and undifferentiated notion of velocity that seems to be sufficient. As a consequence, the teaching focuses on presenting the rules for calculations with the formula $v = s/t$ and relating it to calculating average velocities. The students' vague notion of velocity remains separated from these calculations and the graphical relations.

Thinking in terms of sequences of inscriptions seems to be helpful for planning a learning trajectory that fosters a growth process in learning. This is an alternative for presenting continuous graphs and spelling out the relation between velocity, distance travelled and the mathematics of change. If it were possible to bring students into a position where they understand the need for extending their knowledge, and are able to construct the intended notions, then the dichotomy between their street and school understanding and use of these ideas could be avoided. One question that remains is how do we know that students understand and use the graphical representations as intended? A planned or partly planned sequence should be the subject of a hypothetical teaching and learning trajectory that could be analysed in classroom situations. This analysis should focus on the way in which students' inventions run parallel to, or anticipate, the series of inscriptions, and whether their accompanying reasoning makes good sense. Design research has shown to be a research methodology which can be used for such analyses (see chapter 4).

We are aiming at a process of teaching and learning in which the evolution of student-generated inscriptions and of students' reasoning is part of a process of progressive mathematisation and physicalisation of motion phenomena. The notion of guided reinvention seems to offer opportunities for creating such a process. It aims at a process of teaching and learning that can be characterised as invention, guided by both the teacher and instructional materials.

3.5 Towards a solution: guided reinvention

An alternative approach should aim at a process in which the mathematics remains connected with students' physical and experienced properties of motion, like the relation between velocity and distance travelled, and that emerges from the students' modelling activities. Such a process is also the objective of *Realistic Mathematics Education* (RME), an approach which has its origins in the 1970s, in the work of Hans Freudenthal (Freudenthal, 1973, 1991). According to this approach, instructional design aims at creating optimal opportunities for the emergence of formal mathematical knowledge. The learning of mathematics is characterised by organising phenomena from a mathematical perspective through progressive mathematisation.

Organising phenomena seems precisely what Cunningham (1992) pointed to when he described how we build structures through our experiences. The students' understanding of mathematics should stay connected with, or as Freudenthal would say, should be rooted in their understanding of the phenomena that are being organised. As a prerequisite, students should experience these phenomena as meaningful and relevant, in other words: as experientially real.

Students will not spontaneously be active or produce productive results from a mathematical perspective. A teacher and instructional materials are essential for creating a process of teaching and learning in which students experience a balanced mix of freedom and guidance. The guidance consists of handing out activities (for which there are, as yet, no standard procedures) and motivating students to construct their own solution procedures. These procedures are compared in teacher-guided classroom discussions, and the teacher evokes reflection and instigates the students to contribute arguments concerning elegance, adequacy, and sophistication. The activities and discussions are delineated in a trajectory, which leads to a – hypothetical – guided reinvention learning process.

The main idea of guided reinvention is to allow learners to come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible. As a result, the gap between a street image and school image of science and mathematics should not arise. Various design heuristics have been developed for stimulating productive inventions and realising guidance, among them didactical phenomenology, emergent modelling, and problem posing.

Mathematical phenomenology refers to how mathematical ideas structure and organise phenomena. A *didactical phenomenology* refers to looking for situations that create the need to be organised by the students. Such an analysis investigates how the concepts we want to teach help organise these situations, and how they can be problematised for the students. These phenomena ('that beg to be organised') are the starting point for education, rather than the targeted symbol system. In this research project, we took motion as the phenomenon for developing both calculus and kinematics. History and prior research point at the opportunities offered by motion. In chapter 5 we elaborate this choice and focus on specific motion phenomena that seem to foster the reasoning we target in the students.

The *emergent modelling* heuristic concerns guiding the students with didactical models which connect to their tentative reasoning and which have the possibility of emerging into the targeted models. This heuristic seems helpfull for supporting students in a growth process with a series of graphical inscriptions. The *problem posing* heuristic aims at guidance that focuses on ways to provide students with content-related motives for proceeding in an intended direction. These motives should frame the students' intentions for structuring motion. In the following sections we will discuss these two heuristics in more detail, together with the possibilities of *computer tools*.

Neither guided reinvention nor the design heuristics are new, nor are they a direct consequence of the semiotic topics discussed in this chapter. However, they seem to gain in interest as a result of these semiotic insights and their relation to education. The heuristics of emergent modelling and of problem posing try to offer students guidance that fits with an evolutionary and dialectic relation between framing intention, construction of meaning, and the development of inscriptions.

3.5.1 Emergent modelling

Research on the design of primary-school RME sequences has shown that the concept of *emergent models* can function as a powerful design heuristic (Gravemeijer, 1994, 2004a). Here, the point of departure is in situation-specific solution methods, which are subsequently modelled. First, context problems are selected that offer the students the opportunity to develop situation-specific methods and symbolisations. Then, if they do this, these methods and symbols are modelled from a mathematical perspective. In this sense, the models emerge from the students' activity.

The model first comes to the fore as a *model of* the students' situation-specific strategies. While working within more situations and discussing strategies in the classroom, the model gradually becomes an entity in its own right and starts to serve as a *model for* mathematical reasoning to foster higher level ideas. This shift from 'model of' to 'model for' changes the students' reasoning, and results in the construction of a new (mathematical) reality (Streefland, 1985). In reverse, constructing this reality made it possible for the model to change in character.

The labels 'emergent' and 'modelling' in this context may need some clarification. The continuous progression in this modelling process is emphasised by the label *emergent*. It refers to the fact that the model emerges from the students' activity, together with the mathematical reasoning targeted. Consequently, emergent refers both to the character of the process by which the models emerge and to the process in which these models support the emergence of mathematical knowledge.

The label *modelling* refers to an activity of trying to organise and structure a problem situation or a phenomenon. Traditionally, mathematical modelling is seen as translating a situation into available mathematics, reasoning about it mathematically, and translating the results back into the original situation. In education, this view on modelling requires the students to be able to mathematise a situation, have the appropriate mathematical tools at their disposal, and to be able to step back and judge the adequacy of the model they use. The modelling we refer to precedes this kind of modelling and serves mathematical development (Gravemeijer, 2004b). The inscriptions that students develop in a problem situation where they do not have standard procedures at hand may have a tentative and temporary character. The model that is referred to in this emergent modelling heuristic is shaped by their activities and discussions into a series of consecutive inscriptions and ideas. From a more global perspective, these inscriptions can be seen as various manifestations of the same model. So when we speak of a shift in the role of the model from 'model of' to 'model for', we refer to this global perspective.

While identifying relevant characteristics in a situation and describing patterns, structures or relations, we leave out other aspects of the situation. Consequently, the more mathematical (and general) these descriptions are, the less we will recognise of the original situation. This holds the danger that mathematical notations and manipulations are going to be seen as an artificial game with no connection to day-

to-day life. To prevent this from happening, it seems to be necessary to trace the origins of these notations and concepts regularly.
An important criterion for emergent modelling is in the models' potential to support mathematising in line with the students' thought processes (Gravemeijer, 1994, p. 188). The idea is to look for models that can be generalised and formalised by the students to develop into entities of their own, which can become models for mathematical reasoning. During the students' modelling activities consensus can be reached, or manifestations of these models can be introduced, at well-chosen moments to preserve the connection between mathematical notions and the situations or activities that they describe. Ideally, the students experience this process as if they invented or could have invented the model by themselves.
Note that there will always be tension between a bottom-up approach that capitalises on the students' inventions, the need for teachers to be able to plan instructional activities in advance, and the need to reach certain educational goals. As a consequence, a top-down element is inevitable in instruction. Our key consideration, however, is that the students experience these top-down elements as bottom-up, i.e. as solutions they could have invented for themselves. For the instructional designer, this implies striving to understand students' image of inscriptions and to keep the distance between 'where they are' and what is being introduced as small as possible. In this research project we used an instructional sequence to investigate how this could be realised for the teaching and learning of the basic principles of calculus and kinematics.

3.5.2 Problem posing

The problem posing design heuristic was developed and applied in various studies in physics education (Klaassen, 1995; Kortland, 2001; Vollebregt, 1998). The heuristic addresses the problem of guiding students from a content-related motivational perspective. It contributes to finding an adequate balance between the freedom offered to students for their inventions, and the guidance needed to enable them to construct the targeted concepts in a progressive growth process. If this process is to make sense to them, students must understand what they are doing and why they should extend their knowledge in a certain direction. It then becomes more probable that they are aware of the connections with their conceptual network, will construct the targeted knowledge, and will perform this based on a foundation that they understand.
The emphasis of a problem posing approach is thus on encouraging students to reach a position where they come to see the point of extending their existing conceptual knowledge, experiences and belief system in a certain direction. They should be provided with content-specific motives, so that they control the direction to proceed. The main idea of this heuristic is to introduce a problem situation which students experience as relevant, and which involves more than a few activities.

This overarching problem situation should offer an orientation for students on a new issue, and evoke an interest in, and content related motives for, further investigation of this issue. From there on, students should be led into positions where they feel the need to extend their knowledge in the intended direction (Klaassen, 1995). The students' activities and classroom discussions should evoke reflections, and – when these are successful – the students should be able to pose the problems that have to be solved to proceed in a promising direction with respect to the problem which has been introduced.

> At any time during the process of teaching and learning, pupils should be able to see the point of what they are doing. A problem posing approach attempts to arrive at such a situation by providing pupils with content specific general and local motives for subsequent learning. (Vollebregt, 1998)

As with the emergent modelling heuristic, this heuristic has implications for the teacher's role. The teacher should try to arrange the classroom discussions in such a way that students are challenged to contribute to the direction to proceed.

3.5.3 Computer tools

The use of tools influences the process of students' mathematical sense-making (see p. 47). Cobb (1999) illustrated this by describing an interplay between the students' ways of symbolising and the development of mathematical meaning in terms of chains of signification. In relation to the integration of a series of computer tools in teaching materials for statistics, he used the notion of affordances. The tools afforded the students' reasoning on statistical problems. This implies that instructional designers should take into account how students might reason with representations in such computer tools as they participate in a sequence of mathematical practices (see also Bakker, 2004).

We expect computer tools to afford reasoning and experimenting in various situations and to be useful for realising emergent modelling in classroom situations. Thanks to the visual and dynamic possibilities, students can investigate many situations and work with model representations more frequently for anchoring recognisability, expressing new ideas and supporting level raising (e.g. Drijvers & Doorman, 1999; Pijls et al., 2003; Ruthven, 1990; van Streun, 2000).

Note that the danger of the problems discussed with the discovery approaches – students have to guess the meaning of the representations presented to them – is also present here. At any time in the teaching and learning trajectory, we should be able to argue that the representations will fit students' thinking, and that they are able to trace meanings, possibly through a series of inscriptions that has been built during the sequence and that bears the history of concepts and representations. The affordances of the tools are not just tool characteristics that are objective features for

every user. These features only *become* an affordance for students in problem situations. The use of this affordance is achieved by the student as a result of their preceding learning process.
In addition, research into the use of hand-held calculators and computer tools also points at the importance of the teacher's role for the appropriate use of previously designed tools for education, and as a consequence, for the students' learning. Mathematical ideas and ways of tool use are closely related, and their development is intertwined. Such a process of tool appropriation, learning, and reflection on tool use has both an individual and a collective aspect, and needs guidance by the teacher (e.g. Artigue, 2002; Doerr & Zangor, 2000; Drijvers, 2003; Hoek & Seegers, in press).

3.6 Research questions

The goal of this research project is to find out how students can learn the basic principles of calculus and kinematics. The idea is that the students' problems with these topics arise in a use of graphical representations without problematisation of the concepts that the representations refer to.
An alternative approach of guided reinvention tries to realise a learning process within which graphical representations and concepts develop together in a dialectic growth process. The design heuristics of emerging modelling and problem posing can be used for designing such a reinvention process. In this project we investigated how these ideas can be implemented, how computer tools can support the students, and how they can learn calculus and kinematics in an integrated course in upper secondary education. The questions that arose were: how can students invent and use graphs, and how can we provide them with tools – possibly in computer-environments – that are both meaningful and that foster advanced reasoning? Thus, the main question of this project was:

> *How, and to what extent, can the teaching and learning of the principles of calculus and kinematics be integrated in a guided reinvention course on modelling motion using computer tools?*

The a priori *paradigm* of this research project was posed in chapter 1 and describes learning processes aimed at. Characteristics of these processes are that students experience learning as extending their day-to-day reasoning and that they regard the knowledge they acquire as their own knowledge. These processes are supported by problems which are experientially real for the students and evoke productive solution strategies. The analyses of conceptual problems in calculus and kinematics and the discussion of symbolising point to such an approach for these topics. We therefore need to design a learning trajectory that brings students to invent the principles

of calculus and kinematics, and to experience these inventions as rooted in their day-to-day reasoning.
To design such a reinvention course, building on theories on symbolising and tool use, our *choices* are the design heuristics related to (*i*) emergent modelling and (*ii*) problem posing, and the use of (*iii*) computer tools that afford students ways of reasoning. We try to understand how our paradigm and these choices can be used, to design and analyse a learning process in which students contribute to the constitution of mathematical symbols and understand what they refer to. Moreover, we hope to gain insight into how computer tools can be used for the emergence of symbolisations of motion and change. Finally, we aim at a better understanding of the relation with theories on symbolising and perception.
Consequently, the main question of this research project was split into two research questions. The first research question concerned the aims of the designed process for teaching and learning, and whether we succeeded in preventing the conceptual problems described in chapter 2. The second question concerned the choices underlying the design of our process:

1 *How can students develop the basic principles of calculus and kinematics in a process of teaching and learning that can be characterised as guided reinvention?*

2 *To what extent does the course of this process empirically support the adequacy and the understanding of our choices: the role of computer tools and of the design heuristics related to emergent modelling and problem posing?*

Together with an instructional sequence, we described why we think our design works, how it works, and which observational criteria support these assumptions and our research questions. A global description of these criteria is presented in table 3.1. The first question in the table is related to the first question above. The next three questions are related to the second question and distinguish between emergent modelling (2 EM), problem posing (2 PP) and the use of computer tools (2 IT).
In chapter 5 the design of the teaching and learning materials is described, together with an explication of the relation between the observational criteria and the research questions. In the following chapter 4 we describe the research methodology of design research.

Questions	Observational criteria
1: Do students perceive the problem situations as intended, contribute to the guided reinvention process, and reach the intended goals?	The initial (intuitive) reasoning of the students touches the intended concepts and pinpoints the significant elements in the situation as intended. Their inscriptions and reasoning are shared and form the basic input for classroom discussions and can be used for the way to proceed. Finally, the students master the intended principles of calculus and kinematics.
2 EM: Does the previously planned sequence of graphical tools fit students' thinking and foster advanced reasoning by a shift from *model-of* to *model-for*?	The way they reason with the tools changes from context-oriented to concept-oriented. This change provides the imagery for the following steps in the learning process.
2 PP: Are students aware of a global problem that is being solved, and do the local problem situations provide the students with content-specific motives to proceed in the intended direction?	The students show understanding of the relation between their activities and a global problem. The questions they pose fit the direction to proceed. Students participate in classroom discussions, and the teacher has opportunities to share remarks and pose questions, and use their contributions for evoking the need to proceed in the intended direction.
2 IT: Do the representations in the computer tools fit prior reasoning and how do they afford advanced reasoning and sense-making?	Students initially use and give meaning to the tool with reference to prior activities. During the work, their thinking and reasoning changes to using and discussing the intended mathematical and physical relations. In discussions afterwards, students refer to the (dynamic) representations of the computer tool as signifying a mathematical or physical concept.

table 3.1 Operationalisation of subquestions

4 Methodology

This chapter describes the research methodology for answering the research questions (section 3.6). The relation between the character of the questions and the methodology is discussed in section 4.1. The questions suggest using a *design research* method. Section 4.2 describes the design research for our project and its three phases: the preliminary instructional design phase, the teaching experiments, and the data analysis phase. These phases are addressed in sections 4.3 - 4.5, respectively.

4.1 Consequences of the research questions

The aim of this research project was to investigate (*i*) the process of students' learning in the domain of calculus and kinematics, and (*ii*) the means of supporting and organising that learning process.

We started with analysing literature on problems in the teaching and understanding of the principles of calculus and kinematics (see chapter 2). The main conclusions concern the interplay between using and interpreting graphical inscriptions on the one hand, and organising phenomena on the other. This interplay between graphs and the signified situations became the starting point for our need to gain more insight into the development and use of inscriptions as an aid to learning. In chapter 3 the analysed literature on this topic is discussed. We reasoned that didactical problems in the teaching of calculus and kinematics might have their origin in overestimating the power of graphical symbols, and insufficient attention being paid to students' symbolising activities. We proposed an alternative approach to the teaching and learning of calculus and kinematics. This approach is based on the claim that graphical symbolisations and understanding of motion can co-evolve in a planned process of teaching and learning, a process that is characterised as guided reinvention (see section 3.5). At the end of chapter 3 we formulated the research questions for our project.

The key issues in our approach are the role of computer tools and the design heuristics related to emergent modelling and problem posing. We wanted to investigate students' reasoning with respect to our envisioned process of teaching and learning based upon these issues. Our interest was in the students' contributions, the way they reasoned, and the development of symbolisations and shared understanding in the classroom.

To be able to answer the research questions, we had to create an educational setting with which we could investigate *to what extent* and *how* this dialectic process of symbolising could be fostered. We designed an instructional sequence for creating this educational setting, and serving the how-part of the research question. Consequently, the aim is primarily the understanding of the learning processes in connection with this instructional sequence.

A research approach that consists of planning and creating innovative educational settings, and analysing teaching and learning processes, is precisely what the methodology of *design research* targets. This methodology has proved itself suitable for developing empirically grounded, local instruction theories in the areas of science and mathematics education (Gravemeijer, 1994; Klaassen, 1995; Lijnse, 1995; Streefland, 1991).

4.2 Design research

In general, a design research approach aims at generating empirically grounded theories. The main result is not a design that works, but the reasons how, why and to what extent it works. Firstly, an initial instructional design is developed, and educational settings are created for investigating and generating theoretical conjectures. Secondly, depending on the questions to be considered, the analysis of the teaching experiments focuses on various elements of the design, such as the students reasoning with the tools provided, the classroom discussions, the kind of collaborative work, or the development of specific classroom norms (Cobb et al., 2003; Edelson, 2002; Gravemeijer, 1994; Gravemeijer, 2004a).

The initial instructional design for the teaching experiment aims at a conjectured learning process and is based on prior research and theory. However, during the design research, initial conjectures may be refuted or adapted, and new conjectures can be generated and tested. Design research in this sense has both a hypothetical and a reflective side (Cobb et al., 2003), which leads to a delicate and iterative process of testing, reflection, and redesign. The testing takes place in teaching experiments, and the reflection is – in most cases – based upon qualitative data analyses. We describe the three phases *preliminary design*, *teaching experiments* and *retrospective analysis* for this project in more detail in the following sections.

The literature analysed resulted in initial conjectures for the learning of calculus and kinematics and the means to support this learning. These means of support consist of a sequence of activities for the students, computer tools, and instructions for the teacher. We refer to these means with an *instructional sequence*. In addition to this instructional sequence we formulated testable conjectures and observation criteria. These conjectures, which concerned the major shifts in students' reasoning in relation to the means of supporting and organising those shifts, were shaped by prior research. They were tested and the results are used to empirically support or adapt the emerging teaching theory. Such a theory describes the envisioned learning route, successive patterns in the teaching and learning processes, and the means to support these patterns, and is called a *local instruction theory* (Cobb et al., 2003; Gravemeijer, 1994; Gravemeijer, et al. 2003). The adjective 'local' refers to the topic, in our case calculus and kinematics. The patterns consist of descriptions of major shifts in students' reasoning, specific kinds of activities or contexts, the sequence in these activities, the role of didactical models, and the teacher's role.

Apart from the contributions to a local instruction theory, we analysed our choices

regarding emergent modelling, problem posing, and the role of computer tools. These design heuristics are on the level of a domain-specific instruction theory for the learning of physics and mathematics.
The initial instructional sequence is a first elaboration of our conjectured local instruction theory. The description of this sequence includes student activities, teaching guidelines, together with hypothetical scenarios of the lessons and a justification of the choices made. Such a sequence has much in common with a hypothetical learning trajectory (Simon, 1995). Simon developed the idea of designing, planning and evaluating cycles of one or two lessons by a teacher in his or her specific classroom. Our instructional sequence describes a series of lessons, and aims at testing a conjectured local instruction theory for the teaching and learning of calculus and kinematics (fig. 4.1).

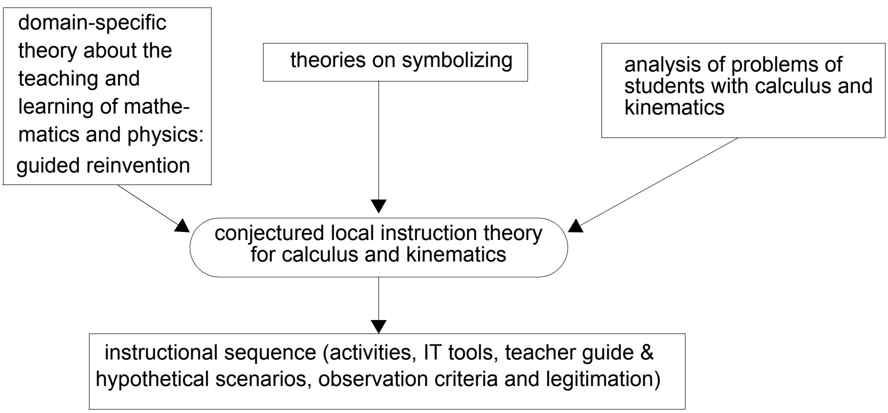


figure 4.1 The constitution of the instructional sequence

The experiments with the instructional sequence would provide empirically based arguments that justify or refute the constituting elements of our conjectured local instruction theory. Moreover, patterns in the experiments might lead to hypotheses concerning emergent modelling, problem posing and the role of computer tools. These hypotheses could be tested retrospectively with the data available. This cyclic connection between conjectures that accompany an instructional sequence, teaching experiments, and emerging hypotheses concerning over-arching themes is also referred to as generalisability of the emerging theory in design research (Gravemeijer & Cobb, 2001; Gravemeijer, 1995). As an aside, note that even theories on symbolising can influence domain-specific instruction theories through conjectured local instruction theories.
Similar use of instructional designs can be found in other research projects within our research programme: on the use of computer algebra for grade 9-10 students

(Drijvers, 2003), and on symbolizing and the learning of statistics with computer tools for grade 7-8 students (Bakker, 2004).

Such a detailed description for a sequence of teaching and learning in design research is in line with the scenario concept as elaborated in physics design research for the teaching and learning of radioactivity (Klaassen, 1995), and for the teaching and learning of the particle model in physics (Vollebregt, 1998). However, in their scenario a heavy emphasis was put on describing the teacher's role in performing the experiment. Our description primarily focused on describing and justifying the shifts in the students' reasoning with respect to the instructional activities and computer tools provided. The performance of the sequence was left to the teacher's skill, although the teachers were all informed about the intended classroom culture and learning processes.

In the next sections we describe our three research phases: the design of the instructional sequence, the teaching experiments, and the data analyses.

4.3 The design of the instructional sequence

The instructional design phase included the development of teaching activities and of a conjectured local instruction theory. This was preceded by an analysis of conceptual and didactical problems in the teaching and learning of calculus and kinematics. The analysis has been described in chapters 2 and 3 and resulted in preliminary ideas about solutions to these problems, and more specifically about the role of graphs in a series of inscriptions while modelling motion.

Next, we delineated our main ideas on the teaching and learning of calculus and kinematics. The ideas were inspired by the integrated history of these topics. Other important aspects were didactical issues, such as the possibilities for mathematising and physicalising meaningful phenomena into the intended learning goals. Such a didactical phenomenology led to our choice of modelling motion and predicting change as central themes for our sequence (see chapter 5).

Parallel to the delineation of disciplinary ideas, we clarified the starting points for the instructional sequence. As starting points we used an *a priori* theory of guided reinvention, and the design heuristics of emergent modelling and problem posing. In addition, we determined what grade was suitable for our experiments, and set up a pilot study to investigate the possibilities within an educational setting.

The analysis, delineation and clarification laid the foundation for our instructional sequence and the conjectured local instruction theory. The theory had several levels of description. It described the major shifts in students' reasoning in connection with activities, tools provided, and scenarios of the intended lessons. The combination of the instructional sequence and this theory led to testable conjectures and observation criteria about the learning processes and the means of supporting these processes.

When the sequence was performed in a teaching experiment, we had to look for evidence that could be used for examining the conjectures. Moreover, patterns or

trends in the data led to hypotheses on the research questions and our choices concerning emergent modelling and problem posing. The data was analysed systematically for such patterns to verify our hypotheses. Here we come across a characteristic of design research: qualitative data are collected for verifying previously formulated conjectures, and for investigating new hypotheses that emerge during the teaching experiments or in the conjecture-verification process. In section 4.5 we address this aspect of design research.
Figure 4.2 describes the role of the instructional sequence in teaching experiments that aimed at a local instruction theory.

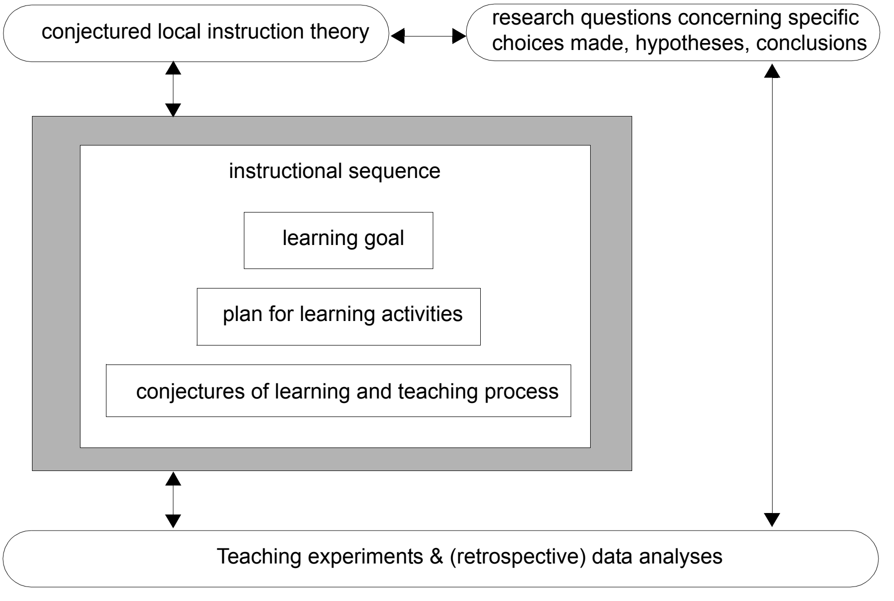


figure 4.2 The instructional sequence in design research

Our reflection after the first teaching experiment led to adapting the conjectures and the teaching sequence, which became the starting point for a second experiment. This cyclic process aimed both at empirically grounded answers to hypotheses concerning the research questions, and at a conjectured local instruction theory.
Ideally, the instructional sequence should converge into a sequence that works best within the constraints of the educational setting, and that illustrates the local instruction theory. The sequence should therefore be tried out and analysed in various situations, as well as be discussed with other parties who play a role in educational innovation, such as teacher training institutes and educational publishers. However, we were not able to go through all these phases for this project.
As an example of how the instructional sequence was documented, the documenta-

tion of the first lesson is printed on page 73. This part does not include the students' activities. The complete teaching sequence, together with crucial activities, is described in the next chapter 5.

4.4 Teaching experiments and data collection

The word 'experiment' in 'teaching experiment' does not refer to a comparison between an experimental group and a control group, but to an experimental classroom setting that is created as a result of the innovative teaching materials provided. In the teaching experiments the instructional sequence is carried out by the teacher and the students. The overall goal is to understand and improve the initial design on the basis of students' reasoning with respect to the created educational setting.

Our research questions concern teaching and learning processes in calculus and kinematics. Our ideas and conjectures about these processes were tested through the instructional sequence in classroom situations. This yielded conjectures such as:

> In their initial (intuitive) reasoning about the weather problems, students refer to the intervals between subsequent positions and relate lengths of these intervals (the successive displacements) with the velocity or change in velocity of the object.

We were interested in the development of students' conceptions in relation to the teaching processes. Did students notice possible patterns? Did they see the relation between these patterns and the need for precise predictions? Did this evoke the need for two-dimensional graphs? Was the teacher capable of guiding the discussion without suggesting the intended directions? These questions were answered by the teaching experiments.

The key issues in our approach were emergent modelling, problem posing and the role of computer tools. The development of models was reflected by the students' invention and use of graphical inscriptions, and the language they used in expressing the underlying concepts.

With respect to the problem-posing-approach, we needed to get a grip on possible motives for students to proceed and how the teacher and teaching materials could evoke these motives. Where we succeeded can be seen in the students' contributions to small group and classroom discussions. We hoped that we would find patterns in the teaching experiments that gave rise to hypotheses concerning emergent modelling and problem posing. These hypotheses could then be verified by analysing the data systematically, or by conducting a revised experiment.

To try to get a grip on the major shifts in the students' reasoning, we collected data that reflected their thinking and the teacher's role. We made audio tapes of group work, and video tapes of both class discussions and of students working in pairs with the computer programs during two computer lessons. We also copied all written materials during activities and the students' final test.

Goal of lesson 1
Goal of the first lesson is to evoke the idea that descriptions of change are useful for making predictions, and that graphs can be useful tools for describing change. Motion is an example of change that can be studied for investigating to what extent we can describe and predict. While reasoning with patterns in successive positions of hurricanes, students should feel the need for comparing displacements in successive time intervals. For doing this, some of the students will come up with the idea of displaying these displacements vertically next to each other. All students should understand this reasoning and how such graphs can be used to make predictions about the moving hurricane. In their reasoning the relation between patterns in displacements and the total distance travelled should emerge.

Hypothetical scenario for lesson 1
The teacher starts with a general introduction to this chapter. In a classroom discussion the two sheets for the weather situation above northern Europe are shown, the first one at 10:30 a.m., the second at 1:30 p.m. A question is posed about a roller blade event in the Netherlands that starts at 5:30 p.m. Should it be cancelled because of the approaching hurricane? This should evoke interest in ways of describing and predicting change. In the students' contributions, the focus on displacements in time intervals should be emphasised and should frame their perception in the following activities.The students should work in small groups. The last activity concerns the trace graph of a hurricane approaching land. The question is: at what time will it hit the coast? This activity is discussed at the end of the lesson. Most of the students will have extrapolated the last displacement, some will propose using the increasing pattern in the successive displacements. During this discussion a consensus should be reached on what you can read from the trace graph and to what extent it can be used for predicting motion. The question of how to display the pattern in the displacements is posed, bringing the vertical positioning of displacements to the fore. We expect students to come up with this idea. If that is not the case, the teacher should provide this possibility of graphing (or better: reformulate the question to direct their thinking towards this idea). The affordances and the constraints of the two-dimensional discrete graphs for predicting motion should come to the fore in the classroom discussion on the hurricane. We should note to what extent graphing is used meaningfully by them. This can be seen in their flexibility in using these graphs in different situations (e.g. a falling ball), and by the way they connect it with previous reasoning with trace graphs (e.g. when they get stranded).

Justification of the scenario (with respect to the conjectured local instruction theory)
We conjecture that one-dimensional trace graphs can be used by the students for describing change of position. Graphs that both support the emerging notion of velocity as a compound quantity, and the relation between successive displacements and total distance travelled. This last relation is a discrete anticipation on the basic principles of calculus. These conjectures are based on experiences described by Boyd & Rubin (1996) in chapter 2, and the historical development of calculus and kinematics.
The one-dimensional trace graphs derive their meaning from the successive measurements. The two-dimensional graphs of the displacements derive their meaning from the trace graphs. A sequence of inscriptions emerges and parallels concept development. The graphical models build on each other, and develop in the activities of the students as a result of the problems posed on describing and predicting the motion of hurricanes.

An observer made notes during all the lessons, evaluated each lesson with the teacher, and participated in students' group work and the work in pairs with the computer to let the students clarify what they were doing. We were aware that this participation influenced the students' learning processes, but we wanted to hear students express their thinking, and to create a classroom culture in which clarifying questions was part of the mathematical activities. The goal of participation moves on to designing interventions for explanations of the teaching practice (Barab & Kirshner, 2001; Cobb et al., 2001).

The teaching experiments took place in Dutch schools. The start of our research project coincided with a nation-wide reform of the secondary school system, which resulted in a new educational organisation, and a clustering of topics into streams. After the reform, secondary school students in grade 10 at pre-university level had to choose one of four streams (roughly described as natural sciences and technology, life sciences, economics, or social sciences). These streams largely determined the subjects they had to take. The technology and the life sciences streams had physics and mathematics as compulsory topics. This reform thus made it possible for us to align kinematics and calculus lessons for whole classes of students. In addition, the reform held consequences for the educational organisation, including an increasing emphasis on the students' own responsibility for their learning, for planning their activities, and for independent work.

This was the setting for our research. In addition, we should mention that, although we had experience in using physics as a context for mathematics education, we had hardly any experience in integrating the learning of physics and mathematics. In general, it seemed that research communities for mathematics and for science education are fairly distinct. We therefore started with a pilot experiment to explore practices in grade 10 in this new educational setting, and the possibilities for integrating calculus and kinematics with these students.

Pilot experiment

The goal of the pilot experiment was to gain insight into the problems and possibilities of teachers and 10th grade students with calculus and kinematics within the intended setting. We expected to learn more about the possibilities for classroom discussion and the students' abilities in reasoning about kinematic and mathematical situations. The teaching materials used during this pilot experiment were based on the students' mathematics textbook together with a few alternative activities. The mathematics book contained a chapter on the basic principles of calculus in eight lessons. For our teaching experiments we planned to use the time needed for this chapter and discussed our ideas with the teacher. The alternative activities focused on modelling motion as a central context, and also addressed the learning of a few kinematic notions. During the experiment we analysed the practices with four case stud-

ies and investigated whether we could line up our goals with the mathematical goals and the teaching of this chapter.
At the beginning of the pilot experiment the students received a course description of the chapter. This description gave suggestions on how to plan their work and which lessons would be used for a plenary discussion of the chapter's central themes. The students worked alternately in small groups, pairs, and individually. After every two lessons the teacher planned a plenary lesson to discuss their activities and the main themes. The teacher determined the content of most of these plenary sessions. This course description was necessary because of the increasing emphasis on students planning their own activities.
The exploratory nature of this first pilot led us to confine our data collection to detailed field notes and audio tapes of classroom discussions and group work, and the written material from four students. These students were selected because of their performance (varying from low to high achievers), and their participation during classroom discussions. The evaluations with the teacher, and the analysis of class discussions and the students' written materials led to initial conjectures concerning the conceptual steps that each of them took, the problems they faced, and how these were related to the new educational organisation and to our alternative activities.
The results of the pilot experiment contributed to our initial teaching design for ten lessons on the basic principles of calculus and kinematics for 10th grade students. We chose two different schools for our first teaching experiment to test design and the accompanying conjectures. We expected to collect enough data in two schools for reconstructing and analysing the learning processes, and we expected to be able to filter out school-specific influences.

First teaching experiment

The first teaching experiment took place in two 10th grade classes in two comprehensive schools in provincial towns (*school A* and *school B*). For the first experiment we wanted to be able to triangulate data within one class, and to compare the two different schools. The comparison enabled us to spot teacher- and school-specific norms and procedures.
In one of the two schools (*school B*), the students had already studied kinematics in their physics classes. We thought that we could still learn from doing this experiment with these students because of our different approach. The chapter in their physics book focused mainly on algorithmic knowledge of how to deal with kinematic equations, while we aimed at a conceptual understanding of velocity and the relation with distance travelled which emerges together with a series of inscriptions. This conceptual understanding had hardly been addressed in the chapter they had already studied. In *school A* kinematics was studied only in the 11th grade. We took this difference between the two schools into consideration when analysing our results.

The student activities and the guidelines for the teacher, together with our intentions, were discussed beforehand with the teachers of both schools in two meetings. During the experiments we made notes and audio-taped all the lessons. During the computer lessons, one of the pairs was video-taped, and the classroom discussions were also video-taped. The pairs were selected with help from the teachers using the criteria for this choice of clear speech and capabilities varying from low to high achievers. We used the video-tapes to be able to analyse gestures and reasoning with graphs on the computer screen and on the blackboard. After the teaching experiments we collected the students' written materials and copied their answers to a test. An analysis of these results was needed to investigate to what extent we had reached content-specific goals with all the students.
First, the analysis of these data provided information on how to optimise the activities with respect to formulating student texts, contexts used, and information provided. Second, conjectures that paralleled the instructional sequence could be verified as far as the students were taught as intended. This led to adjustments to the sequence and our conjectured instruction theory. Third, this analysis led to new hypotheses concerning the choices made with respect to the research questions. The adjustments and the new hypotheses were objects of study in the second teaching experiment.

Second teaching experiment

The second teaching experiment was confined to eight lessons in one 10th grade class. During these lessons we focused our data collection on adjusting the instructional sequence and the tools and computer tools used. As a result of this focus, we expected one classroom experiment would provide us with enough data for analysing the teaching and learning processes in the specified situations.
The second experiment took place in a comprehensive city school (*school C*) with a teacher who was experienced in discussing mathematical problems with students without presenting them with the intended approach or answer. We expected him to understand the teacher role we aimed at in our problem-posing approach, but still took more time preparing the experiment with this teacher. We discussed the planning of the activities and especially the scope of the classroom discussions.
We wanted to analyse the development in reasoning shown by both weak students and high achievers with the computer tools. We audio-taped a weak pair, an average pair, and a high achieving pair during the computer lessons, and video-taped the average pair. Classroom discussions were video-taped and field notes were made as in the first teaching experiment (table 4.1). In these experiments we expected to collect enough information to reconstruct and to understand the whole process of the teaching and learning of calculus and kinematics.

School **grade: no. of students**	**lessons (date)**	**data collection**	**characteristics**
School A grade 10: 17 grade 10: 22	8 (Oct. 1999)	notes and audio tapes, student materials	exploratory **pilot experiment**
School A grade 10: 19 *School B* grade 10: 18	10 (Nov. 2000)	notes, video and audio tapes, student materials, test results	**first teaching experiment** testing initial teaching sequence
School C grade 10: 24	8 (May 2002)	notes, video and audio tapes, student materials	**second teaching experiment** testing revised instructional sequence

table 4.1 The teaching experiments in chronological order

4.5 Data analysis

The relation between theory development and teaching experiments emphasises that hypotheses are created and modified while interpreting the data available. The interpretation of the data depends on our ability to understand the students' reasoning, to understand on which ideas their reasoning builds and by which perspectives it is guided. Consequently, it depends on our ability to reconstruct the learning and teaching processes. Analysing such processes differs from analysing isolated statements in deciding whether students hold misconceptions or need conceptual change (Klaassen & Lijnse, 1996).

It was impossible for us to reconstruct the learning processes of all the individuals in the experimental setting. However, we wanted to understand and reconstruct the classroom progress based on the data available, which should result in an empirically grounded understanding of what happened in the classroom. In doing this, we had to overcome a two-sided problem: how to collect enough data to ensure that we could *reconstruct learning processes* and verify emerging hypotheses, and how to select and organise the data into pieces which could be analysed rigorously? This data organisation was mainly influenced by what we considered important with respect to our previously formulated research questions, together with our conjectured local instruction theory. The data were organised into case studies of class discussions and of students' work during the computer lessons. We interpreted these case studies in terms of what preceded the lessons, the student activities, the teaching, and the tools provided. Interpretations were compared with other available data, such as students' written materials and data from another experiment in our research project.

The *interpretative framework* for the teaching experiments was primarily an instructional design perspective. Interpretations of classroom events guided the instructional design decisions and aimed at understanding and improving the teaching and learning processes. In addition, we tried to incorporate both a social and an individual perspective. We tried to assign such meanings to students' expressions that they came out as consistent with their history. The behaviour and the students' contributions were related to classroom norms and students' beliefs about what was expected from them (Cobb, Yackel & Wood, 1992).
Using these case studies we tried to understand what had happened and also to reconstruct a consistent view of the class's progress. The validity of our reconstruction was determined by our ability to find explanatory constructs that underpinned our interpretations of the classroom learning process (see also section 4.6 on validity). We will first describe the analysis of the data of the pilot experiment in more detail. This analysis differs from the analyses of the other experiments because of its exploratory character.
The notes and audio tapes of *the pilot experiment* were written up as lesson reports. These, together with the written materials of four students, formed our basic material for analysing the learning processes of these students. We looked for trends and notable incidents that were informative about the choices made. Here we also analysed the students' reasoning with respect to our alternative tasks on modelling motion. The results shaped our ideas about the activities, the level of the students, and the possibilities in the educational setting.
In *the two teaching experiments*, we distinguished two levels of data analysis. The first level was testing the conjectures that accompany the instructional sequence. Did the major shifts in the students' reasoning occur as we conjectured, and did the supporting materials have the intended effect? This was verified by the observation criteria that were documented with the conjectures. The data collection – for trying to reconstruct the shifts in students' reasoning – included notes made during the lessons, audio and video tapes of student contributions in classroom discussions, and students' written solutions to tasks and in the final test. The field notes were written up into lesson reports, which identified notable episodes varying from classroom discussions, to the learning process of a particular student over a few lessons. The audio and video tapes were used to work out these episodes as case studies; these concerned mainly the classroom discussions. In addition, the lesson reports yielded case studies of group work and the progress of individual students.
The second level of the data analysis was the search for patterns or trends related to the underlying design heuristics and the conjectured role of computer tools. We identified regularities and patterns by comparing and contrasting notable events with other available data (e.g. written materials) and other situations within the experiment. For instance, when we noticed events that could be characterised as successful, and similar situations which had less success, we analysed whether and how

these differences were the result of our design, and whether they could be related to our implementation of the design heuristics. During this analysis, hypotheses emerged concerning the choices we had made. This internal comparison provided answers on our research questions and the limitations of their realisation in educational settings.

As an example of the data analysis, we show how we translated research questions and conjectures into observation criteria for the first lesson, and how these criteria acted as a framework for analysing data. We start this example with two of the questions formulated in chapter 3 (page 65):

– Do students perceive the problem situation in the intended way?
– Do students reason and contribute to the reinvention process in the intended way?

In the description of the first lesson (as presented in section 4.3) the following conjectures can be found:

– Students experienced the weather prediction problems as relevant and were motivated to find an answer with the information provided. In their initial (intuitive) reasoning students referred to the intervals between subsequent positions and relate lengths of these intervals (the successive displacements) with the velocity or change in velocity of the object.
– Students used (one-dimensional) trace graphs to describe the motion of the hurricane. The displacements started to signify velocity for the students. Some students suggested placing the displacements vertically next to each other – in a two-dimensional displacement graph – to display a pattern for better predictions. These inscriptions and the corresponding reasoning were shared, compared, and formed the basic input for classroom discussions and for the way to proceed.

These conjectures were connected to criteria related with specific activities in the sequence and previously planned classroom discussions. The lesson reports on both schools indicated that during the classroom discussion, the majority of the students was motivated by the context and participated actively. The video tapes were used to work out these discussions and to analyse the students' contributions. It showed that these contributions were not limited to a few remarks by someone who 'knew' the answer, or to who was asked a specific question by the teacher. The weather-context evoked reasoning about successive positions by creating opportunities for a discussion that supported reasoning about motion in terms of displacements, during which inscriptions (trace graph), concepts, and accompanying language emerged. Change in velocity became related to change in displacements. Moreover, the students' written materials in the weather activities supported these conjectures. These findings were used to formulate hypotheses concerning (*i*) the instructional sequence ("the weather-context has the potential to interest students in ... and to evoke reasoning about ..."), (*ii*) the level of the local instruction theory ("situations with discrete measurements, in which it makes sense to reason about change, are useful starting points for the teaching and learning of calculus and kinematics"), and (*iii*) about our choices ("the emergent modelling heuristic is a valuable one because developing the notion of change appears to parallel a sequence of inscriptions from trace graphs to two-dimensional graphs").

We were able to answer our research questions with this data analysis. The conjectured local instruction theory was investigated, and the analyses led to hypotheses which were related to our questions. These hypotheses could be tested with the data available and supplementary data analyses, or could be starting points for the second teaching experiment. Finally, causal patterns and relations became stable in the experimental setting, and evolved into explanatory constructs for a large part of our instructional sequence.

This contribution to the local instruction theory of the teaching and learning of calculus and kinematics is empirically grounded in the teaching experiments and underpins an instructional sequence. However, these results still have a temporary character because they are based on one research project. Recurrent experiments, which build upon what we learnt and which use improved sequences, should result in a robust theory for different educational situations.

4.6 Validity

In this section we discuss four criteria of our design research methodology which should validate its scientific basis (building on Gravemeijer & Cobb, 2001). The *first* is the formulation and verification of testable conjectures about the students' development with respect to the educational environment created. These conjectures were formulated together with observation criteria and related to the instructional sequence and the tools provided. The teaching experiments and data analysis resulted either in verification of the conjectures, or in adjustments or new conjectures for subsequent experiments. We described this process systematically to offer other researchers the possibility of virtually replicating it, and retracing our conclusions through the cycles of data analyses and teaching experiments. This methodological norm is referred to as *trackability* by Smaling (1992) and was emphasised by Freudenthal (1991).

The *second* criterion is based upon the *theoretical foundation* of the interpretative framework that guides our data reduction and interpretations (Gravemeijer & Cobb, 2001). This framework was shaped by the discussion on invention oriented and discovery learning approaches (see section 2.2). Consequently, we took an instructional design perspective and drew upon a potentially revisable, conjectured local instruction theory, which was rooted in the domain-specific theories characterised by guided reinvention, emergent modelling and a problem posing approach. These theories have proved themselves in various other topics of mathematics and physics education (see section 3.5).

The *credibility* of the instructional design, the data interpretation, and our argumentation is the *third* criterion that validates our methodology. Peer review supports its credibility. We discussed our initial instructional design with colleagues in the Centre for Science and Mathematics Education. Intensive discussion focused particularly on: (1) the didactical models, (2) the contexts that should evoke reasoning and inscriptions that fitted these models, and (3) the students' motivation to proceed in an intended direction. The discussions marked out and deepened the conjectured local instruction theory, and optimised the preliminary instructional sequence.
We compared some of our protocol interpretations with those of others, although comparing interpretations with respect to the students' history was not easy to do. This required other researchers to know about what happened during the teaching experiments. Interpretations were discussed by the author and his supervisors, and in our research team (see section 1.1). In addition, interpretations of three 15-line protocols were discussed in a working group (2 × 90 minutes) at a PME-conference (Pijls & Doorman, 2001). The discussions led to interpretations of both failures and successes being less biased by the author's individual perception. We do not consider that we achieved totally unbiased interpretations in this project, because not all the interpretations were shared, and the author's information on what had preceded the experiments also played a role in these discussions.
Finally, our *fourth* criterion was on a different level of validation: the engagement of teachers and students during the teaching experiments (Gravemeijer & Cobb, 2001). In preparing the experiments, we had discussed our ideas and the instructional sequence with the teachers. Our ability to explain our intended goals for the experiment, their willingness to participate, their contributions to performing the teaching experiment, and their engagement during the actual experiments validate, to a certain extent, the teaching experiments in our project. A similar argument holds for the participating students. Moreover, performing the instructional sequence in 'normal' classroom settings has its limitations for the researcher, but can provide indications of the practical value and relevancy of our results.

5 The instructional design

The instructional design described in this chapter is an envisioned learning route of a guided reinvention approach to calculus and kinematics. It is the level on which we test our assumptions in classroom situations. The specific elements of the design concern the integrated learning of physical and mathematical concepts through the students' modelling activities. We conjecture how intended models emerge, how computer tools support this process, and how the students can be motivated to perform these activities. These conjectures are formulated together with a description of the teaching activities and of the envisioned classroom learning process. This chapter describes our initial design for the first teaching experiment; it is inspired by the historical development of calculus and kinematics.

A question for the design of a guided reinvention trajectory is: 'How could I have invented it?' In the design, we should try to forget our domain knowledge and look at the main problem situations from the students' point of view. This is especially difficult in mathematics, since it is *the* discipline through which we structure the world around us. One of the designer's tasks is to 'unstructure' this world, and try to understand students' perspectives and the footholds they might have available given their perception of the situations presented.

It is useful to look at the history of a topic to gain insight into this issue, to investigate how certain concepts developed, and how and why people tried to organise certain phenomena without having any notion about calculus or kinematics. We are interested in a historical study of these topics as a starting point for the initial design, rather than in an analysis of the systematics of the subject matter itself (de Lange, 1987). A historical study may indicate possibilities and clues for a guided reinvention approach. Especially, we search for the possibilities for emergent modelling, for framing the students' view of problem situations, and for the use of tools that can afford the development of symbols and meaning.

Section 5.1 sketches a few stages in the history of calculus which were important to the context of the historical perspective on our design problem. In section 5.2 we describe the pilot experiment which we carried out to investigate the possibilities with respect to the content and to the school and classroom organisation for our teaching experiments. Section 5.4 starts with an exposition of an initial conjectured instruction theory for the basic principles of calculus and kinematics by modelling motion, as a situation that can be organised to make certain predictions, and specific situations that stimulate students to develop specific and productive ideas and strategies.

Then follows our first design of the teaching sequence as a starting point for our teaching experiments.

5.1 The emergence of calculus and kinematics in history

In this section we focus on the historical development of calculus and kinematics. The history of these topics shows us which problems, tools and methods enabled scientists to develop these theories. The interest in this conceptual development is framed by an instructional design perspective, while most historical analyses focus on the resulting theories according to Confrey & Costa (1996). Gulikers & Blom (2001) gave an extensive survey of research on the use and value of history in mathematics education, especially for geometry. In addition to conceptual arguments, they listed arguments concerning the didactical repertoire of the teacher, the nature of mathematics as a developing discipline, cross-curricular aspects, and the role of mathematics and mathematicians in society.

In the following description we have tried to select crucial problem situations and important developments of the tools, primarily from a concept-development perspective. This historical review up to Galileo is mainly based on Dijksterhuis (1980) and Clagett (1959). In section 5.2.2 we reflect on this description from a didactical perspective.

We will focus mainly on the historical period up to Leibniz and Newton, during which time the basic concepts and models for calculus and kinematics were shaped. The description covers a period of 2000 years. This could give an impression of a development by fits and starts, but one should realise that it was a long and gradual process, in which the breakthroughs can be localised in the work of a few, brilliant scientists.

5.1.1 *A historical sketch*

Questions about falling objects were essential for the development of calculus and kinematics, and one could say that these topics emerged from modelling forced motion and free fall. This historical sketch starts with Aristotle (c. 350 BC). He formulated laws on motion according to his everyday experiences and common sense understanding about the nature of objects. Whether an object falls to earth, or floats, depends on its properties. In Aristotle's cosmology, each object could be characterised by form and matter. Matter can be described as a mixture of elements, and is that which can make a form (e.g. a certain form made in clay). The type of mixture determines the natural place of an object, which is part of its form. Form expresses the essential nature of the object and its constant velocity during free fall.

Aristotle's ideas remained almost unchanged until the late Middle Ages. In the thirteenth century, scholars were convinced that a falling object increased its speed. They tried to improve Aristotle's theory and developed the impetus theory of motion. It is in the nature of the object to have a propensity, or impetus, to move towards its natural place, depending on the mixture of elements. If a mover moves an object, its artificial motion is the result of an additional impetus in the object, which is communicated to the object by the mover. This impetus will decrease if the

bodily contact between mover and object is stopped. Decrease of the impetus results in a decrease of the forced movement. However, there is no theory about the particular way in which the object loses its impetus.
The striving towards its natural place gives an object an impetus that determines its velocity in the first time-interval of free fall. After that moment, the object has both an impetus (its striving towards its natural place) and a velocity. This causes an increase in the object's velocity in the second time interval, etcetera; which explains the increasing velocity of a falling object. The scholars did not have the means of observing that the increase in the velocity of a falling object is proportional to the time elapsed.
Until Galileo's time, the impetus theory could be recognised in explanations of the trajectory of an object thrown into the air. The motion of a thrown object decelerates until the impetus, which it received from the throw, has decreased to zero. After that moment, the object's striving for the ground will cause the object to fall to earth vertically with an increasing velocity.
Scientists started using variables and formulas in the fourteenth century. This was the time of the so-called *Calculatores*. Thomas Bradwardine, for example, tried to describe the velocity of an object when the proportion between a force F that causes motion and the resistance R is changing. He based his description on the theory of proportions, which states that the addition of proportions equals the multiplication of the corresponding fractions (e.g. when proportion $a : b$ equals 1 : 2 and $b : c$ equals 1 : 4, then $a : c$ equals 1 : 8), and the multiplication of a proportion by a parameter n equals the corresponding fraction to the power n (three times the proportion 1 : 3 equals a proportion of 1 : 27). Bradwardine argued that the velocity v of an object was determined by the proportion $F : R$. If this proportion became n-times bigger ($F^n : R^n$), then the velocity became n-times bigger, or the two velocities were proportional as 1 : n; in modern notation: $v \sim \log(F/R)$. Bradwardine gave several examples to illustrate his theory and to explain why it described motion better than preceding theories.
According to Dijksterhuis (1980) this example of a mathematical formula shows how scientists tried to find mathematical laws in nature. Hence, we learn from these examples that their view on the role of mathematics differed from that of Aristotle, and they show what kind of difficulties had to be solved in order to describe phenomena in a mathematical language. These difficulties not only originated from problematic physical assumptions, but also from limitations in the mathematical language available. The Calculatores could not describe velocity as a proportion of distance to *time*, because then they would have had a fraction of two different types of quantities. They still followed the Euclidean tradition (c. 300 BC) and worked only with proportions of the same quality.
In the first half of the fourteenth century, logicians and mathematicians associated with Merton College (Oxford, UK) investigated velocity as a measure of motion.

They theorised about changing qualities like temperature, size, and even a human quality like charity. The types of change they identified were uniform, difform (changing), and uniform difform (constantly changing). One of their problems was to describe a uniform difform motion, i.e. to describe the distance travelled by a body moving with a uniformly accelerated motion. This problem is not easy because the velocity changes constantly during such a motion. The interpretation of motion as change of place became one of the central issues studied. Clagett (1959) gave a detailed account of this emergence of kinematics at Merton College.

The scientists at Merton College used a notion of instantaneous velocity and descriptions of the velocity of a moving object, but there was still no definition of velocity as a compound quality (the distance travelled divided by the traversal time), and certainly no definition of instantaneous velocity as a limit of this division. Scientists and mathematicians would still have to work for several more centuries to gain this last insight. However, three important results were achieved at Merton College:

1 A definition of the notion of instantaneous velocity. The velocity at a certain moment in time can be described by the distance that would be travelled if the object would move on with that very velocity, unchanging during a certain time interval. As Dijksterhuis noticed, this is a circular definition, because when you ask what *that very* velocity is, you can only say 'the velocity at the fixed moment' which is still to be defined. However, it should be noted that the idea of a *potential* distance travelled in a certain time interval, represents the instantaneous velocity of the object. This is exactly what velocity in our everyday language means. Driving at a speed of 70 km/h is interpreted as: if you were to continue at this very speed for one hour, you would have travelled 70 kilometres.

2 A description of the notion of a constantly changing velocity: the velocity increases by equal parts in equal time intervals (and *not* in equal distances travelled!).

3 'The Merton rule': if the velocity of an object is constantly changing from zero to a velocity v in a time interval t, then the distance travelled is equal to half the distance travelled by an object that moves with a constant velocity v in the time interval t. In modern notation $s(t) = \frac{1}{2} \cdot v \cdot t$

Notable is the central position of the quantity time in these results. One of the proofs of the Merton rule was given by Richard Swineshead (c. 1335). He assumed an object A moving with constantly increasing velocity from zero to v, and an object B moving with constantly decreasing velocity from v to zero. At every moment t, the sum of their instantaneous velocities equals v. So, together they travel the same distance as one object moving with a constant velocity v. From this he concluded that A and B each travel the same distance as an object moving with a constant velocity $v/2$.

In this period, Nichole Oresme (c. 1360) invented a new element in these arithmetical descriptions: he introduced the graphic representation. He worked at the Uni-

versity of Paris and studied changing qualities. He was not primarily interested in what actually happens, but in how you could generally *describe* what happens. For instance, he described ways to display the distribution of the heat in a beam: think of a line along the beam and imagine at every point of this line the heat at that position in the beam represented by a line perpendicular to the beam. The length of this second line displays the heat at that position in the beam. These perpendicular lines constitute a geometrically flat shape. This shape denotes the distribution of the heat and its area is a measure of the total heat in the beam. A constant temperature is displayed by a rectangular shape, while an uniform change from low to high is displayed by a triangular shape (or a trapezoid).
Oresme reasoned and compared changes in qualities with geometrical shapes and found that the configuration of a geometrical shape determined the properties of a quality.

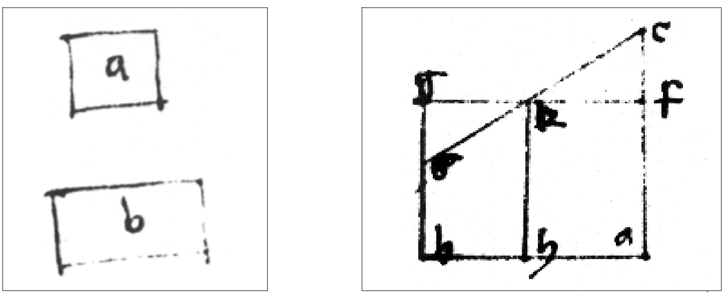


figure 5.1 Drawings from a fifteenth century copy of Oresme's 'De configurationibus qualitatum'

Oresme also applied this technique to motion. His remarkable way of thinking can be seen by the way in which he defined velocity as a quality of objects that can be pictured against time (the dimension over which the velocity of the object varies). Thanks to this choice, the area of the geometrical shapes of this quality had many similarities with the current velocity-time graphs. The perpendicular lines denote instantaneous velocities and the area of the shape can be interpreted as total distance travelled. Oresme compared velocities by the proportions between the different areas of the rectangles (see figure 5.1 left).
If an object moves with uniformly accelerating velocity in a time interval, the distance travelled equals the distance travelled by a constantly moving object with the same velocity at the middle of the total time interval (see figure 5.1 right). With this reasoning Oresme proved the Merton rule. Although Oresme did not write the original formula: $S_t = \frac{v_0 + v_t}{2} \cdot t$, Dijksterhuis attributed this formula to him because he implicitly used it to solve kinematic problems.

These graphs support an interpretation of velocity as a quantity with instantaneous values and they simplify, conceptualise and illustrate theorems about motion. This graphical method was applied to various types of motion but it is remarkable that all these motions concerned more or less theoretical situations (fig. 5.2). The reasoning was not applied to real-life motion phenomena nor to free fall.

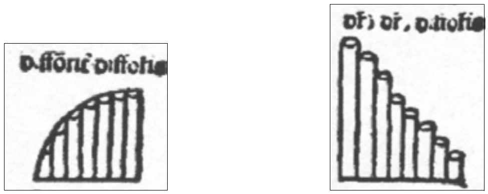


figure 5.2 Drawings from Oresme's 'Tractatus De Latitudinibus Formarum'

Some mathematicians argue that Oresme's proof of the Merton rule is not valid. First, he should have defined instantaneous velocity as a differential quotient and then deduced the distance traversed by graphical integration. Dijksterhuis discussed this and defended Oresme by stating:

> It is a situation which occurred regularly in the history of mathematics: mathematical concepts are often – maybe even: usually – used intuitively for a long time before they can be described accurately, and fundamental theorems are understood intuitively before they are proven.
> (Dijksterhuis, 1980, p. 218)

Oresme visualised the Merton rule in a way that could be extended to understanding more complex problems. His graphs made it possible to visualise these problems and to acquire kinematic insights that were not yet accessible through calculus in those days.

Until the sixteenth century, it was commonly accepted that the time needed for an object to fall to the ground was proportionally reversed to its weight. This was still a heritage of Aristotle's theory. In 1586, Simon Stevin published his *Beghinselen der Weeghconst* (Principles of weighing). Stevin opposed this theory and described an experiment with two falling lead balls of different weight that touched the ground at exactly the same time. In this experiment he tested Aristotle's assertion and concluded that it was contrary to this experience.

Stevin also argued that a proportionality between weight and falling time in a medium like air or water is impossible: take two objects, one floating on water and the other sinking, a proportionality between their weights exists, but there cannot be a proportionality between their falling times. During this period, the need emerged for experimental settings to investigate motion, and for specifications of variables to look at.

In 1618, Isaac Beeckman proved a new relation between elapsed time and falling distance that is independent of the weight of the object. He approximated a continuous force that pulled the object as if with little tugs. After each time interval τ, such a tug increased the velocity by a constant amount γ. This process was visualised by the graph below, in which the distance travelled in a time interval τ is represented by the area of the corresponding bar (fig. 5.3).

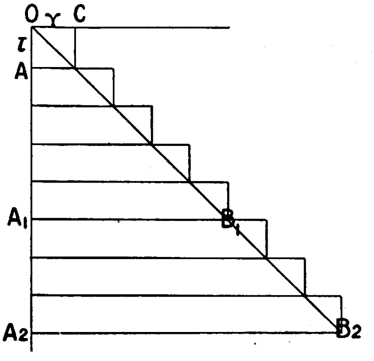


figure 5.3 Graph showing Beeckman's reasoning with areas of bars

When the length of the time interval τ approaches zero, the distances travelled in total times OA_1 and OA_2 are represented by the areas of the triangles OA_1B_1 and OA_2B_2. These distances are proportional to each other as the squares of the time intervals OA_1 and OA_2. He also used this reasoning in proportionalities between similar quantities. In his time they were still not able to formulate the relation between time and distance travelled in one formula: $s(t) = c \cdot t^2$.

The difference between Oresme and Beeckman is that Beeckman used a discrete approximation of the area. Such approximations were related to Archimedes' methods to determine centres of gravity (c. 200 BC). This is no surprise, because Archimedes' work was translated in the sixteenth century. Stevin, Kepler and Descartes also used his methods in their publications. Interest in the work of Archimedes was the result of a rising prominence for the mathematical disciplines and of the practical utility of mathematical methods in other disciplines in the sixteenth century.

There was another remarkable element in Beeckman's work: he did not use velocity as caused by an intrinsic property, but as a result of a force that pulls the object. This cause does not affect the actual value of velocity, but it does influence the increase of velocity. He claimed: "no change of velocity without a cause", while many, following Aristotle's natural philosophy, believed: "no change of position without a cause". Beeckman's claim may be self-evident to us but it was revolutionary at that time.

Galileo (1564-1642) is one of the most famous scientists who worked on these kinematic problems. In his time, the role of mathematics in scientific research was discussed. Two possible views were recognised: (*i*) mathematical regularities lie at the very heart of reality (Platonic); and (*ii*) mathematical regularities are invented abstractions of surface appearances. Galileo advocated the Platonic view and argued that visual phenomena were the result of, and should be described with, mathematics[1]. One of the phenomena which Galileo studied is free fall. In his *Dialogue Concerning Two New Sciences* he wrote about the Aristotelian view on this topic and why this view must be wrong in a dialogue between Simplicio and Salviati:

> Simplicio (...) he [Aristotle] supposes bodies of different weight to move in one and the same medium with different speeds which stand to one another in the same ratio as the weights; so that, for example, a body which is ten times as heavy as another will move ten times as rapidly as the other (...).
>
> Salviati (...) I greatly doubt that Aristotle ever tested by experiment whether it be true that two stones, one weighing ten times as much as the other, if allowed to fall, at the same instant, from a height of, say, 100 cubits, would so differ in speed that when the heavier had reached the ground, the other would not have fallen more than 10 cubits.

Simplicio represented the Aristotelian ideas on motion, and Salviati the new ideas of Galileo. Galileo used graphs to explain the quadratic relationship between distance travelled and falling time. He drew 'velocity-time graphs' in the same way as Oresme and Beeckman, but he reasoned differently. He followed Swineshead's proof of the Merton rule, and used a collection of instantaneous velocities that are represented in the following graph by the lengths cc_1 and dd_1 (fig. 5.4).

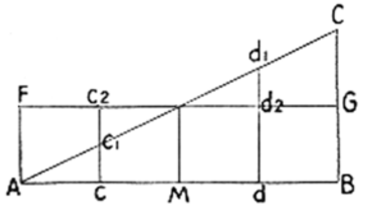


figure 5.4 Graph from Discorsi III 1, Opere VIII 208 by Galileo.

These lengths do not represent actual velocities because he did not identify velocity as a quantity. We assume that Galileo reasoned with these lengths as potential displacements, as in the Mertonian definition of instantaneous velocity. We can

1. The Book of Nature is written in the language of mathematics. From: 'Il saggiatore' (The Assayer) by Galileo, Accademia dei Lincei in 1623.

therefore add these lengths and place as many of them as we want next to each other. The lengths cc_1 and dd_1 are symmetrical around moment M, and $cc_1 + dd_1 = cc_2 + dd_2$ applies everywhere. From this he concluded that the distances travelled by movements according to the graphs AC and FG are equal. Intuitively, all these lines together are equal to the area, which Oresme had already used. From this graph we can immediately deduce, by using areas, that the distance travelled until moment M is one-third of the distance travelled in the second half of the total time interval.
As an example of Galileo's reasoning we reproduce some of his notes on motion. This part (Discorsi Proposition 3/03-th-02) concerns the determination of the speed of a projectile following a parabolic path. The lengths of horizontal lines that Galileo used in his reasoning below are precisely what we call potential displacements.

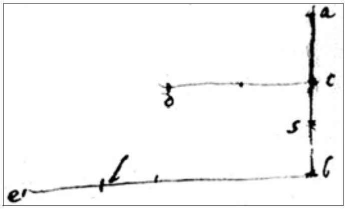


figure 5.5 Drawing by Galileo which accompanies this proposition (retrieved from http://www.mpiwg-berlin.mpg.de/Galileo_Prototype/index.htm)

{281} SALV. Our Author next undertakes to explain what happens when a body is urged by a motion compounded of one which is horizontal and uniform and of another which is vertical but naturally accelerated; from these two components results the path of a projectile, which is a parabola. The problem is to determine the speed [impeto] of the projectile at each point. With this purpose in view, our Author sets forth as follows the manner, or rather the method, of measuring such speed [impeto] along the path which is taken by a heavy body starting from rest and falling with a naturally accelerated motion. (fig. 5.5)
Let the motion take place along the line *ab*, starting from rest at *a*, and in this line choose any point *c*. (...) The problem now is to determine the velocity at *b* acquired by a body in falling through the distance *ab* and to express this in terms of the velocity at *c* (...) Draw the horizontal line *cd*, having twice the length of *ac*, and *be*, having twice the length of *ba*. (Condition 2/23-pr-09-schol1) It then follows, from the preceding theorems, that a body falling through the distance *ac*, and turned so as to move along the horizontal *cd* with a uniform speed equal to that acquired on reaching *c* {282} will traverse the distance *cd* in the same interval of time as that required to fall with accelerated motion from *a* to *c*. Likewise *be* will be traversed in the same time as *ba* (...)
(retrieved from http://www.mpiwg-berlin.mpg.de/Galileo_Prototype/index.htm).

Galileo tested his hypothesis on the quadratic relation between time and distance of a falling object with experiments. He knew that sequences of successive odd numbers, starting with 1, add up to a square, and he used ratios of odd numbers between the distances travelled in equal time intervals. This ratio must be 1 : 3 if you divide time into two equal intervals (fig. 5.6).

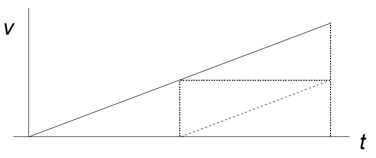


figure 5.6

If you divide the time into four intervals the ratio is 1 : 3 : 5 : 7, et cetera. With this property he tested the formula that is based on the conjecture that the acceleration of a free falling object is constant. An important step which Galileo made was to reason that the motion of free fall is similar to (in terms of proportions) and can be delayed by an object rolling down an incline. He probably designed a slide with nails on one side. The distances between the nails were in the same ratio as the successive odd numbers, thus a rolling ball should need the same time to pass each following nail (fig. 5.7).

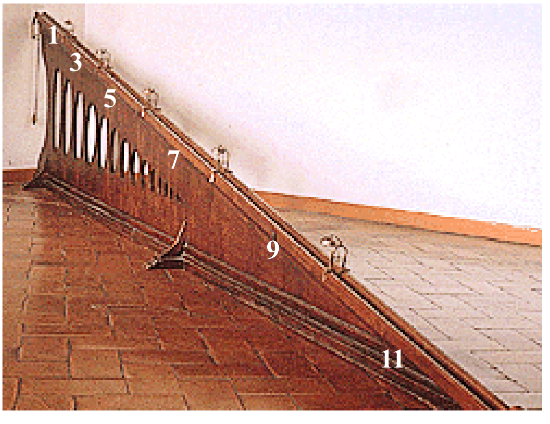


figure 5.7 A 19th century instrument for illustrating Galileo's experiment (IMMS, Firenze)

Many scientists commented on Galileo's reasoning, for instance, Fermat (1601-1665) believed that an object must have a velocity at the moment of falling, otherwise it would not start moving. This is yet another example which illustrates that their ways of thinking about velocities of falling objects and about instantaneous change were not trivial. It shows that even famous mathematicians during the time of Galileo had problems with the idea that, at the moment of starting to fall, the object could have acceleration while its instantaneous velocity is zero.

Two scientists, Leibniz and Newton, were crucial in the development of calculus and kinematics; they discovered and proved the main theorems of calculus. In the seventeenth century, methods were discovered for calculating maximums and minimums in optimization problems. These methods concerned mainly polynomials, but many problems could not be described with polynomials, such as the breaking of light. The conceptual understanding of the mathematics of instantaneous change developed, how to calculate it was a topic of interest, and Leibniz's and Newton's contributions concerned precisely this issue. Their invention of a literal symbolism was essential for the rapid progress of analytic geometry and calculus in the following centuries. It permitted the concepts of change to enter algebraic thought.

Newton formulated the ideas of Oresme, Beeckman and Galileo more accurately. His work mainly concerned a search for assertions that could be starting points for a systematic organisation of force and motion. This search led to a description of force as a product of change of speed and 'bulk' of a body. Bulk means something like heaviness, but he was unable to give a correct definition of the concept of mass. Force became an invented cause for explaining motion. Newton restricted himself to finding forces that determine motion (of planets, falling bodies, etc.), like Oresme and Galileo who first wanted to describe phenomena before looking for explanations that governed them. After finding these forces, Newton tried to explain how they work.

The language of Newton was closely related to motions of geometrical entities in a system of coordinates. The y-coordinate denotes the velocity of a changing entity (e.g. an area or a length) and the x-coordinate denotes time. Such a geometrical approach fitted the research tradition in the seventeenth century and might have supported his findings (Thompson, 1994b). The embedding of motion and time in geometry is one of the most characteristic features of Newton's dynamical techniques. Newton used the context of motion to give intuitive insight into the limit process of the proportion between two quantities that tend to zero (Pourciau, 2001). He argued that the ultimate proportion of two vanishing quantities should be understood as the velocity of an object at the ultimate instant when it arrives at a certain position. The two quantities are position and time, and he defined the limit or vanishing proportion between change of position and change of time as the instantaneous velocity. Similarly, this ratio of vanishing quantities is to be understood not as the ratio of the quantities before they vanish or after they have vanished, but as the very ratio at which they vanish.

In Newton's symbolism, quantities without a dot, such as x, are called fluents. Velocities by which fluents change are called fluctions: $\dot{x}$. These fluctions represent instantaneous rates of change as proportions; in modern notation: $x = \mathrm{d}x/\mathrm{d}t$. Newton defined infinitesimals as moments of fluctions, and represents dx with xo, where o is an infinitely small quantity. In calculations you can leave out the terms that are multiplied by o, because they can be neglected with respect to the other terms.

Struik (1987) noticed an aspect of vagueness in Newton's symbols. The vagueness in his symbolism is the use of 'o'. Is it a zero, an infinitesimal or an infinitely small number? Newton tried to denote its meaning by means of a theory on first and final proportions in the context of motion. From this we can see that he had intuitively mastered the limit concept, but did not formulate it very clearly; certainly not for his contemporaries.

The roots of Leibniz's work were in algebraic patterns in sums and differences and their properties. In 1672, he published his work on the sequences of sums and differences of sums, before he formulated the fundamental theorem of calculus.

Leibniz noticed that with a sequence: a_0, a_1, a_2, ... , and with a sequence of differences $d_1 = a_1 - a_0$, $d_2 = a_2 - a_1$, ... $d_n = a_n - a_{n-1}$, he could conclude: $d_1 + d_2 + ... + d_n = a_n - a_0$. Therefore, the sum of the consecutive differences equals the difference of the first and the last term of the original sequence. According to Edwards (1979), Leibniz refers to this inverse relation between the sequences a_n and d_n in his later work as his inspiration for calculus. The mathematics of change in algebraic structures was developing into a general calculus. From phenomena like motion, attention moved to studying these structures, formulas, and their graphs.

From algebraic roots, Leibniz introduced a more accessible symbol system for calculus than Newton, a system which we still use today. Maybe this is the result of his abstraction of real situations, and his goal of creating "a symbol system that would codify and simplify the essential elements of logical reasoning" (Edwards, 1979). Edwards added to this that it was precisely in mathematics that Leibniz fully accomplished his goal: "It's hardly an exaggeration to say that the calculus of Leibniz brings within the range of an ordinary student, problems that once required the ingenuity of an Archimedes or a Newton." Or as Kaput (1994a) formulated Leibniz's invention: "This is the genius of Leibniz's contribution. One can mechanically 'ride' the syntax of the notation without needing to think through the semantics."

Leibniz did not write much about the limit concept as a foundation for his symbol system. He illustrated his method in his first article *Nova Methodus* on calculus in 1684 with a graph of a formula that did not have any relation with a context. After the 'abstract' exposition of the method he illustrated the power with some applications. Leibniz did not define 'infinitely small'. He interpreted a tangent as a line through two points on a curve that lie at a distance to each other which is smaller than every possible length. Leibniz did not publish this definition in the article, because he thought this to be too revolutionary. He only published the rules to 'ride' the calculus and the convincing applications, without a foundation for his symbolism (van Maanen, 1995).

The historical development until Leibniz can be summarised in the following timetable (table 5.1). It is remarkable that our current secondary education reveals hardly anything of this struggle for mathematising change. The methods of Leibniz are taught as an obvious, or natural, way to treat change in a mathematical way. Calculus came to be considered as an independent discipline at the beginning of the 18th century, independent of geometry, as a result of Euler's work. The objects of investigation in mathematics were all known analytical expressions (Koetsier, 1987).

Timetable		
c. 350 BC	Aristotle	falling speed ~ heaviness
c. 200 BC	Archimedes	calculations of areas with rectangular approximations
13th century	Albert of Sachsen	falling speed ~ falling distance
14th century	Oresme	time graph of a changing velocity
15th century	Leonardo da Vinci	struggle with concepts like force, velocity and acceleration
16th century	Simon Stevin	experiment with two lead balls
beginning of 17th century	Isaac Beeckman followed by Galileo	through Archimedes: falling speed ~ falling time; and law of squares
end of 17th century	Newton and Leibniz	invention of calculus

table 5.1

The remaining work consisted of laying a rigorous foundation of functions, differentials and infinitesimals. This took the mathematical society almost one century.
Early in the 19th century, Bolzano, d'Alembert and Cauchy defined infinitesimals as dependent variable quantities. Cauchy defined the limit concept and finally eliminated all misunderstandings. The problems that made Cauchy formulate this unambiguous definition concerned functions of real variables: why they behaved so differently according to their Taylor and Fourier series, and in what respect could these functions be seen as functions of complex variables? Like the notations of Leibniz, his formulation of a differential quotient, is still used nowadays in calculus education.

5.1.2 *Looking at history through a didactical lens*

This historical study provides us with indications how models and more sophisticated mathematical knowledge evolved from informal knowledge, and how to use tools to afford model shifts.
Aristotle's main ambition was to organise matter into basic elements and their proporties. From this view it is not surprising that he defined a relation between falling speed and matter. Oresme's intention was to describe and value changing qualities, one of which was velocity, in order to be able to compare them. He used graphs for displaying and reasoning about changing qualities. He did not define velocity as a compound quantity, nor did he use scales along his two-dimensional graphs. Nevertheless he interpreted areas as distances travelled and used the geometrical figure to compare different kinds of motions.
The graphical method made it possible to illustrate the middle-speed theorem and to

investigate the relation between change of velocity and distance travelled in many theoretical motions. The method was successful thanks to Oresme's choice to draw a graph with a horizontal time axis. Possibly, his choice was influenced by his trying to image potential distances travelled.

The graphs of Oresme derive their meaning from the situations they describe. These graphical tools afforded him and his contemporaries a way of describing relations between velocity and distance travelled. During the time of Beeckman and Galileo, reasoning with characteristics of graphs became a method, almost independent of the situation described. Together with these methods, reasoning about the meaning of instantaneous velocity and acceleration at the very beginning of free fall began to emerge.

In this history we recognise a dialectic process of the development of meaning and of graphical methods, a process from two-dimensional discrete graphs for describing motion to reasoning about slope and area, and about the relation between velocity and distance travelled. Kinematic and mathematical concepts emerged, first used intuitively, while later on they were objects of study (see quotation of Dijksterhuis on page 88). This might have implications for a trajectory of teaching and learning the basic principles of calculus and kinematics. Instead of starting with velocity as a compound quantity and reasoning with two-dimensional continuous graphs, history indicates that we might start with discrete graphs that derive their meaning from the situation being modelled. These discrete graphs would provide students with meaningful graphical tools that afford them both a way to reason about characteristics like area and slope, and to invent the relation between velocity, time and distance travelled.

The methods of Leibniz opened up the possibility of symbol manipulation without examining these symbols and understanding their meaning. This symbolic writing seems to replace conceptual thinking by substituting calculation for reasoning, the sign for the thing signified. However, we note that Leibniz's symbol manipulations were built upon extensive experience with numerical patterns in sums and differences. We assume that his experience underpinned a meaningful use of these manipulations.

This process, where thinking with concepts is replaced by symbol manipulations, might have advantages for efficiency, but can have limitations in flexibility. What should students do in a new situation, or if they do not remember the exact algorithm? Reasoning with a symbol system according to Leibniz's methods can only be meaningful when it draws upon conceptual understanding. The symbolic methods of calculus can be applied, but carry the danger of degenerating into abstract methods if there is no underlying idea about the meaning of the calculations. For Leibniz, the underlying meanings were mathematical and based on sums and differences, whereas for Newton they were mainly physical and related to motion. For both Leibniz and Newton, the graphical reasoning of Oresme, Beeckman and Galileo was an important starting point. This suggests – as Dijksterhuis already noticed – that an intuitive understanding of reasoning with graphs of motion precedes formal methods such as integration and differentiation of functions. Moreover, we notice a process-object development from arithmetical prescriptions to reasoning with formulas. In the end $s(t) = c \cdot t^2$ can be understood as an object, while Beeckman's work still had an arithmetical character. We can speak of a process of reification in relation to this (Sfard, 1991). However, it is not the graph but rather the activity of summing and taking differences that is reified into the mathematical objects of integral and derivative. The inscription – the graph – visually supports both the activity and its reification. To emphasise these related aspects of the mathematical object that is developed, Tall used the term 'procept' (Tall, 1996).

We should not take this notion of reification too literally here. In education, and also in history, the result at a certain moment will often be something in between a process and an object. It should also be acknowledged that the development will not be as linear as our description suggests. Like researchers, students may shift back and forth between process and object conceptions, depending on the problems they confront.

Choices for an instructional design

Looking at this presentation of history from an emergent modelling perspective (see section 3.5.1), we see a development of calculus that starts with modelling problems about velocity and distance. Initially these problems are tackled with discrete approximations, inscribed by discrete graphs (see Oresme's graphs at page 88). We could say that discrete graphs come to the fore as *models of* situations, in which velocity and distance vary, while these graphs later develop into *models for* formal mathematical reasoning about calculus. In the 17th century, graphs as inscriptions – initially discrete and later continuous – formed the basis for more formal calculus. This use of graphs in an emergent modelling approach seems useful for our teaching trajectory. It might be a natural step to use discrete graphs for describing motion, signifying measurements or theoretical motions (as Oresme did) and to take that as a starting point for reasoning about these motions. We assume that, in this reasoning, the use and understanding of graphical characteristics will emerge, together with kinematic understanding.

What kind of problems evoked reasoning on motion? One of the central problems in history was grasping the concept of free fall. Apparently, the proportional relationship between falling speed and falling time is not a trivial one. We teach students that at the first moment of a free fall, i.e. at the moment the object is not yet moving, the object instantly has an acceleration of 9.8 m/s^2. This beginning of a free fall is largely explained by our reasoning, not through experiment or intuition. Moreover, day-to-day experiences suggests, and hardly seems to contradict, $v \sim$ *weight*. This should be kept in mind when the context of free fall is used in teaching. Still, motion, in general, and free fall in particular, appear to be contexts that are suitable for secondary school students. They still grapple with the notions of instantaneous velocity and acceleration, and the relations with average velocity and distance travelled.

The grappling of students with velocity as a compound quantity is described in chapter 2 and can be understood from the history presented. It is remarkable how many centuries it took before velocity was defined as the division of two different quantities. Teachers can try to bring students to a position where they can see that their notion of velocity should be extended to a compound one. Another possibility is to let this compound notion of velocity emerge from reasoning about displacements and potential displacements in successive time intervals. This last choice can parallel a development in graphical inscriptions that emerge during modelling motion. This seems to fit well with prior ideas on a trajectory for teaching and learning the basic principles of calculus and kinematics.

Looking at this history, Kaput's (1994a) characterisation of calculus as 'the mathematics of change' comes to mind. In the process of trying to get a handle on change, the method of approximating a constantly changing velocity with the help of discrete graphs plays a key role. The relations between sum- and difference-series can be seen as predecessors of calculus. These ideas can be exploited in an instructional design by starting a learning sequence with investigating discrete patterns in displacements.

In addition to this didactical analysis of history, we have to analyse the knowledge and reasoning of modern 16-year-old students, and whether this can be connected to the didactical findings outlined above. Some of these findings, together with the organisational possibilities for such an approach, were explored in a pilot experiment.

5.2 Pilot experiment

Here we describe a pilot experiment which was performed early in this research project, parallel to the literature survey. This experiment had an explorative character and involved a preliminary case study of possible teaching practices in grade 10. We wanted to investigate two types of alternative activities concerning modelling motion, and the possibilities of sums and differences in functions as a topic. In this section the teaching materials and experiences are summarized. The materials and four case studies of students in this pilot experiment are described in more detail in the appendix.

The first activity concerned an orientation on modelling motion using a series of photographs of successive positions of a cat walking (inspired by Speiser et al., 1994). We wanted to create the need to draw graphs, and to foster initial reasoning about the relation between distance travelled and changing velocity. This activity could also provide insight into the students' ways of reasoning for modelling motion.

The topic of sums and differences was inspired by Leibniz's work, and was based upon a teaching sequence concerning the basic principles of calculus (Kindt, 1996). Students investigated the properties of sums and their increments, and the relations between series, in the context of mathematical formulas. In this process, the students were supposed to develop mathematical reasoning with intervals that would lead to the difference quotient. It was also a first introduction to the relationship between sums, summation symbols, increments and difference symbols.
The second activity, on modelling motion, concerned the transition from reasoning about velocity with continuous time-distance graphs to the mathematical notion of a difference quotient. This activity, about a comic strip character (see p. 115), was inspired by teaching sequences that fitted this line of thinking (Kindt, 1979; Kindt, 1996). Students had to determine velocities from distance-time graphs. The intervals in these continuous graphs, which were necessary for the difference quotient, probably derived their meaning from the preceding discrete work.
We stated earlier that this research project paralleled a nation-wide secondary school reform, incorporating both a new educational organisation and a clustering of topics into 'streams'. This had consequences for the teacher's organisation of the lessons. The teacher planned the content of all the lessons beforehand in a course description. Students were advised to follow this schedule, but *they* were responsible for their pace. The teacher rarely planned any interactive classroom discussions about their activities, because he assumed that differences in pace would arise quickly.
The analysis of classroom discussions and the students' written materials contributed to intuitive conjectures about the students' conceptual steps, and how these steps were related to the new organisation and to our alternative activities. The results of this pilot experiment were primarily based on anecdotal descriptions with the data available, and were used for defining conjectures operationally for our first teaching experiment.
The first alternative activity, on the walking cat, resulted in a variety of graphs, which could be used for discussions on the main issues of the chapter. The diversity in the students' solution strategies indicated that they did not have a standard procedure for displaying and reasoning with motion measurements as presented in the task. Tables with total distances and displacements, and different graphs (time horizontally or vertically) appeared productive elements for a discussion with respect to both representations of change, and the relation between displacements and total distance travelled. With respect to the specific context of this activity, we noticed that the students appeared to find it difficult and time-consuming to take measurements from a series of photographs. In each photograph they had to find an anchor point for the previous position. We concluded that such an activity would be more useful if the measurements were easier to make.
We saw hardly any connection between the students' work with sums and differences, and their work on modelling motion. We doubted whether the mathematical

relations in sums and differences really contributed to a teaching sequence on motion and for problematising instantaneous change. It seemed to be isolated from kinematic problems and might better precede the sequence or be dealt with afterwards.

The second alternative activity, on the comic strip character, evoked the intended graphical interpretations of instantaneous change. We expected the graphical reasoning with this activity would be connected with reasoning about graphs that resulted from a mathematical formula, and with calculations with such a formula. However, the pilot test did not establish a connection between students' graphical reasoning and working with a formula. The notion of how to determine instantaneous change when a graph has a large curvature, and how a formula can be used, was not problematised. Neither was the difference between average velocity and instantaneous velocity explicitly discussed. As a consequence, we saw that many students were only interested in the instrumental skills of how to perform the algorithm for the standard mathematical problems and how to use the graphing calculator to do this. In addition, due to limited time before the final assessment, the teacher demonstrated the use of the graphing calculator for answering typical questions on average and instantaneous slope. Traditional importance of instrumental skills force teachers under time pressure to use transfer-methods of teaching (Bauersfeld, 1995).

In a sequence for the teaching and learning of calculus and kinematics, we need to consider the transition from reasoning with discrete data to reasoning with continuous graphs and formulas. Drawing discrete graphs, based upon data, is time-consuming, and difficult to achieve when all the students need to have the opportunity to discover how graphical characteristics help to find patterns in the relation between displacements and total distance travelled. Especially these patterns can be problematised to motivate making continuous models, and discussing average and instantaneous velocity. We could, therefore, provide the students with tools to investigate more situations graphically. Computer programs can afford students ways of focusing on reasoning with graphs. Graphical characteristics can then emerge in meaningful contexts and can lead on to work with continuous models. This knowledge might be useful in problematising instantaneous change. Consequently, the development of a series of inscriptions from discrete graphs to continuous models parallels the students' conceptual development.

We identified crucial problem situations for giving the students the opportunity to invent solution procedures, and discussing them in classroom interaction. These situations, together with the instructions for the teacher, should prepare the teacher to discuss the students' contributions in line of the intended trajectory. However, we noticed that, as a result of the students' responsibility for the planning of their own work over a few lessons, there were big differences between the students' level of work. These differences in reasoning made it difficult to discuss an activity with a specific purpose with the whole class. We advocated such discussions so that the teacher could create a guided reinvention process for the whole class. This finding had implications for the teaching materials and for the use of course descriptions in our instructional design and teaching experiments.

5.3 Modelling motion as a conjectured local instruction theory

Concurrently with the pilot experiment, we performed a literature study on conceptual problems and possible solutions, which, together with our experiences in the pilot experiment and the history of calculus and kinematics, underpinned our conjectured instruction theory. Graphs played a central role in the conjectured trajectory in which we tried to overcome the conceptual and didactical problems described in chapter 2.

In chapter 3 we argued how the notion of emergent modelling can function as an educational design heuristic for a process of progressive structuring of motion from fragmentary student knowledge to an intended organisation of motion with physical and mathematical models. In these activities the development of inscriptions and their characteristics parallel the students' conceptual development. In addition, we pointed out the problem posing design heuristic in order to create opportunities for these developments for students. These opportunities were necessary to guide their thinking and their perception of problem situations and inscriptions that structure these situations.

Here we present a hypothetical development of models that describe motion, from context-close discrete models to the intended mathematical and kinematic models of change and motion, and how this development can be underpinned for students in a classroom situation. In section 5.3.2, the teaching sequence is described along with our conjectures on the teaching and learning of the basic principles of calculus and kinematics. This is the level at which we tested our research questions in classroom situations.

5.3.1 *Concept development through emergent modelling*

We aimed at a trajectory on modelling motion that encompasses the notion of velocity as a compound quantity, the difference between instantaneous and average velocity, and the relation between velocity and distance travelled. This trajectory should prevent conceptual problems such as sketched in chapter 2; we list: the velocity concept, instantaneous change, differences in notations between physics and mathematics education, a too rapid formalisation into quantitative methods, and problems with the use of graphs. It seems possible to develop kinematic notions together with the mathematical characteristics of graphs from contextual discrete graphing to reasoning with graphs of continuous models and difference quotients.

Analysing motion in an appropriate context should evoke an interest in grasping change and instantaneous change, in being able to predict, and in an initial orientation on change of position. In chapter 2 we described how Boyd & Rubin found how

the intervals between successive positions in time series appeared to be a basic structure element for reasoning about motion: a structure element both for describing aspects of motion, and for its representation in graphical inscriptions like trace graphs and two-dimensional discrete graphs (see section 2.2.2). Reasoning with displacements might result in graphs that have the potential for leading to a discussion on the relation between change in velocity and change in the total distance travelled. Therefore, we tried to induce reasoning with patterns in displacements in successive time intervals to underpin the benefit of illustrating these displacements in two-dimensional graphs. The teacher could play an important role in problematising these patterns while discussing the students' contributions.
Reasoning about velocity and its changes is still restricted to reasoning about displacements and their changes in fixed time intervals. We think that students can invent such graphical inscriptions and contribute to the intended trajectory. Moreover, the discrete graphs that might emerge in the activities can be a starting point for reasoning about the graphical characteristics that play a key role in understanding velocity as a compound quantity and leads on to the uses and characteristics of continuous graphs. For the instructional sequence, we confined the velocity concept to a scalar quantity and paid no attention to a frame of reference. These choices were the result of the limited number of lessons available and our focus on a trajectory along a series of graphs.

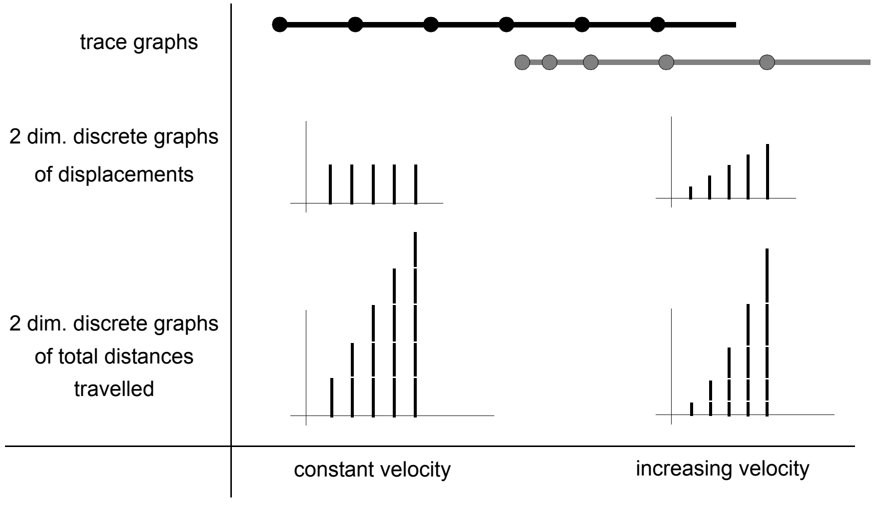


figure 5.8 Discrete graphs of measurements in fixed time intervals

This introduction to modelling motion should result in the students' understanding of the relation between displacements and distances travelled, which can be inter-

preted mathematically as a relation between sums and differences in graphs (fig. 5.8). The model in the emergent-modelling heuristic is shaped by this sequence of consecutive graphs. From a more global perspective, these graphs can be seen as various manifestations of the same model: a discrete motion-graph. Students can be expected to contribute to the invention of this model. It can be used to model their own 'informal' activity, and can gradually develop into a model for more formal mathematical and kinematical reasoning.
With this sequence of graphs, the connection between two-dimensional graphs and displacements in trace graphs is preserved, and leads to reasoning with increases. We also expected to prevent interpretations of graphs that have pictorial resemblances with the shape of the actual trajectory. Drawing dots too rapidly in a two-dimensional graph (and connecting them) might result in such interpretations (see chapter 2).
This seems the moment for the transition to continuous models. Requests for more precise predictions play a part and such questions evoke answers that involve more measurements and smaller time intervals. Students experience displacements becoming very small, and patterns that are more difficult to illustrate, which should create the need for them to overcome the problem. Two solutions might arise: the first one is scaling the vertical axis; this seems simple, but makes the comparison between different motions more difficult. The second one is scaling the displacements themselves. The observation that corresponding time intervals play a key role in this scaling can be motivated by trying to compare different displacement graphs with different time intervals. The second solution is the one we were aiming at. Scaled displacements become constant (average) velocities in the corresponding time intervals. This transition from graphing measured distances to displaying a (piecewise constant) compound quantity was historically, and still is, a conceptual leap.
Looking back at the history, we see that Beeckman used a kind of bar graph of piecewise constant velocities to display the hypothetical motion of a falling object. Such a graph could be a topic in the transition from discrete graphs of displacements to continuous velocity graphs. We think that the historical problem on free fall can be used for students to investigate the consequences of models of free fall. These bar graphs can be used for approximating a hypothetical, continuous velocity-time graph of free fall. From this reasoning, bar graphs should come to the fore as a way of approximating distance travelled with velocity-time graphs. These activities allow the meaning of an area under such a graph to emerge. The area of each bar in the graph represents a displacement and the bar graph is an intermediary step between discrete two-dimensional graphs and continuous graphs of motion.
The underlying concepts develop parallel to these graphical models. In the beginning of this sequence, velocity was associated with a displacement. Then the notion of velocity developed into a compound quantity involving the corresponding time

interval, and finally, in working with continuous models, the difference between constant velocity, average velocity and approximating instantaneous velocity emerged and was connected to various graphical characteristics.
What remains is the transition to the meaning of slope and difference quotient. Interpretations of graphical characteristics of continuous distance-time graphs, like the relation between linearity and constant velocity, are prepared in the discrete case. Problematising instantaneous velocity – e.g. by posing the question whether someone exceeded a speed limit – could evoke the targeted reasoning with chords on the graph for approximating velocity. Students might come up with the idea to use this calculation of average velocities on small time intervals for approximating instantaneous velocities.
As soon as these assumptions are developed sufficiently, we can start working on the solution. As a result of the preparation with discrete graphs, we assumed that this would give fewer problems than in the pilot test. The discrete experiences should support and give meaning to reasoning in the continuous case. The compound quantity velocity appears to be a measure for the slope of a distance travelled graph. The composition of time and distance travelled can be related to intervals of increase in the graph. In this way we expected the assumption to arise that the slope at a point on the graph can be approximated for determining the instantaneous velocity at that very moment.
Proceeding in this way, in the next activities and lessons, graphical models should begin to function as models for mathematical reasoning about extrapolating and interpolating patterns in these graphs and the use of the time intervals. Eventually, the graphs should be used for reasoning about integrating and differentiating arbitrary functions. Consequently, a shift is made from problems cast in terms of everyday life contexts to a focus on the mathematical and physical concepts and relations. To make such a shift possible, a mathematical and physical reference framework must be developed that can be used to look at these types of problems mathematically and physically (see also Simon, 1995).
Computer programs can be used to investigate many situations with graphs in order to afford students' reasoning about graphical characteristics and to develop their understanding of the relation between velocity, time and distance travelled. It is exactly the emergence of such a framework that this approach tries to foster. The next section gives an idea how this shift could be presented to students and achieved in a classroom situation.
Elements of such a development of calculus in the context of modelling motion can be found in many curricula (Hughes-Hallet et al., 1994; Kindt & De Lange, 1984; Polya, 1963; Sawyer, 1961). Nevertheless, we have not seen a systematic development of both kinematic *and* mathematical notions based upon a sequence of inscriptions, together with attempts to let students pose the problems that have to be solved with respect to a global problem and in the intended direction.

5.3.2 *The instructional design for modelling motion*

Here we describe an instructional design for modelling motion to learn the basic principles of calculus and kinematics. It contains characterisations of student activities, guidelines for classroom discussions, and our conjectures concerning the way in which the classroom learning processes will develop.

The guidelines are intended to help the teacher organise the lessons. In these guidelines we describe what can be discussed in classroom discussions, what input we expect from the students, and what the outcomes of these discussions should be before the students can proceed with subsequent activities. We do not want to compel the teacher to act as we describe here, but rather to give clues to enable him or her to deal with the presented materials in the intended way.

The conjectures that accompany this instructional design link back to our educational paradigm and the choices we made (see chapter 3). This section reflects our current instructional theory for the learning of calculus and kinematics, although it may be revised in the future. The principal theme of the sequence is grasping change in order to make predictions. The sequence starts by considering weather forecasts, since change and predictions are well-known notions in this context.

Weather forecasts to evoke an initial orientation on change of position

A situation in which it makes sense to describe motion is the weather forecast. The sequence starts with two satellite photos taken with 3 hours between them, and the aim is to predict whether the clouds, that have clearly changed position, will reach the Netherlands in the next 6 hours. We expect students to measure displacements and extrapolate from them in making their predictions. Next, the students are shown successive positions of a hurricane on a map, with fixed time intervals between the positions, and asked to predict when and where it will hit the coastline. These questions should lead to opportunities for discussing the changes in successive positions. A context is a story about the hurricane Olivia (fig. 5.9) with the accompanying question:

> The map shows a hurricane approaching land. It is Hurricane Olivia heading towards the west coast of Mexico. The last five positions of the hurricane were determined on 9th, 10th and 11th October 2000 at 6 a.m. and 6 p.m. Predict when the hurricane will reach land and describe how you worked this out.

This problem is posed as an over-arching question and returns throughout the unit as an example for the need to grasp change, and to reflect on what tools have been developed (fig. 5.9). We suggested the teacher to discuss the students' predictions and we expected some of them to use the pattern of the hurricane's increasing displacements. Discussing this pattern should encourage the students to proceed by drawing the displacements vertically next to each other in a two-dimensional discrete graph. After being introduced to time series and trace graphs, the students worked with situations described in stroboscopic photographs.

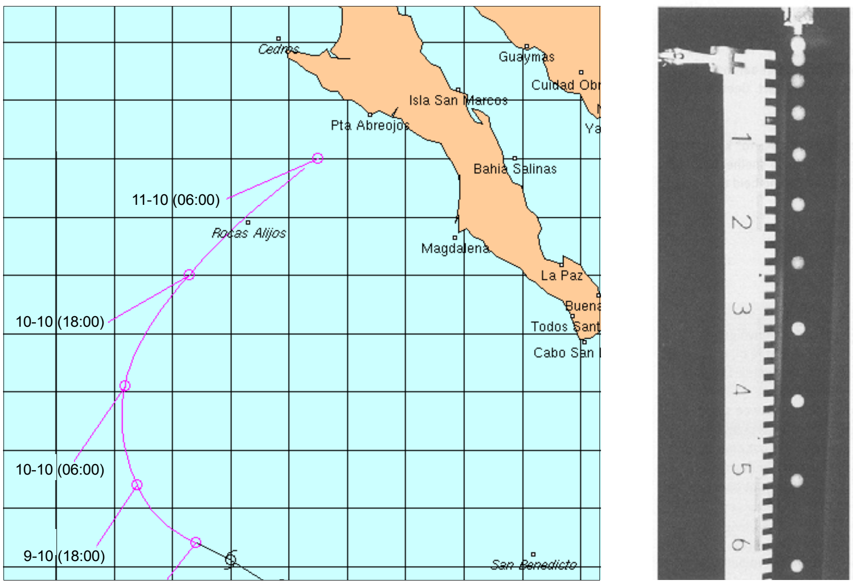


figure 5.9 Time series (left) Hurricane Olivia and (right) the falling ball

One of the questions was to display the motion of a falling ball using a graph to describe *how* the ball is speeding up. The idea was that students think of intervals as a measure of change of velocity (based upon Boyd & Rubin, 1996). Students should realise that it makes sense to display the measurements graphically for investigating and extrapolating patterns in the measurements. The time series and their graphs evoke reasoning with patterns in displacements and the relation with change of velocity. The weather context is what Noss & Hoyles (1996) called a situational relation between understanding and representations.

After working with the hurricane and the stroboscopic photographs, and conducting the classroom discussion, two types of two-dimensional graphs emerge: discrete graphs of intervals between successive positions, which we call *displacements*, and discrete graphs of *total distances travelled*. The classroom discussion should lead to a consensus about the use of these two-dimensional graphs for describing motion, and that drawing such graphs is a sensible way to proceed. In addition, the students have experienced that drawing graphs can be a time-consuming activity.

At this point, the use of the computer tool Flash, is introduced. The extensive care in introducing two-dimensional graphs might appear exaggerated, however, the conceptual development of velocity and the characteristics of graphs are tightly interwoven. We considered it necessary for the students to start again by studying situations that make it clear which situational characteristics lead to certain graphical characteristics. Note that a key element of the notion of reinvention is that the models first come to the fore as models of situations that are experientially real for the students. It is in line with this notion that graphs are not introduced as an arbitrary symbol system, but as models of discrete approximations of motion that link up with students' prior activities or experiences, and afford the intended reasoning.

An attempt with ICT to induce reasoning with patterns in discrete graphs of motion and the relations between them

The idea was that a computer tool provides the students with opportunities to investigate many stroboscopic situations. They were offered a variety of problems that aid contextual independence and supported their ability to invent and use graphical reasoning. The students could click on successive positions of an object in a stroboscopic picture, and the program showed the distances between these positions in a table, and displayed them in a displacement graph or in a graph of total distances. During these investigations the students moved on from measuring and situation-specific reasoning, to reasoning about graphs and their relations. The use of the computer tool should enable them to invent properties like the relation between average displacement and total distance travelled, and to find the relation between the linearity of a distance travelled graph for a motion with constant displacements.

A picture of the Flash computer screen is given here (fig. 5.10). The tool shows a stroboscopic photograph by Marey of a stick that has been thrown and which rotates through the air (Frizot, 1977). In the photograph, successive positions of the middle and of one of the endpoints of the stick can be located by clicking on the photograph. The clicking signifies measuring distances between successive positions. Next to the photograph is a table giving with the lengths of the displacements, and below the photograph is a graph of the displacements. Students could select one of the two discrete graphs (displacements or distances travelled) and the graph is constructed simultaneously with their clicking. The lengths are displayed in a two-dimensional graph as bars instead of dots to preserve the link with the displayed measurement. Consequently, we expected the lengths of the vertical bars to signify the distances between their measurements (clicking) in the photograph.

The distance is represented in the graph, not as the height of a dot, but as the length of a vertical bar. This representation is inspired by the historical development, described in section 5.1.2, where geometrical figures were used to represent quantities long before they were abstracted to dots in a graph. These graphs are assumed to afford reasoning within the problem situation, i.e. about patterns in displacements, change of velocity, and about the relation between displacements and distance travelled. These vertical bars again come to the fore in reasoning about difference quotients in continuous graphs.

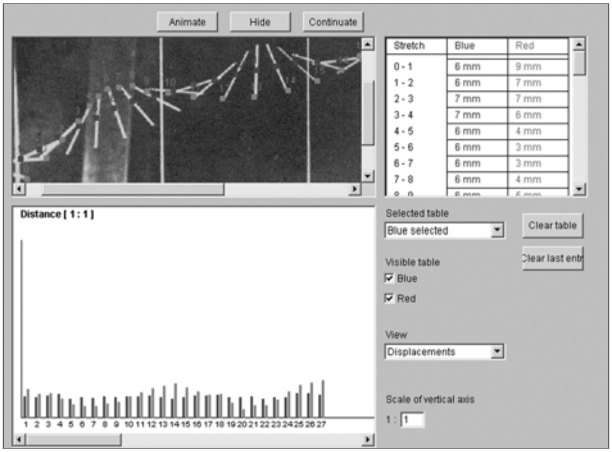


figure 5.10 Flash screendump of a thrown, rotating stick

Questions about the thrown and rotating stick are ‘Describe the difference between the two motions’ and ‘Does the total distance travelled differ between the two motions?’

We point out the reflexive relation between a model and the way one uses it (see chapter 3). While reasoning about the motion in the photograph with Flash, the use and interpretation of the graphical models change during the activities. This change concurs with a shift in the way the students think about the model, from a model deriving its meaning from the modelled context situation, to thinking about mathematical relations. First, these graphs are used to describe the situations and are related with measurements in the photograph. The image underlying the graph is that of the subsequent intervals between the dots. Second, the use of the graphs is dominated by thinking about graphical and conceptual relations between displacements and distance travelled (e.g. linearity in distance travelled is related to constant displacements). What used to be a record of measurements is now used as a tool for reasoning about patterns in measurements.

After these computer activities, students should be familiar with:

– The crossing of lines of summit of displacement graphs implies that the velocity of one of the objects exceeds the other, and not that one of the object passes the other.
– The crossing of lines of summit of graphs of total distances, implies one object passes the other.

– Constant velocity is related to constant displacements and to a linearly increasing distance travelled graph (and vice versa) (fig. 5.11).

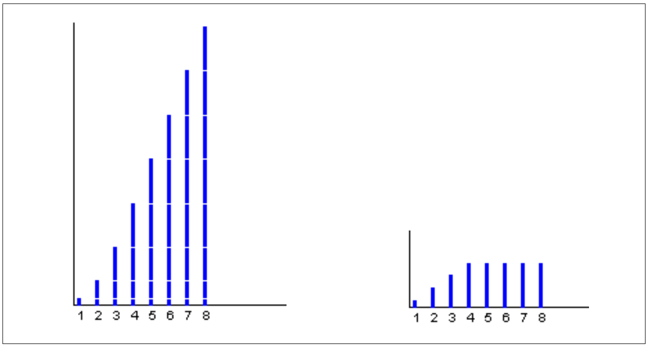


figure 5.11 Constant displacements from the 4th measurement

– The discrete case of the main theorem of calculus is implicitly touched on in this kinematic context where the sum of the displacements equals the total distance travelled, and the difference between two successive values of the distance travelled equals a displacement (fig. 5.12).

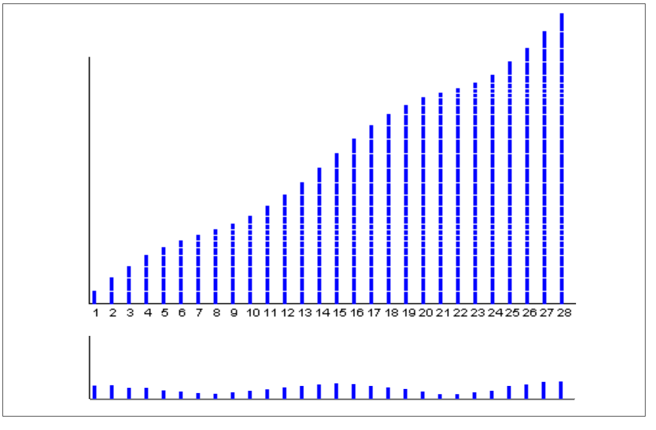


figure 5.12 A graph of total distances together with a graph of displacements

The students' reasoning with these graphs is important from a didactical point of view, because we conjectured that they support understanding in the continuous case and prevent iconic interpretations. During the activities with the graphical tools in Flash we expected students to develop their understanding and their language about changing velocity with graphical characteristics. Discussions among students should tell us whether they really invent meanings, or use superficial resemblances and a strategy of trial and error.

These activities prepare for the transition towards the notion of velocity as a compound quantity and the difference between average and instantaneous velocity. A key question concerning this difference is:

This is a graph of the distance travelled by an animal starting to run (fig. 5.13). Assume the animal does not change its speed after 25 seconds. Draw the displacement graph and then draw the graph of the distance travelled. If you are not certain about your drawing, explain why not.

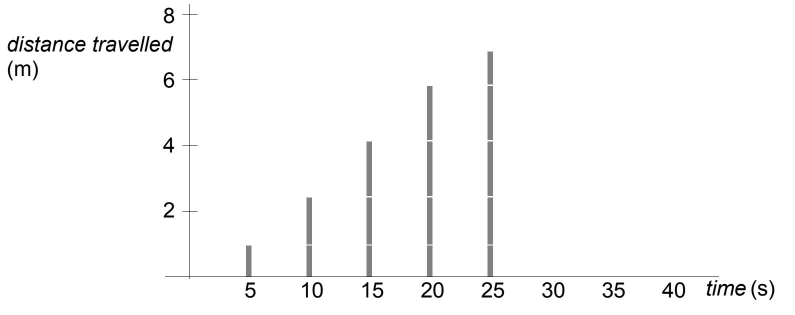


figure 5.13 A graph of distance travelled

While working with these discrete graphs, students should come up with the problem that in order to be precise about predictions they need more measurements. Consequently, time intervals decrease and it becomes more difficult to measure, to picture, and to view patterns in the displacements. We think this is a way to make the step from discrete graphs to a graph with a continuous time axis depicting average velocities.

The displayed average velocities are supposed to derive their meaning from the corresponding calculations and displacements. The shapes of graphs of average velocities look like those of displacements. This understanding of velocity is related to a medieval interpretation of velocity as a potential displacement (e.g. see p. 91).

Our final question in this section reflects on the starting situation of Hurricane Olivia. It is whether the tools developed enable us to make better predictions? We expected the students to comment that you could measure the successive positions of the hurricane at shorter time intervals to gain a better view of the pattern in the displacements. In addition, they should note that you can never be sure about what happens between measurements. They should differentiate between changes in average velocities based upon the measurements and the actual velocity after the last measurement. This can be used by the teacher as a content-related motive for introducing hypothetical continuous models for predictions.

Introduction of a continuous model: Galileo and free fall

The transition to continuous models was introduced in the context of a narrative about Galileo's work. Students were asked to interpret Galileo's hypothesis that the velocity of a falling object increases in proportion to the time it falls. This proportionality between velocity and time was underpinned by the students' investigation of a stroboscopic picture of a falling ball with Flash.

We chose the story about Galileo because we thought that it would be a relevant problem for the students, and would give them a view on a milestone in the history of this topic. We did not necessarily think that all students were interested in history, but the problem might interest students in the possibility of making predictions based on a hypothetical continuous model. The central problem in this section is shown in figure 5.14:

There is an anecdote that Galileo dropped two lead balls of different weights from the leaning Tower of Pisa to see whether they reached the ground at the same time.
The tower is 55 m high. The balls reached the ground after about 3.5 seconds.
Assume they did indeed fall according to his theory $v = constant \times t$.
How large would this constant be for the falling balls?

figure 5.14 Leaning tower of Pisa

We expected the students to come up with the idea of approximating the increasing velocity by linearly increasing displacements, with a total distance travelled of 55 metres. The number of displacements depends on their choice of a time interval. This strategy was expected to follow on from their preceding activities.

The displacements represent – in accordance with the medieval notion – the distance covered if the moving object maintains its instantaneous velocity for a given period of time. With these displacements they could calculate constant average velocities for the chosen time intervals. The graph of the average velocities will also increase linearly, and the slope of the graph then represents the constant value they need to find. This last notion of the relation between the slope of a linear graph and the constant value in the formula has been addressed in previous years, and we expected the students to be able to use that notion in this context.

In the following activities, the students worked with the possible linear relation between falling time and falling distance; for instance, they could verify it with measurements in the stroboscopic photograph.

Further development of the continuous model with discrete results

In the previous subsection, we expected the students to have found that linear increasing displacements approximate a linear increasing velocity-time graph and in this section we stated that Galileo derived a continuous model for the relation between falling time and falling distance. This model can be represented with a linear velocity time graph. The central activity is about a discrete approximation of this graph (Kindt, 1996; Polya, 1963). We let the students experience Galileo's line of reasoning.

We have not yet thought of a way of encouraging them to invent the discrete approximation with piecewise constant velocities. The multiplication between a time interval and a constant velocity results in a displacement in the corresponding time interval. From there on, they should see the connection with the discrete case: adding displacements gives the total distance travelled, and the use of the 'middle' displacement.

The teacher can introduce this situation by presenting the continuous graph and asking for a way to solve the problem of predicting distance travelled. The teacher should try giving the students the opportunity to think about and discuss possibilities before presenting Galileo's reasoning. If they describe their thinking, it shows us to what extent it is in line with Galileo's reasoning, and how far Galileo's reasoning can further be revealed to them. In a subsequent activity, a discrete approximation is presented to the students.

> From his formula for velocity, Galileo found a formula to determine the distance travelled. His reasoning was more or less as follows.
> Below you see the graph of $v(t) = 10 \cdot t$.
> The time is divided into 10 intervals for calculating the distance travelled after 5 sec ($\Delta t = 0.5$ s.). The velocity in an interval is chosen as constant and equal to the initial velocity of the interval (fig. 5.15).
>
> For each 5 sec you can now calculate the displacement. Work out the displacement in the interval [2.5 ; 3].
> By summing these calculated displacements you get an approximation of the total distance travelled.
>
> By summing these calculated displacements you get an approximation of the total distance travelled.
> Is the approximation you have worked out too large or too small?
> How can you make a better approximation of the distance travelled?
>
> Every calculation of the displacement in a time interval can be seen as working out the area of the accompanying grey bar. If you make the time intervals smaller, the bars approach more closely the area of the triangular shape bounded by the sloping line, the line $t = 5$ and the time axis.
> Explain this.
>
> What is the exact distance travelled?

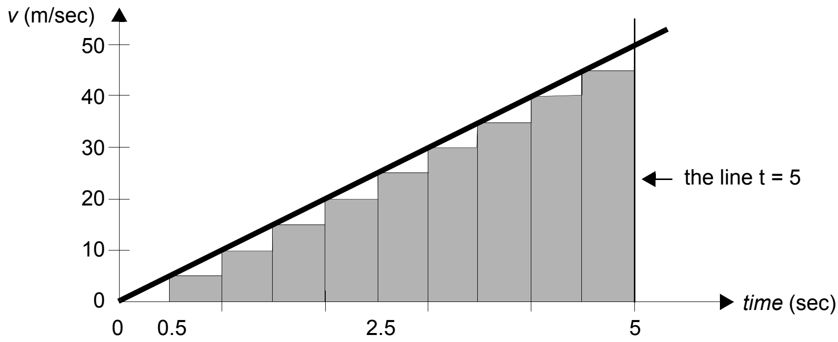


figure 5.15 A discrete approximation of the proportionality between velocity and time

After calculating and improving discrete approximations, the students were expected to make the connection between the area of the bars in the discrete graph, and the area of the triangle that is created by the continuous graph:

$$s(t) = (t \times 10 \cdot t) / 2 = 5 \cdot t^2$$

This resulting formula reveals the quadratic relation between time and falling distance that Galileo used to test his hypotheses empirically. All students should be able to connect this reasoning with the discrete case. This is what we should observe during the activity. The students might be surprised by the calculation of an area. However, we expected them to experience this as valuable, because the resulting formula is far more useful than calculating and adding displacements

There is a danger in our presentation of Galileo's reasoning that the students might solve the questions that follow without understanding the main concept. In the following activities, they have to use this reasoning, and to adapt it to new situations. If they do this correctly, we assume that they have understood what happened. If they do not succeed, it should be possible for them to trace the meaning as a result of the emerging understanding supported by a series of graphs.

A following activity is about the velocity graph of a cyclist (fig. 5.16). In tackling the question (approximate the distance travelled), we supposed students would approximate the graph with bars of constant velocities at suitable time intervals.

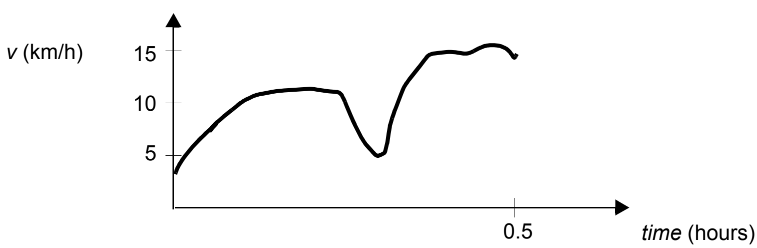


figure 5.16 Question about the *v*-*t* graph of a cyclist

By approximating changing velocities with bars, the first step is made towards creating an experience base for the process of describing motion and leading to integrating functions. The calculations with intervals in continuous velocity graphs should also be helpful for reasoning about velocity and instantaneous velocity with continuous distance travelled graphs.

Evoking the need for determining instantaneous velocity

In the previous section, the students were introduced to continuous models and graphs and they found a method of determining distance travelled from velocity-time graphs. In this section, we posed the problem of whether it is possible to determine velocity from a distance travelled graph. In the preceding activities, the students used a time interval Δt for calculating displacements Δs and total distances travelled ($\Sigma\ \Delta s$). These intervals are structuring elements for reasoning about distance travelled with velocity-time graphs.

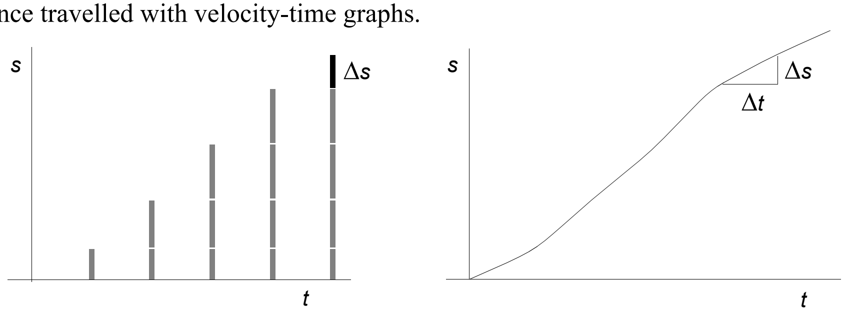


figure 5.17 Displacements Δs in a discrete and in a continuous graph

The relation between discrete graphs of displacements and of distances travelled, and the use of these structuring elements should support students to start reasoning about a relation between the slope of an *s-t* graph and velocity (fig. 5.17). As in the history of this topic, we assumed that the association between area and distance travelled would be more accessible for students than the association between the slope of a chord and the corresponding average velocity. Therefore, in our approach, reasoning about continuous velocity graphs preceded activities with continuous distance travelled graphs.

A situation about a Dutch comic character who drove his car through a village (inspired by Kindt, 1979) is presented together with a continuous time graph of his distance travelled (fig. 5.18). We expected students to reason about velocity with discrete approximations of time and distance (Δt and Δs) in this graph. The first questions were: what would the graph look like if he travelled 10 km in 15 minutes at a constant velocity? Do you think he broke the speed limit? We expected the students to come up with reasoning in which they calculated quotients of displacements and corresponding time intervals.

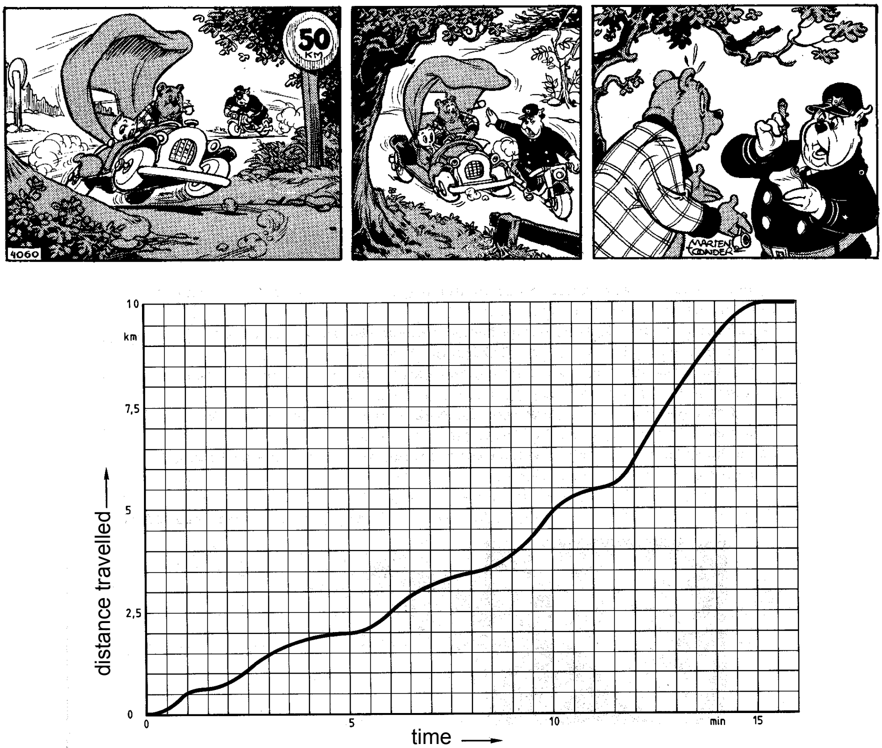


figure 5.18 Was Mr Bommel breaking the speed limit?

The next questions addressed the problem of predicting velocity with the information given in the graph. What would this graph look like if the velocity did not change after the first 6 minutes? This idea referred to the reasoning in the discrete case where potential displacements signified instantaneous velocity.
The students were supposed to use graphical strategies with tangent-like results 'on sight', and come up with different slopes and velocities. This should result in a motive to find a way of being more precise and to reach a consensus. After discussing this activity, the tangent-like continuation is called a linear continuation at a certain point of a graph. To be more precise about the velocity at any instant, it was suggested they model the situation where a part of the graph is approximated by a graph of a function (in line with Galileo's reasoning). The students were asked: Can you now be more precise about the velocity after 6 minutes? Can you be more precise about a linear continuation?
At this point we thought of using a connection between strategies 'by eye' and a strategy for calculating average velocities by using intervals, signifying displacements and time intervals, and difference quotients. For making this connection, we

let the students use the Slope computer program (inspired by Van der Kooij & Goris, 2000). They can let the program draw a difference quotient on the graph as a chord, and can zoom into a part of the graph (fig. 5.19).

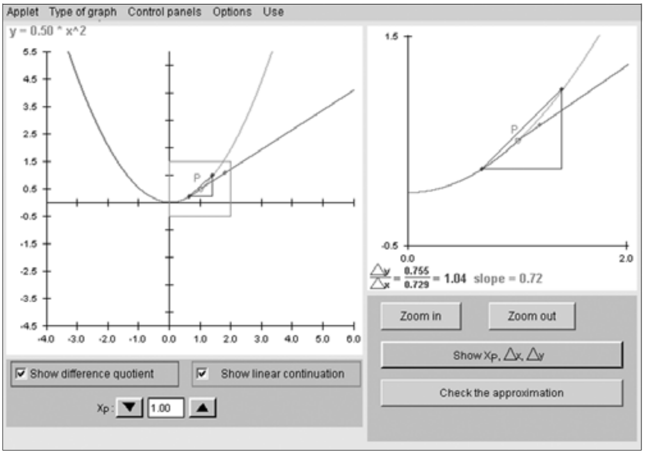


figure 5.19 A screen dump of Slope

By investigating various situations, and using the chord and zoom-in tools, we expected some of the students to invent ways of being more precise in their approximation of the linear continuation from a point of the graph. This builds on the ideas of potential displacements, and local straightness of a graph (Tall, 1996). A typical question accompanying the program was: what would the graph look like if the slope (i.e. speed) did not change from point P? For the students, this should be connected to their work with Flash (fig. 5.11). With the Slope program, students could approximate this continuation by directly manipulating a linear continuation and a chord on the original graph that represents a difference quotient.

After some introductory tasks, students could play a game with the Slope program. The goal of the game was to determine the linear continuation of a graph at a point by approximations with the difference quotient. The program presents five random situations, and for each situation they get five points if their first try is correct. For each wrong try, the number of points is decreased by one. In this game – depending on the level they are playing at – the students have a number of tools for approximating the slope of the linear continuation. They can determine at which level they want to play but the game should challenge them to be as accurate as possible, and consequently, to think about the kinds of tools they can use.

The first level of the game with Slope is called ‘On sight’. It offers the possibility to rotate the continuing red line to create a transition as smooth as possible. At this level they cannot zoom in. It becomes difficult to determine the linear continuation on sight, especially if the graph has a large curvature.

The second level is called 'Using all tools' and here the students have two extra tools: a zoom-in option and a blue chord on the graph that can be changed. The slope of the chord is displayed. We assumed that the students will change to this level when they feel the need for being more precise. However, we are conscious of the danger that students may go to this second level and try to discover how to use these tools by trial and error. This is an important observation criterion for this lesson.
We expected that, using Slope, the students would develop a strong graphic and dynamic image to support the relation between the slope of a chord, the difference quotient, local straightness of graphs, and approximating instantaneous change. The graphic-dynamic image of approximating a linear continuation with Slope should function as a generic organiser (Tall, 1996) for finding slopes of tangents and for discussing these notions in subsequent lessons. The images in Slope can start to function as a generic example, which embodies the general property of approximating the value of instantaneous change with a difference quotient.
During these activities we expected the students to change their reasoning from situation-specific, e.g. about breaking speed limits, to mathematical reasoning with graphs, difference quotients and instantaneous change. This mathematical reasoning will support their understanding of instantaneous velocity and its relation with average velocity.

5.4 Summary

The instructional sequence is supposed to create a process of teaching and learning in which students develop the basic principles of calculus and kinematics. This process will enable them to shift from context-closed reasoning to a reasoning with graphs and calculations with intervals. The development of concepts and related representations can be traced back and is supported by a series of graphs. This series should reflect students' contributions and inventions during their activities, and the guidance provided by both the teacher and the teaching materials.
The instructional design is an initial implementation for a conjectured local instruction theory and can be used as an operationalisation of the research questions. The guided reinvention approach is realised by the design heuristics of emergent modelling and problem posing. The shifts presented, from 'model of' to 'model for', should concur with a shift in the way students perceive and think about the model, from models that derive their meaning from the context situation modelled, to thinking about mathematical and physical relations. Students' reasoning is supported by a global problem, which should evoke content-related motives to proceed in a certain direction. This problem leads to graphical reasoning, to posing problems that have to be solved, and to reflections on the results of activities.
The central model in this learning route is that of a discrete graph. This model is the basis both for integration and differentiation through sums and differences, and for the relation between velocity and distance travelled.

We described a general operationalisation of the research questions (see page 65). With the instructional design we can make these questions operational. We demonstrate this for the first lessons on *Weather forecasts to evoke an initial orientation on change of position* (table 5.2). In the table we quote the research questions from chapter 3, and describe how they should be answered within the educational setting that we created.

Questions	Observation criteria
1: Do students perceive the problem situations as intended, contribute to the guided reinvention process, and reach the intended goals?	In their initial (intuitive) reasoning about the weather problems, students refer to the intervals between successive positions and relate lengths of these displacements with velocity. Students invent ways to describe and investigate patterns in displacements. These inscriptions and the corresponding reasoning are shared, and form the basic input for classroom discussions and for the way to proceed with two types of discrete graphs.
2 EM: Does the previously planned sequence of graphical tools fit students' thinking and foster advanced reasoning by a shift from model-of to model-for?	The way students reason with the graphs changes from context-oriented (referring to distances in the stroboscopic pictures) to an orientation on characteristics of, and relations between, the graphs of displacements and of total distances travelled.
2 IT: Do the representations in the computer tools fit prior reasoning and how do they afford advanced reasoning and sense-making?	Initially, students use the stroboscopic pictures and prior activities to signify the graphs in Flash. During work with Flash, students increasingly use the graphs offered for solving the posed problems. As a consequence, they simultaneously invent use of and relations between these tools.
2 PP: Are students aware of a global problem that is being solved, and do the local problem situations provide the students with content-specific motives to proceed in the intended direction?	Students point out that there are not enough measurements for being precise about the hurricane. They note that more measurements make it harder to display displacements. The teacher can share these remarks in a classroom discussion and evoke content-related motives for the way to proceed. Students experience that this way is a promising one with respect to the global problem of describing and predicting motion.

table 5.2

This instructional design is elaborated for the lessons in the teaching experiments. The teaching experiments are described in chapter 6. For each part of the sequence in chapter 6, we first recapitulate the content and the observation criteria, and secondly, we present the results and give illustrations of these results.

6 Teaching experiments

This chapter presents the two teaching experiments with the instructional sequence based upon the conjectured local instruction theory (see chapter 5). It begins with a description of the first experiment in section 6.1 and a reflection on this in section 6.2. The reflection led to a revision of the teaching sequence, and to adaptions and a better articulation of the underlying conjectured theory. A second experiment was required to collect more systematic data about the role of computer tools in the learning process. The changes in the teaching sequence for the second experiment are discussed in section 6.3. The second experiment and a reflection on the experiences gained are described in sections 6.4 and 6.5.

6.1 First teaching experiment

In chapter 5, the conjectured local instruction theory was elaborated in an instructional sequence for ten lessons. This sequence was tested in parallel in two pre-university classes, grade 10, at different secondary schools (*school A* and *school B,* more details about this choice can be found in chapter 4). In this section we describe our experiences with the sequence at both schools. The description is divided by the units of the sequence which comprised two or three lessons. The description of each unit is introduced with a short review of the contents (discussed more extensively in the previous chapter) and the observation criteria in relation to the research questions. This introduction is followed by the results and illustrations of the results.

At one of the two schools (*school B*), the students had already studied kinematics during their course in physics, a treatment of the topic that focused primarily on using formulas. At the other school (*school A*), the students had not yet studied kinematics. This difference between the two schools remains a point of attention in the analysis of our experiences.

6.1.1 Weather forecasts to introduce the concept of change of position

The introductory part of the learning trajectory comprised one lesson. Predicting and describing change were introduced in the context of weather. The central question for the students was: how can you describe change so you can make predictions? Our research questions were: can we foster the notion that change is an important issue and requires tools for its description, and can we make students aware that graphing displacements is a possible way to proceed?

The weather context is meant to focus the students' attention on displacements in constant time intervals (time series) as a way to keep track of change. First, students will be given two satellite weather photographs with three hours in between and a question about weather prediction. Our intention is that their reasoning will develop so they use time series to organise the change in weather. One representation of a time series is a trace graph of points and connecting lines of a moving hurricane drawn on a geographic map. In relation to the central question this representation is the starting point for the learning trajectory. Our expectations for this lesson were described in section 5.4.2. The lesson is successful if the students refer to the importance of gaining insight into patterns of trace graphs in the weather context. Moreover, we expect that at least a few students will, on their own initiative, draw two-dimensional graphs of the displacements in order to better analyse and extrapolate the pattern. The class discussion based on these solutions should result in a consensus on proceeding with two types of two-dimensional discrete graphs: the displacement graph and the distance-travelled graph.

Results

The lessons at the two schools were conducted differently. The central aspects became rapidly clear at *school A* from the class discussion immediately following the first task. At *school B*, the tasks were discussed at the end of the lesson and there was no class introduction. One of the reasons for this was that the lesson took place on Monday during the first hour, a time when many students arrive late. This was an unexpected phenomenon at this school. The emphasis during the activities at *school B* was therefore placed more on independent work (in pairs) than on class discussion. This had the consequence that the students worked through the tasks more hurriedly since they did not entirely understand what was expected of them.
The activity with the two satellite photos prompted the students to use displacements to make predictions about the movement of the clouds. This was apparent from the students' worksheets and the first class discussion at *school A*. The students reasoned about the change in velocity of the hurricanes with the changing lengths of displacements and the patterns of these changes. In their reasoning, they described changes in patterns with speeding up and slowing down. Only one student misinterpreted the trace graphs and confused, for example, points that were close to each other with a higher frequency, which suggested a higher velocity. The iconic nature of the graphs did not cause any confusion at this point because the trace graph followed the track of the hurricane on the map.
Especially at *school A*, the students contributed ideas that provided direction for the solutions and for the sequel to the lesson. The question 'How can you improve the prediction?' was linked in the class discussions and the written work primarily to the answers 'with more data' and 'with more information about the phenomenon' (for example, about the difference in velocity of a hurricane over water and over land). The students did not independently state that better mathematical tools could be useful. Only one student's notebook contained the statement that more knowledge about graphs could aid this process. The teacher did not focus the discussion on the importance of gaining insight into the patterns of the displacements. This would have evoked the need for a different way of organising displacements.

In the planned activities and the instructions for the teacher, we apparently had not made it sufficiently clear that the teacher should have problematised precisely this aspect: understanding motion by understanding patterns in displacements. The problem-posing role of the teacher when evoking questions and ideas was more important and more subtle than we expected. The teacher at *school A* did not focus the discussion about the hurricane on how acceleration occurs, i.e. the pattern in the intervals. But he did address the request for more data. This became the direction in which the students looked for the answer, while we had intended the discussion to go in the direction of understanding patterns and reasoning with various graphs and their global shapes.
During the class discussion about a time series, the teacher at *school B* asked the students about the change in 'displacements', even though this concept had not yet been explicitly introduced and had not yet been linked in this lesson to the distances between the points. This is a good question about one aspect of organising the information in a different way, but it was asked before the problem had been discussed for which it could have offered a solution. Despite the fact that these students had already studied kinematics during their physics course, they then appeared to guess at his intentions and the desired meaning of the displacements, which was exactly what we wanted to avoid.
We were of the opinion that the importance of a graphical description for making predictions had started to become clear to the students. Students contributed to the idea that time series play a role in this process. With the contribution of the students, consensus was established about the model of the time series: the trace graph. In their reasoning, changes in velocity signified changes in lengths of displacements in subsequent time intervals. Nevertheless, the importance of gaining insight into patterns and trace graphs did not emerge sufficiently. As a result, the intended two-dimensional discrete graphs did not come up for discussion. During the students' homework, this might cause a problem with the activity involving patterns in displacements of a stroboscopic photograph of a falling ball (see page 106).
We have the feeling that in the present teaching sequence – with the accompanying instructions to the teacher about when to leave the students free to articulate their own ideas and what guidance is needed – this type of activity (a context problem about weather) has the potential to interest students in the issue of change and supports them to develop the idea that displacements between successive positions are elements that provide structure to their reasoning about motion. At the level of local teaching theory, we concluded that situations with discrete measurements, beginning with patterns in trace graphs in a context problem about making predictions, are useful starting points for teaching calculus and kinematics.

Illustrations of the results

While working on the question about the satellite weather photographs, the students brought up the problem of velocity: change in the velocity and direction of the clouds. During the class discussion, two satellite photographs taken at different times were placed on the overhead projector. The discussion concerned velocity, change in velocity and the direction of the displacements. The students concluded from the photographs that in three hours the clouds travelled the distance from Brittany (F) to Belgium, and they extrapolated the distance from their measurements. At this point, Mathias remarked:

> You can now measure the velocity, but you don't know if it goes faster or slower later on.

We suspect that Mathias is referring here to the limitations of measuring and extrapolating. Many of the students participated in this class discussion. This is how the introduction to the central problem of understanding change took place. In the following task, the students interpreted the given time series without any trouble, and they focused on the change in displacements when describing the motions and making predictions. Inge's written answer is representative of what most students answered, and illustrates an interpretation of a given time series where she used the notion of 'slowing down' signifying a decrease in displacements.

> Many hurricanes change direction and go slower and slower, in 24 hours they don't go as far. When they are turning, they slow down.

As part of the tasks about a hurricane (named Olivia), a time series was provided as a model for the trajectory of the accelerating hurricane (see page 106). Students extrapolated the line smoothly and repeated only the last displacement, or tried to use the increasing pattern to determine the moment when the hurricane will reach land.

During the discussion of these solutions, only one student suggested that you could use the latter method to extrapolate a pattern of increasing displacements. However, the teacher at *school A* did not formulate a problem about the progression of the pattern, but instead focused on the necessity for more data. The transcript below illustrates this:

1 Veerle: Extrapolate the line.

2 Teacher: Yes, that's the first thing you can do. [He extended the line on the sheet.] And then?

3 Sabine: Measure the distances between the points. [She has not measured precisely, but estimated only the final trajectory.] You put that at the end.

4 Teacher: So you are assuming that the hurricane is always going at the same velocity?

5 Other student: But it keeps going faster.

6 Teacher: Exactly.

7 Jarno: But it goes slower on land. [This was the result from studying trace graphs of various hurricanes.]

8 Teacher: Yes, they could cancel each other out.

9 Inge: But it's still about when the hurricane hits land.

10 Teacher: But you need this to estimate the following point.

11 Student: Could I have more data?

12 Teacher: Exactly, you don't have enough data. Unfortunately this happens frequently in the real world. Therefore you just have to reason with this data and then make estimations for what you don't know.

On line 6, instead of agreeing with the student's remark, the teacher could have asked 'How do you know?' So far the discussion focused on the precision of making a prediction. It is striking that at the other school (*school B*), a comparable discussion about a previous task proceeded with more difficulty. There the teacher emphasised that there was something going on with the displacements, but this was before it became clear to the students how the displacements could offer a solution to the problem. The transcript illustrates the students' attempts to find the answer which the teacher desires:

1 Teacher: Exactly. One of the hurricanes appears to go over land and the other does not. That's all there is to it with this type of hurricane, of course: do they go over land or do they not? I think this is the essence of what you are trying to predict. So you try to say something here about the direction the hurricane would take. This is based mostly on experience. Another element [...] also plays a role here: the displacement. What is going on with the displacement?

2 [silence]

3 Teacher: Is nothing happening to the displacement? … Has anyone looked at this? With hurricanes it's also important what time they hit land. Not only *if* they hit land, but also when. So they still have time to take precautions.

4 Student: But isn't that already on the map? Next to the dots it shows how many hours….

5 Teacher: And does something change there?

6 Student: Something changes with the dots, they're always 24 hours apart...

7 Teacher: They're always 24 hours apart ... I thought that the size of the displacement played a role. Has anyone seen this at all? Does it vary or doesn't it?

8 Gwen: It slows down...

9 Teacher: Where do you see that?

10 Gwen: The points are closer together. Maybe that's caused by the land.

Gwen's response on line 8 illustrates that she does not yet understand exactly what the teacher was intending with his question about what is going on with the displacements.

6.1.2 Using IT to induce reasoning with discrete graphs of motion

During the first lesson we did not successfully address the patterns in displacements and the use of two-dimensional graphs. The aim of this second unit was that students should begin to understand the need to display patterns in trace graphs and understand the relation between these patterns and the characteristics of displacement graphs and distance-travelled graphs. Subsequently, both two-dimensional graphs should emerge during the discussion of the students' answers to the question about the stroboscopic photograph of the falling ball.

We gave students the opportunity to investigate various situations with the computer program Flash (see page 108). We wanted them to construct the relation between patterns in trace graphs and graphical characteristics during their investigations. Our main question was whether or not the activities with this computer tool were compatible with prior activities, and supported the students to reason in the intended way. We expected their reasoning to change from context oriented – where lengths and patterns of displacements refer to distances on the stroboscopic pictures – to a more abstract orientation involving characteristics of the two graphs in Flash.

During the lesson after the computer activities, our intention was that these relations would be made explicit during a class discussion of their experiences with Flash. Afterwards, the students worked on tasks aimed at familiarising them with the intended reasoning with two-dimensional graphs and to create opportunities for developing models of motion that were more continuous.

With respect to the central question about predicting motion, we expected that students would experience the limitations of discrete models for displaying motion, and would then feel the need for other graphic tools. One limitation is the measurement problem: to be more precise one needs more measurements, the displacements decrease and it becomes more difficult to see a pattern. To solve this problem, the teaching material guides students to graphs of average velocities (scaling measurements to the corresponding time intervals). Our expectation that such graphs emerge from their reasoning should be supported by the way the students presented the advantages of these graphs during the class discussions.

At the end of this unit, the teacher reflected with the students on the task about the hurricane. We expected the students to realise that they could not be any more precise since they did not know what happens between the measurements. The teacher should use this uncertainty to start a discussion about the difference between average and instantaneous velocity and the use of continuous models of motion.

Results

The lessons in this unit were only partly carried out as planned. The computer lesson did not take place at the *school A* due to technical problems. During the next lesson, two pairs of students went through the activities using laptop computers. The results concerning the computer activities are based on the work of one of these pairs and on one pair at *school B*, where the computer lesson was completed by the entire class.
The introductory tasks about making a graph to show how the falling ball accelerates did result in the intended diversity of graphs. In the students' workbooks, we observed that three variables had been used for the vertical axis: sequential displacements, total distance travelled, and height. For the horizontal axis, we saw two choices: numbered measurements and time. The students alternated between drawing the graph with points or with a continuous line (no bar graphs were used). In short, this freedom to choose the kind of graph to draw provided input for a discussion about possible methods of description as an introduction to the discrete graphs of Flash. However, the teachers spent little time on this discussion. They wanted to give the students plenty of time for the computer activities. Moreover, the computer lab did not lend itself very well to a class discussion.
The limited preparation resulted in there being no consensus beforehand about why and how to proceed with the discrete graphs in Flash for modelling motion. The first task was used by the students to explore the program. The substantive yield of this activity was low. However, measuring by clicking and seeing the graph appear did have the intended development as a result. Due to this building upon stroboscopic pictures and trace graphs, the process of learning did not remain to trial and error. Most students deduced that the lengths of the vertical bars signified displacements on the trace graph.
During the computer lesson, the two pairs of students at *school A* developed their reasoning and their use of the program. At the beginning, their language primarily referred to successive positions in the stroboscopic photograph. As the lesson progressed, their language increasingly involved characteristics of the graphs of displacements and of distance travelled. Our intention was that the students' emerging reasoning would concern these graphs and how they are related. The students appeared to develop insight into the relation between a graph of constant displacements and constant velocity and the accompanying linear graph of distance travelled. However, it is unclear how explicitly this took place. The only indications we could find in the records concerned the participation of an observer who requested clarification about what the students thought. Although the observer tried to be as restrained as possible, requests to make things more explicit do influence the learning process (i.e. reflection) of the students.
During the discussion about the computer lesson at *school B*, the teacher only asked what the advantage of one graph was above the other. There appeared to be no ques-

tions about this. This could have been because the students had the feeling that they had already learned this in physics. However, the learning process that we observed with the pairs of students at this school showed many similarities with the process of the pairs at *school A*. We were unable to find any systematic differences due to the students having more previous knowledge about kinematics. In retrospection, a more extensive discussion about the computer lesson was required to emphasise the relation between the two graphs and the specific limitations of their discrete character. After all, in the next lesson, problems will be designed about precisely these aspects.

At *school A*, the computer lesson was not discussed afterwards mainly because most students did not participate in the computer activities. At this school, the teacher reflected first on the state of affairs resulting from the activity on the hurricane. After this he had the students work on several tasks from the instructional materials with pen and paper and discussed the various graphs, their mutual relation and the relation with descriptions and predictions of motion. During these discussions, the students made fewer contributions than at *school B*. We conjectured that they were less familiar with the graphs and the relevant reasoning because they had missed the computer lesson.

Our intention was that at the end of this lesson (the one after the computer lesson), an activity would be discussed where the students would extrapolate a displacement graph and a distance-travelled graph (page 110). It turned out that the students at both schools had correctly extrapolated the graphs (linear increasing distance travelled and constant displacements). There was, especially at *school B*, an extensive discussion about the value of the constant displacements. In this way the distinction between instantaneous velocity and average velocity in relation to the characteristics of these graphs emerged; characteristics which anticipated the activities in the computer program Slope (page 116).

The fact that at *school B* the students were more easily able to discuss continuous changes with the discrete graphs is probably due to their previous knowledge of kinematics (the relation with tangent lines). Based on the available data, however, we could not conclude this with certainty.

Due to the schedule for the experiment, neither of the teachers had any time to discuss the activities from the end of this unit about the limitations of discrete graphs. In the students' workbooks, however, we found starting points for such a discussion, but most of their answers were incomplete. Finally, no problem was designed for the transition from a discrete horizontal axis to a continuous time axis and no consensus was achieved about the relation between displacement graphs and graphs of average velocity. In retrospection, we see that we did not emphasise the importance of this for the students or the teacher sufficiently, nor did we describe this transition in sufficient detail.

After the relevant lessons, we discussed this with the teachers. We decided that dur-

ing the task about Galileo, we would focus extra attention on the transition from discrete measurements to a continuous model for velocity, by initially reasoning as much as possible with displacements, and then hold another class discussion about the transition to graphs of average velocity.
Based on these experiences, it was not an easy task to formulate conclusions about this unit and we will return to this in our reflection on this first teaching experiment in section 6.2.

Illustrations of the results

The two students who were observed at *school B* during their work with Flash used the first activity of the falling ball to explore the program. They looked at what happened on the screen and they saw the creation of the table and the displacement graph (fig. 6.1). They interpreted the graph successively as (*i*) the height of the ball, (*ii*) the distance to the bottom of the screen, and (*iii*) the distance to the top of it.

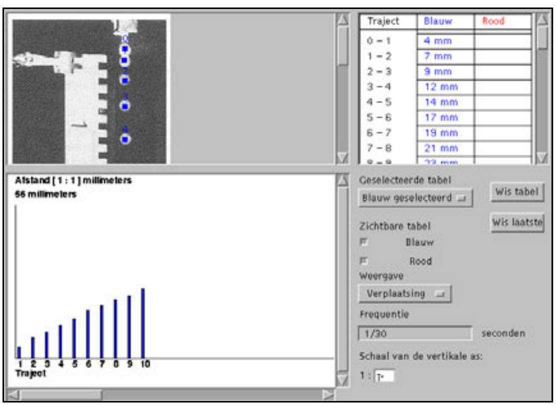


figure 6.1 Flash and the falling ball

They did not appear to have noticed that the distances between the sequential positions were on the graph. They did not check their earlier predictions, but completed this task and stated they 'got it', probably meaning that they understood how to operate the program.
After this they moved on to the activity about the stick. First they followed the middle of the stick (with blue) and then the end (with red)[1]. They initially interpreted the graph once again as a description of the distance to the ground. One student explained to the other that this was the reason for the waves on the graph. Then they realised that their interpretation was incorrect because the waves did not correspond with the motions on the photograph (fig. 6.2).

1. Coloured screen dumps can be found in the appendix *Computer tools*.

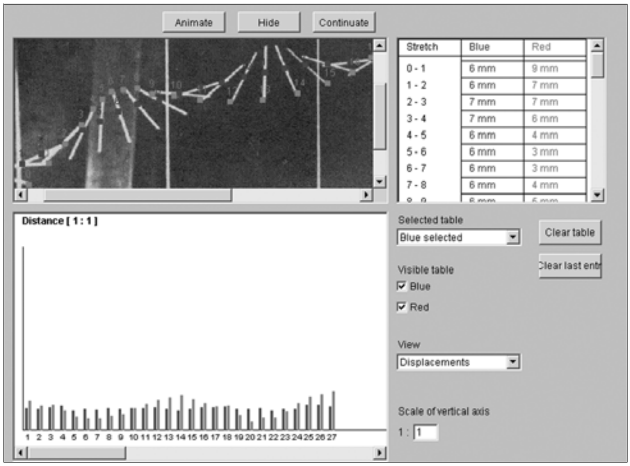


figure 6.2 Flash and the rotating stick

The observer asked them what the graph could mean in that case. By comparing the bars on the graph with positions on the photograph, they related the bars with the distances between the sequential click points, i.e. the displacements. During the discussion that followed, the language about velocity became related to characteristics of the two graphs.

1 Jan: That’s right, then, hey. What does it mean then, that the red is wavy?

2 Observer: That’s the distance between the dots.

3 Ayla: That sometimes the ends go faster and sometimes slower.

4 Observer: Yes. [turns to Jan] Do you see that too?

5 Jan: Yes.

6 Observer: Yes, that’s what you really see on the photo, that - can you see that on the photo? That the end sometimes goes faster? … How can you see that on the photo?

7 Ayla: Then there’s less space between two sticks. I mean, between two photos.

8 Observer: If it goes faster?

9 Ayla: When it goes slower. [Jan confirms this.]

10 Observer: ... Can you tell, from this table, if the centre moves further than the end in total?

11 Ayla: No... the red one does. [the endpoints of the stick]

12 Observer: Why do you say that?

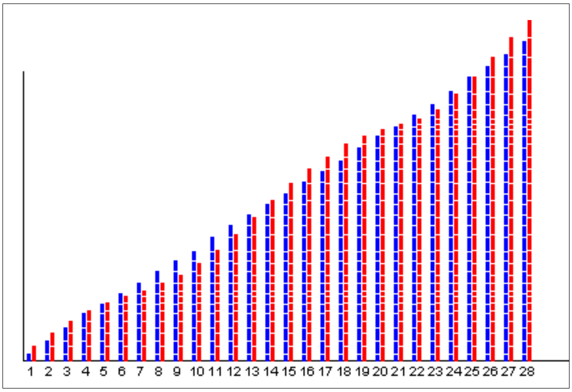


figure 6.3 Distance-travelled graphs of the rotating stick

13 Ayla: Why do I say that? Well, go down one time interval. [Jan moves the mouse to the graph with displacements under the photo]. The red trajectory, that's more if you add it all up than the blue one. [She changes the displacement graph to a graph of total distance travelled (fig. 6.3)].

14 Jan: It doesn't get any higher... [he points to the blue one]

15 Ayla: The red one has travelled further. [Observer and Jan confirm this]

16 Observer: So your estimate was correct. Do you understand what's happened now?

17 Jan: Yes, yes. It added all these things together.

18 Observer: What you just saw was that the red fluctuated around the blue and that the blue was fairly constant. How can you see that the blue is fairly constant here, for example?

19 Jan: Because if you drew a line, it would be straight. [Ayla makes a straight line on the screen.]

Apparently, the iconic interpretation of graphs is also very stubborn in this case. While reasoning about the graph, the students first used a trace graph on the photograph. However, when asked about a possible difference in the distance travelled between the middle of the stick and the end of the stick, they limited themselves to the displacement graph. Patterns in this graph were linked to the description of velocity: 'sometimes faster, sometimes slower' and 'less space corresponds with slower'. Moreover, the straight progression of the tops of the discrete distance-travelled graph became linked with the constant forward displacements of the middle of the stick (shown by making a linear gesture).
The two girls at *school A* went through a similar process as the two students described above. Initially, they tended to interpret the graph in terms of the path fol-

lowed, and later in terms of displacements and velocity. Floor (one of the girls) regularly used her fingers to describe displacements. The displacements became larger and she then talked and gestured about differences between sequential positions that became larger and about a velocity that increased (fig. 6.4).

figure 6.4 Displacements on the photograph and on the graph

The same linking of language between velocity and graphic characteristics was also seen with the two students at *school B,* who were observed during an activity about the cheetah and the zebra ('Would the cheetah still catch up with the zebra if it started 10 seconds later?'), they found the answer with Flash. The remark in line 11 was – at least implicitly – going towards instantaneous velocity.

1 Ayla: Yes, the cheetah would still catch up.

2 Observer: Really? How did you reach that conclusion?

3 Ayla: Uhhm, you put two here together and then you can see here that they're both equal.

4 Observer: Yes. And which of the two graphs is that?

5 Jan + Ayla: That's the total distance travelled. [They point at the left-hand graph in fig. 6.5.]

6 Observer: Oh yes. So why did you choose the one for the total distance?

7 Jan: Because it's the total distance that they cover and then you can…

8 Ayla: Then you can see if they catch up with each other.

9 Observer: And can't you see that in the other graph? [The right-hand graph in fig. 6.5.] On that one you can also see that the red catches up with blue, can't you?

10 Jan: Yes, but...

11 Ayla: Yes, but that's at just one moment. That only means that it's going faster at that moment, but not that it'll catch up with the zebra.

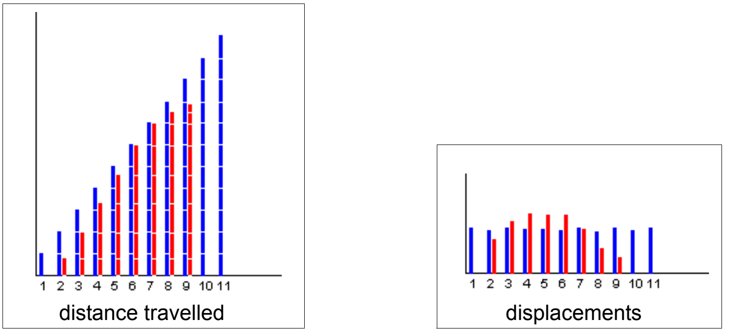


figure 6.5 Flash graphs of the cheetah and zebra activity

The discussion of the computer lesson at *school B* took place two lessons after the lesson itself (during the intervening lesson, a test on the previous chapter was discussed). The teacher asked if the students still knew which graphs were present in the program.

1 Teacher: You saw two types of graphs...

2 [silence]

3 Students: Bar graph ... line graph.

4 [The teacher lets the students know that this is not what he meant.]

5 Teacher: You were showing two things in these graphs, what were they?

6 Student [now responding quickly]: Displacement and distance travelled. Another student: *x-t* and *v-t*. [The teacher addresses the remark of the first student.]

7 Teacher: Does one graph have an advantage over the other?

8 [No one speaks.]

9 Teacher: In a specific situation, does the cheetah catch the zebra for example?

10 Student: The distance travelled.

11 Teacher: Why?

12 Student: Then you can see that he really catches up with him.

13 Teacher: And the displacement graph?

14 Student: That something goes faster.

15 Teacher: Good. You have picked up on this aspect very well. Check your sums and then go further. We will talk about this later in class.

The teacher was satisfied when the students answered his questions correctly and there were no further remarks. However, the silences during the discussion indicated that the students did not understand his questions immediately. Moreover, it is unclear whether all the students actually understood the difference between the two graphs. Neither the relation between the two graphs nor the specific discrete characteristics were discussed. It appeared that the students were attempting to use their knowledge from their physics lessons. At least one student explained the link with the terminology that is customary in physics (line 6 in the protocol above). It was unclear whether they were simply trying to satisfy the teacher's expectations or were actually thinking along with the teacher.

At *school B* the teacher discussed the activity on the extrapolation by asking: to what extent can you extrapolate a discrete graph from one point in time? The question was how the graph would appear if the velocity no longer changed after 25 seconds. The teacher drew both graphs on the blackboard and asked the class where the distance travelled graph should go from this point.

1 Gwen: Uhh, horizontal.

2 Student: But then it would be standing still, wouldn't it?

3 Student: Draw a line through it. [draws a sloping line in the air]

4 The teacher drew a sloping line through the last peaks on the graph.

5 Teacher: So there are two answers. Which one is correct?

6 Chris: I think that horizontal is also correct. That's a constant velocity, isn't it?

7 Student: But it's still not the same velocity as at 25?

8 The teacher again indicated the horizontal continuation [places another mark on the blackboard]. Is this possible?

9 Student: But it can't just suddenly stop, can it?

10 Teacher: How can we get there? Would it have a velocity of 0 after 25 seconds?

11 Student: Yes. Then it slowed down very quickly.

12 Teacher: So it is possible. You can imagine a situation where that would be possible. Can you also imagine a situation where the line would just continue on?

13 Student: Then you would use constant velocity.

14 Teacher: And something in between, is that also possible?

15 Student: Yes. If it slowed down a little before the 25.

16 Teacher: So, we actually have to conclude that at this moment [points to the board at 25 seconds] we do not know the velocity from this graph. We can imagine it, within certain limitations that we can think of. And then there are many possibilities (fig. 6.6).

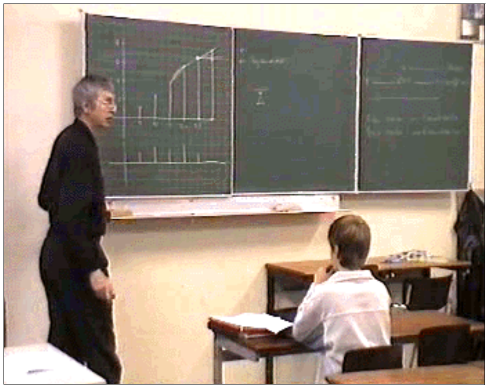


figure 6.6 How should the graph be continued?

This discussion concerned possible instantaneous velocities based on data on the graph. The students arrived at various possibilities. Gwen said ‘horizontal’ (line 1). However, from her notebook it turned out that she actually meant a horizontal displacement graph (fig. 6.7).

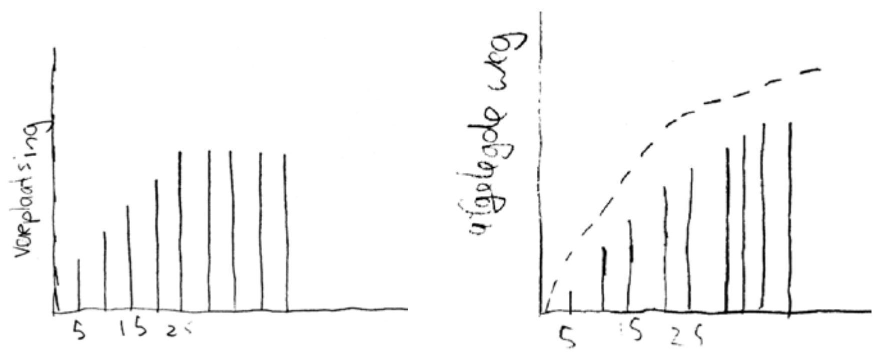


figure 6.7 Gwen’s displacement graph (left) and distance travelled graph (right)

The student’s comment on the left-hand displacement graph was ‘The motion becomes constant’, and on the right-hand graph ‘The distance travelled increases regularly’. One student interpreted Gwen’s remark about ‘horizontal’ on the distance-travelled graph as standing still and suggested a sloping, linearly increasing graph. Gwen did not correct this during the class discussion; nor did the teacher ask if this was what she meant. The teacher summarised the uncertainty.
The topic of this discussion was a graph. The students and the teacher discussed this graph in graphic terms such as sloping and horizontal, but at the same time they were discussing constant velocity, standing still, and moving faster or slower.

The concept of velocity took shape in the students' language simultaneously with the characteristics of graphs. Although these students had already taken the physics course, it turned out that these activities were not trivial. Apparent slopes on discrete distance-travelled graphs signify the value of displacements in the accompanying time intervals. This reasoning created building blocks that will later be needed to interpret the difference quotient. Moreover, the idea was established that average velocities can be calculated and that instantaneous velocity can only be approximated from this data.

At *school A*, the teacher took more time for the class discussion because they did not do the computer lesson with Flash. The teacher did not begin the class discussion as a direct result of an activity, but he first looked back at the task about the hurricane Olivia (page 106). He once more showed the students the map of the time series of the hurricane, asking them how they would now tackle this task. A discussion started about the usefulness of the two types of graphs with this problem.

1 Teacher: We tried to predict the time and location of the hurricane. But what we've done until now, couldn't we do it better? Could you tackle this in a different, perhaps handier, fashion? Previously we more or less guessed where the hurricane would end up. We lacked data and mathematical material.

2 Martin: I think that you could draw a graph of the distance travelled and that you could then place a horizontal line at the point where it, uh, hits the coast.

3 The teacher walked to the blackboard: So, you say a... [also writes on the board] distance-travelled graph. Do you mean a graph of the total distance travelled or ...?

4 Martin: Total.

5 Teacher: I'll sketch that graph, then you get something like this. And then? He has drawn a parabolic continuous graph.

6 Martin: And then you calculate the distance from the starting point to the coast.

7 Mathias: Maybe you could also put the accelerations, uh, the displacements on a graph and map them out and then calculate how much influence the …

As a result of this discussion, the two graphs were again placed in the context of understanding change. The teacher discussed the advantages and disadvantages of both graphs. Unfortunately, he drew a continuous graph as if it went through the peaks of the discrete graphs. We had seen the students do this as well, but thought it was better to not yet discuss its meaning. Only Martin and Mathias contributed to this discussion; both were good students. The other students listened and took notes. After this the teacher had the students work on the extrapolation task. He announced that he would discuss this activity at the end of the lesson.

The discussion of the extrapolation activity had the same elements as the discussion at *school B*. In this case, however, the teacher did not focus the attention of the class as much on the uncertainty about what happened after 25 seconds. The teacher guided the discussion towards the question of how to determine the velocity after 25 seconds? Inge was the first to respond with a reference to the displacements.

1 Inge: The bars will stay the same height as the last one.

2 Teacher: So you assume that the velocity will stay the same, which is …?

3 Inge: One metre per 5 seconds.

4 Teacher: Has the velocity always been the same?

5 Various students: We don't know that; that's possible; no.

6 Teacher: How could you determine the velocity more precisely at that instant?

7 [Inge has no idea. No one responds.] Teacher: Mathias?

8 Mathias: Use smaller time intervals.

9 Teacher: Good, measure smaller intervals. He sketched on the blackboard.

10 Lisanne: Yeah, ok, but you don't have that information. How do you know that?

11 Teacher: Yes, I'm just making a sketch.

12 Lisanne mumbled: But that's not any more precise… you couldn't think that up yourself, you have to get that information some place, don't you?

The teacher discussed this task at the end of the lesson, where he also had a substantive aim: measuring more often is more precise, but yields very small displacements. A conclusion was that for approximating the velocity at one instant with distance-travelled graphs or displacement graphs, the smallest possible steps are required. Lisanne apparently understood the method, but for her this was not an option in this context (line 10). As an aside, the horizontal axis was already presented in the activity as a continuous time axis, while this could not have been the case.
The question was whether the students were ready to use continuous models after this preparation, although the students had now experienced the limitations of discrete models. This provided a stimulus for making a distinction between average velocity and instantaneous velocity. The latter can possibly be approximated, but can never be precisely determined using measurements of displacements.

6.1.3 Introduction of a continuous model: Galileo and free fall

This unit comprised two lessons on the transition to continuous models in the context of Galileo's hypothetical continuous model for free fall. The central questions for the students during these lessons were: what does his hypothesis mean, how can it be tested, and how can it be used to make predictions?
In the previous lesson, there was no explicit discussion of limitations of discrete graphs and of the possibilities of continuous models for predicting change. Unfortunately, due to long-term planning at both schools, there was little time to spend on

an extra review of several activities. Together with the teachers, we decided to begin the lesson with a class discussion about Galileo's problem. As a result, not all students had the opportunity to think about the consequences of the proportionality between falling time and falling velocity taking. We therefore had less understanding of whether or not the sequel to the lesson linked up with their ideas, but nevertheless this appeared to be the only option for making progress.
The first activities concerned working with a hypothetical progression of displacements that increase linearly; then a link was made with 'piecewise constant average velocities' that increase linearly. After this, the constant of proportionality between falling velocity and falling time of a free fall from the Tower of Pisa became a topic for discussion (see page 111) In any case, the teacher should discuss this task with the class.
The research question for this lesson was whether it was possible to make clear to the students the significance and the consequences of the transition from discrete to continuous models. We studied whether the students used discrete tools for approximating a continuous model, and if they did so, how they used these tools and how they gained insight into the characteristics of a continuous model and the relation with discrete models. We assumed that the students would use vertical and horizontal intervals during the interpretations of continuous graphs. A choice of intervals which would enable them to build their reasoning upon the discrete models. Moreover, these activities should strengthen their insight into the meaning of instantaneous velocity and the relation with average velocity.

Results

Following the class's introduction to Galileo's problem, the students started with an activity about free fall from the Tower of Pisa. Hardly any of the students were able to complete this task independently at home. They entered the classroom rather irritated. We had underestimated the complexity of the task; during the class discussions it turned out that this was due to the large number of steps that the students were expected to complete.
It was striking that the students contributed actively to the class discussions. The use of the 'middle' displacement was unexpectedly strong (the distance travelled with displacements that increase at a constant rate is equal to the distance travelled with constant displacements that have the value of the middle displacement). At both schools, the students independently arrived at this principle during the discussion. This insight supports the idea that average velocity is indeed an average of *all* velocities and not just a quotient of two intervals.
The type of reasoning used by the students is similar to what actually happened during the historical development. Calculating with displacements made it possible to think about change in velocity. Oresme was concerned about potential displace-

ments and not about actual displacements, but nevertheless this was an idea that the students came up with themselves.
This observation confirmed our assumption that reasoning with displacement graphs works well for the students in this situation. Although we have some indications for the supporting role of the computer program Flash, we cannot say whether or not Flash played an essential role in this process. We ascertained too few differences between *school A* and *school B* to draw any conclusions. The same applies to the previously acquired knowledge of kinematics of the students at *school B*. For that matter, we did see that several students at *school B* tried to apply formulas for the free fall task, although mostly these attempts were futile.
Unfortunately, we observed hardly any students making a contribution to the link with the continuous model. The distinction between a pattern in discrete displacements and a continuous change in velocity appeared to be difficult for the students to make. This is because the students often had already used 'velocity' when reasoning about the displacement graphs, and the peaks were linked with the continuous line. At the end of the class discussion, the teacher had the opportunity to problematise the relation between the 'middlemost displacement' and the 'average instantaneous velocity', but he did not make use of this opportunity.
We concluded that working with discrete graphs of displacements helped students focus on and think about an average displacement. However, the link with average velocity and the relation between these graphs and continuous *v-t* graphs turned out to be more complex than we had expected. The essential difference with regard to a continuous horizontal time axis was not problematised explicitly, but was experienced by the students in their work with the various graphs.

Illustrations of the results

The first illustration concerns Galileo's problem and the students attempts to solve this with discrete approximations. At *school B*, the teacher introduced the task about the Tower of Pisa by first drawing a graph with constant displacements on the blackboard.

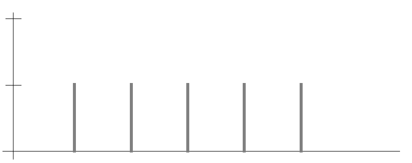


figure 6.8 Constant displacements adding up to the total falling distance

While doing this, he helped students choose the number of displacements (and therefore a Δt). His question was how would the graph look if the displacements increased evenly and the total falling distance remained the same (fig. 6.8).

1 Student: A sloping line.

2 Teacher: A sloping line… sloping how?

3 Student: Sloping to the right going up.

4 Teacher: Could we be a bit more precise? In the beginning the bars are therefore a bit smaller, and later on they become a bit larger. Could you now calculate exactly how big the bars should be?

5 Student: The middle bar is the average.

6 Teacher [points towards the student who made this comment]: The middle is the average…

7 Student: Then you draw the line through 0 to the end of the middle…

8 Teacher: And then continue the line, like this? Can we still check if this is correct? Or if the total displacement is correct?

9 How can you then see if the total displacement is correct?"

10 Student: Add everything up (fig. 6.9).

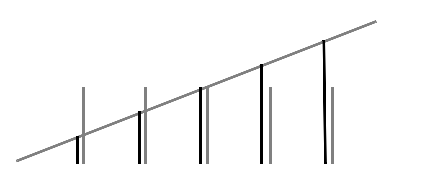


figure 6.9 Constantly increasing displacements

With the displacements, the students could see that what was lacking in displacements at the beginning with respect to the average equals the leftovers at the end. This extrapolation of the line from the origin along the 'middlemost' displacement also took place at *school A*. It turned out to be a natural form of reasoning (in the mathematics lesson) for virtually all the students. We also ascertained during this that the teacher had taken an important role in guiding the discussion. The teacher asked what was going on with the velocity. A student answered that it got faster and faster. Going faster is displayed with increasing displacements.

1 Teacher: What is actually going on with the velocity?

2 Inge: It's going faster and faster.

3 Teacher: How?

4 Student: Linearly.

5 Teacher: Exactly, the velocity increases linearly with time. What do you do with this now?

6 Mathias: Can I draw it?

7 Teacher: Tell me how to do it.

8 Mathias: Draw a sloping line going up through the middle one.

9 [The teacher draws this (see fig. 6.10).]

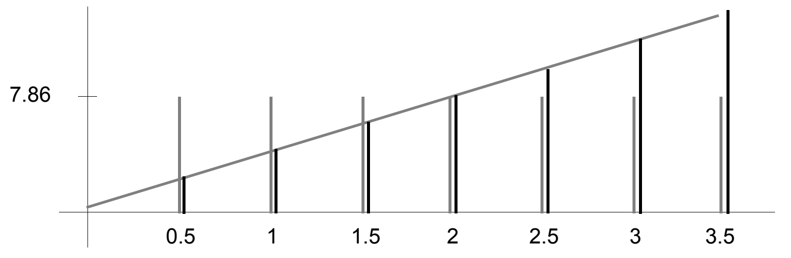


figure 6.10 Teacher draws a graph with units on the axes

10 Teacher: Why is what he says correct?

11 Student: If you add up all the displacements, that's also 55 metres.

12 Teacher: Can anybody explain this in a different way?

13 Jochem: At the halfway point, you are going at the average velocity.

The idea of 'adding up all displacements' for figuring out that both motions have covered the same distance, followed from the remark in line 11. However, there was hardly any emphasis placed on the distinction between displacements and velocities. The teacher asked his questions while regularly using the word 'velocity' without talking about displacements.

As a result of the graph of displacements, another discussion took place between ourselves and the teacher about the 'intersection point' of the constant graph and the linearly increasing discrete graph. Shouldn't these graphs actually intersect each other after 1.75 seconds? At this point, the previous, unclear transition to a continuous time axis began to cause difficulties. In fact, a continuous axis had not yet been addressed at all. The graph shows that during the interval from 1.5 to 2 seconds, the same distance is travelled in both motions. The two motions have the same velocity at the halfway point of that interval.

The teacher then proposed that the transition to a graph of average velocities must be made to find the constant of proportionality. After all, Galileo formulated a hypothetical proportionality between falling velocity and falling time. The falling time from a height of 55 metres was 3.5 seconds. The question was therefore: is it possible to determine the constant of proportionality between falling time and velocity?

In retrospection, we realised that the teacher could have asked the students about the constant at this point. For example: could you say something about the constant from this displacement graph? On this basis, the distinction between a graph of displacements and one of average velocities could have been brought up for discussion.
A student at *school A* asked once again if the stone actually fell from the Tower of Pisa in this way. The teacher then emphasised that we do not know this, but that we are describing the free fall of the stone according to Galileo's model.

1 Jochem: Are you sure about this? Did it actually fall this way?

2 Teacher: No, this is Galileo's assumption.

3 Teacher: OK, so it would have probably happened this way. We must get to the formula *v = constant * t*. We're almost there.

4 Student: You know the displacement and the time, so you can calculate it.

5 Marianne: For every 0.5 s, you add 1.96...? [7.86 / 4 = 1.96]

6 [The teacher confirms this and points to the units]: So that is 1.96/0.5 = 3.92 m/s [and he finishes the graph (fig. 6.11)].

7 Teacher: Now we get what we wanted to know, a straight line. What is the slope?

8 Inge: 7.86. Is the average velocity always the constant?

9 Teacher: Is it a coincidence that we already had that 7.86 previously? What is that exactly? The real value is 9.81. This is what you find if you do these experiments with falling objects on the earth. On the moon you get a totally different value. The fact that this 7.86 is a bit lower is probably due to friction.

In this discussion, we wanted to accomplish too much in a short time. Inge's remark showed that there was some confusion about velocity and displacement. The value 7.86 was the average displacement during a time interval of 0.5 seconds. However, the teacher did not address this, but discussed the slope of the graph and the relation with an acceleration constant during free fall.

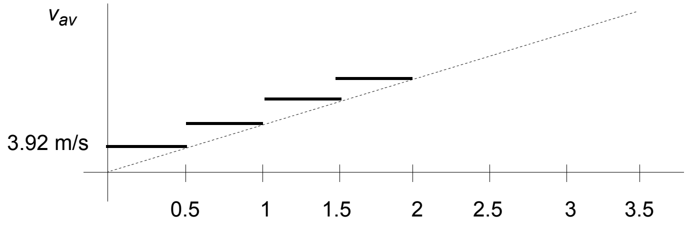


figure 6.11 Graph of average velocity with continuous time-axis

A cause of Inge's confusion (line 8) could be that the students had worked on an earlier occasion with displacements in time intervals of 1 second. In that case, the value of the displacement equals the value of the average velocity at the relevant interval. The question is whether or not this class discussion had an adequate effect. Did the students understand the transition from discrete graphs to continuous models by using bar graph approximations as an interim step? Does the bar graph approximation derive its meaning from the discrete graphs of displacements? This could be shown by the way in which they dealt with the following activities. We determined if these students could still use the mathematical and physical concepts flexibly and if they could base the meaning of sequential steps on what they had learned in the above lesson.

6.1.4 Further development of the continuous model

The work in the previous lesson on the problem of free fall was an initial impulse for understanding the relation between displacements and a continuous change in velocity. In this unit (comprising one lesson), continuous velocity graphs were the starting point for questions about the distance travelled. The homework for this lesson was an exercise where students were led step-by-step through Galileo's reasoning. The idea was that the students approximate velocity-time graphs with pieces of constant velocity, resulting in a bar graph approximation with which they can calculate the displacements. Adding the displacements together gives the answer to the task: the distance travelled. This is a well-known procedure in the discrete case, so that we expected that the average displacement would be linked with average velocity. Moreover, this should offer the students the possibility of interpreting average velocity as the average of all assumed values, and not, for example, as the average of the initial and terminal velocity.

The calculation of a sum of displacements appears to correspond with the calculation of the area under the v-t graph. This is a first step in the use of Δv and Δt as elements to structure a graph for approximating a not-represented quantity. It anticipates the calculation of velocities (slopes) with an s-t graph. The question that concerned us was whether or not this context was sufficiently intriguing to actually get students involved in Galileo's problem.

From a modelling perspective, the bar graph approximation is an intermediate model between displacements and reasoning with continuously changing velocity-time graphs. Could the students now work with continuous velocity-time models in a flexible and meaningful fashion, and did they understand that the distance travelled could be determined with a bar graph approximation of average velocities? Is the more general issue still sufficiently in the picture to give meaning to the situation and to indicate to the students how they should proceed?

We expected that during this part of the learning trajectory, the students at *school B* would work differently to those at *school A*. A number of tasks concerned uniformly accelerated motion for which the students at *school B* would probably show less variation in their answers and would make more immediate use of the accompanying formulas. Because they did not reason as much using the bar graph approximation method, we thought perhaps they would have more difficulty with the task about the cyclist with a non-uniformly varying velocity graph (see page 113).

Results

At *school A*, all the students took part in the discussion of the task about Galileo's reasoning. Many students had completed this task as homework and this resulted in a lively discussion after the previous, rather strenuous, lesson. During the discussion, the students unexpectedly did not indicate that they could improve the approximation by dividing the graph into narrower bars. Our expectation was probably based too much on our overview of the mathematical intention. Our perception of these activities was framed by our knowledge of Riemann sums for approximating areas and how these approximations can be refined. The students, however, saw many more possibilities. Their contributions resulted in a discussion of the geometrical aspects of the graph and the kinematic aspects of the situation described. Their strategy was not framed by a bar graph approach of area.

The questions resulted in a large variety of solution methods from the students. This led to discussions about differences and similarities, which again emphasised the origin of the area method and the relation with the bar graph method and displacements. The compensation strategy for areas was present, however. This was probably because the students were prepared for the strategy of compensating around an average (middlemost value) by the discrete graphs of displacements. From the available data, we could not determine if the students could actually explain this strategy using a bar graph approximation where the area of every bar signifies a displacement. Students made little use of notations such as Δs and Δt. They were concerned primarily with concrete situations in which they could quantify the intervals immediately and use them correctly for calculating and for geometrical reasoning.

In the following activities, the students at *school A* used the area method flexibly, and they made their own modifications with the constant accelerations. The task about the cyclist was not discussed, unfortunately. When we examined the students' worksheets, we found mostly good answers. One of the students had made notes next to the graph on the instruction sheet, which gave us an idea of his strategy. He appeared to be approximating the distance travelled in an insightful fashion using a constant velocity. We were unable to ascertain this strategy in other students because they did not hand in any copies of the instruction sheet.

The students then worked industriously on the tasks in this unit. These tasks are variants of constant accelerations and were intended to ensure that students utilised, and became familiar with, the strategies for constantly increasing velocity. This was something that had happened insufficiently so far. Our impression was that this improved the atmosphere in the class at this time.

At *school B*, the teacher first discussed the distinction between displacements having the relation 1 : 3 : 5 : 7 : ... and displacements having the relation 1 : 2 : 3 : 4 : ... He showed how it was possible to derive a graph of average velocities from the first case that approximated a uniformly accelerating graph through the origin, while in the second case it did not. At that point, a discussion began that took up almost the entire lesson. The activities about the bar graph approximations were not discussed.
On the students' worksheets at *school B*, we could see what they had done on the task about the bar graph approximation and several following tasks. In accordance with our expectation, we found good answers to the questions concerning constant acceleration. These students improved their approximation by using smaller intervals, and they regularly used formulas such as $x = v \cdot t$. Unfortunately, none of the students got as far as the task about the cyclist. The question is whether or not they understood that average velocity cannot always be calculated with $(v_{initial} + v_{terminal}) / 2$.
We were of the opinion that the students contributed to giving meaning to and reasoning with the area under a *v-t* graph (perhaps due to the progression from displacement graphs). The idea of compensating from an average appeared to still function here. As yet there was no confusion caused by calculating average velocity by dividing intervals of various dimensions. None of the students came up with the idea of dividing Δv by Δt to calculate the average velocity. Once the students were working with areas, we saw hardly any link with discrete displacement graphs being made and we noticed that in their reasoning there was only little attention for the kinematic meaning. Perhaps we had emphasised this insufficiently during this phase of the material. The teachers discussed the mathematical aspects of the approximation strategy and spent little time reflecting on students' problems or approaches regarding the wider issue of understanding change.

Illustrations for the results

We start with an illustration of the reasoning of students at *school A* with the area method. The object of study for the students was a linearly increasing velocity graph. The question was: can you use such a graph to predict the distance travelled? During the class discussion, it seemed to become a technical situation. However, it turned out that many students had already done their homework independently. To determine the distance travelled for a given *v-t* graph, most students used the bar graph approximation (fig. 6.12).They answered the teacher's question about improving the approximation of distance travelled during constant acceleration:

> Suzanne: Calculate the area of the triangle and add it to the bar.
>
> Mathias: But can't you also divide the terminal velocity by 2 and multiply that by the time?
>
> Another student: Place the middle of the bar exactly on the graph.

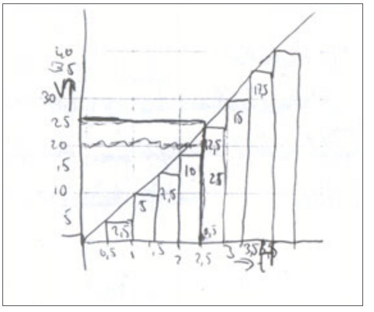


figure 6.12 Student's bar approximation of a velocity graph

They answered the teacher's question about improving the approximation of distance travelled during constant acceleration:

> Suzanne: Calculate the area of the triangle and add it to the bar.
>
> Mathias: But can't you also divide the terminal velocity by 2 and multiply that by the time?
>
> Another student: Place the middle of the bar exactly on the graph.

None of the students referred to using smaller intervals, which we had expected, but working with areas did emerge, apparently. Moreover, Mathias suggested a solution that probably used a constant 'average' velocity. After discussing the possibilities the teacher gave the students 5 minutes to complete the next exercise on determining distance travelled during constant acceleration. The students worked industriously; they eagerly applied a technique that they thought they had mastered. This is a characteristic aspect of practising to acquire confidence in one's own ability. Various methods emerged during the discussion of this task:

- Ben first filled in 10 and then obtained a velocity of 25 m/s. After this he calculated the area of the triangle (20 x 10 / 2 = 100 m) and the rectangle (5 x 10); adding these together yielded 150 m.
- Floor initially had 10 x 25 = 250 and then subtracted the small triangular part, 250 – 100 = 150.
- Mathias calculated a sum of displacements: 6 + 8 + 10 + 12 + ... + 24 = 150.
- Wendy used a compensation strategy for areas: "The graph does not begin at 0 but at 5. So you add 5 at 25, this gives you 30. From 0 to 30 is the same area: $\frac{1}{2}$ x 10 x 30 = 150" (see fig. 6.13).

The solutions by Ben, Floor, Mathias and Wendy showed that they were able to use the area method flexibly in various ways. Only Mathias returned to a bar graph approach that led to a sum of displacements. The teacher wrote all the solutions on t he blackboard and expressed his appreciation of the students' creativity.

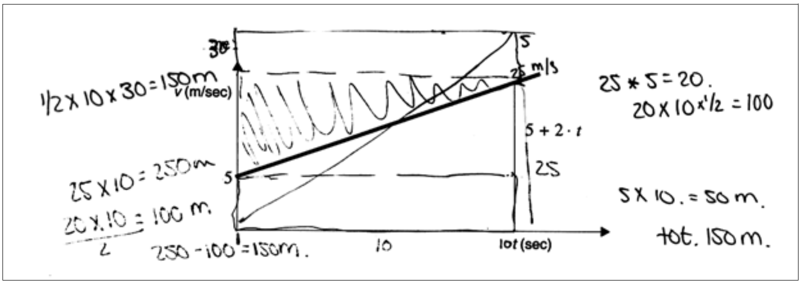


figure 6.13 Wendy's solution by compensating areas

Then the teacher assigned homework for the following lesson, and the students went to work. Most of the students had a good answer for the task about the cyclist, but we could not determine their strategy. With Mathias' answer, we observed drawings in the graph by which he again appeared to use constant velocities (fig. 6.14).

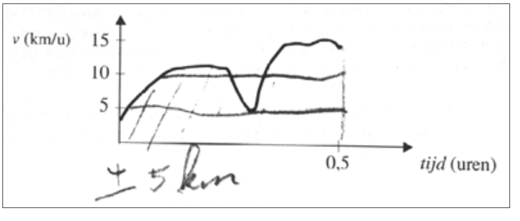


figure 6.14 Mathias' approximation of distance travelled with a velocity graph

Unfortunately, the above activity was not discussed in class at *school B*. From the worksheets of several students, it appeared that these tasks (except that of the cyclist) did not present any difficulties. The following solution to the question about what the exact distance travelled is according to Galileo's reasoning, is representative for the use of formulas by students at *school B*. This student had first calculated the average velocity and used this to determine the total distance travelled after five seconds:

$\bar{v} = 25 \qquad \bar{v} \cdot t = x \qquad 25 \cdot 5 = 125$

This strategy is in contrast with the bar approximations used in *school A*.

6.1.5 Evoking the need to determine instantaneous velocity

This unit comprised three lessons. A central problem was posed in the context of a car trip made by Mr Bommel, who is suspected of committing a speeding violation (see page 115). A distance-travelled (*s-t)* graph of the trip was provided.

In this context, the graph was intended to create opportunities for the students to use Δs and Δt and give meaning to the slope of an *s-t* graph and the difference quotient $\Delta s/\Delta t$. In line with the discrete reasoning, Mr Bommel's velocity at a specific moment is equal to the velocity that he would have if his velocity would no longer change from that moment. In the *s-t* graph, this results in a tangent-like continuation which the students had also seen with the discrete graph. With the quotient $\Delta s/\Delta t$ they can approximate this instantaneous velocity if they keep Δt small. The idea was that the computer tool Slope (see page 116) would help the students in this process. Then they worked on various situations, also using the graphing calculator. Research questions in this unit were:

- Are students motivated by the context problem, and do they have the feeling that reasoning with the *s-t* graph and horizontal and vertical intervals is helpful for solving the problem?
- Do these intervals signify time intervals and displacements, and do the students relate linear pieces of an *s-t* graph to the concept of constant velocity, which builds upon the characteristics of discrete distance-travelled graphs?
- Does the computer program Slope afford all students to understand and use approximations with small intervals of instantaneous change at one point on the graph by using a graphic and dynamic representation of a difference quotient?

The preparation during the previous lesson did not proceed as smoothly as we had hoped. The computer lesson with Flash at *school A* did not take place, the usefulness and aim of continuous models was only discussed incidentally, and little time was spent on working with intervals during the previous unit. Nevertheless, we expected that the students would be able to see the difference between calculating average velocities using a *v-t* graph and using a *s-t* graph. With a *v-t* graph this was based on the association with displacements and a compensation strategy around an average value, and with a *s-t* graph we expected that working with intervals (or quotients of intervals) would be supported by the problem of Mr Bommel and the previous use of 'linearity' in discrete distance-travelled graphs.

Results

The Mr Bommel activity supported students in starting to reason with intervals and to use a difference quotient. At both schools, students worked on this task during the lesson. Nearly all the students worked in pairs and discussed the time interval for which it is relevant to calculate the average velocity according to a quotient of a displacement and the accompanying time interval. The students stated that on a nearly straight piece of the graph, the velocity during the accompanying time interval could be approximated. They interpreted the fluctuating parts of the graph as situations where Mr Bommel's velocity changed a great deal and you could therefore only calculate an average velocity. Students did draw tangent-like continuations there as a response to the question about the extrapolation of the graph if the velocity at that point would remain the same. This work provided ingredients for a class discussion about the problem of approximating instantaneous velocity but, unfortunately, there was no time to hold the discussion.

The transition from the Mr Bommel task to a continuous model with a given formula then proceeded too quickly. The idea of a linear continuation was prepared by using a discrete model, such as the Continue-button in Flash, but did not emerge here sufficiently as a means for determining an instantaneous velocity linked to the approximation of a local slope. A problem based on this determination should have been discussed extensively with the class. As a result, a number of students did not have a good idea of the usefulness of the tools in Slope.

It turned out that during the computer lesson, many students did little substantive work and did not know exactly what they were looking for. They attempted to follow the questions and to determine what was expected of them regarding the program. It was striking that there were no rapid links made with the previous lessons. The teacher and the observer helped many students by explaining the aim and usefulness of the program. It was surprising how most students ultimately began working eagerly on the game in the computer program.

Besides our hope that students would independently recognise the problem for which the program offered solutions, we expected that the graphic-dynamic image in Slope would offer support for approximating a local slope with a difference quotient of a chord. While working with the program, the students talked about the approximation process based on a 'blue triangle' that was used in the program for approximating a linear continuation in red. The difference quotient with the chord on the graph is drawn as a blue triangle[1]. The reference to the blue triangle, of which you can move two angles on top of the graph towards each other to approximate the instantaneous slope, turned out to be a functional framework of reference during the discussions in the following lessons.

In this way, a graphic-dynamic picture had formed while the students were working with Slope, which later supported the model of instantaneous change. The blue triangle was a shared concrete and dynamic image, with which the students referred to the underlying concepts of the model, i.e. the underlying limit process for approximating a slope with a difference quotient. At the end of this unit the students applied the technique of the difference quotient in several situations and learned how they could use the graphing calculator to accomplish this. Technical skills did not dominate this process. Students spoke about increases, the difference quotient, velocity and characteristics of the graphs in terms of motions. The meaning of the concepts appeared to be rooted in the description of motion and the work with the blue triangle in Slope. For that matter, during this lesson we saw no difference between *school B*

1. Coloured screen dumps can be found in the appendix *Computer tools*.

and *school A* resulting from a difference in kinematic background or the cancellation of the computer lesson with Flash.
During the final discussion, both teachers addressed the most important skills of the last few lessons. Unfortunately, there was not enough time to place everything in the light of descriptions and predictions of change. A reflection on the more overarching questions and problems did not take place. We concluded that the Mr Bommel context and the speeding violation problem supported productive reasoning for the students. They used the *s-t g*raph and horizontal and vertical intervals for solving the problems. These intervals signified time intervals and displacements, and the students related linear pieces of an *s-t g*raph to constant velocity. However, the computer program Slope did not afford all students to be more precise in the approximation of instantaneous change on a point on the graph.

Illustrations of the results

The work of Inge at *school A* can be used to illustrate the students' solutions for the task about Mr Bommel. Somewhat more neatly than most students, she drew her answers on one graph. It can be seen below how she indicated the progression of the graph if Mr Bommel drove at a constant velocity and how the graph would proceed if such a velocity no longer changed after six minutes. She drew a tangent-like continuation from $t = 6$ (fig. 6.15).

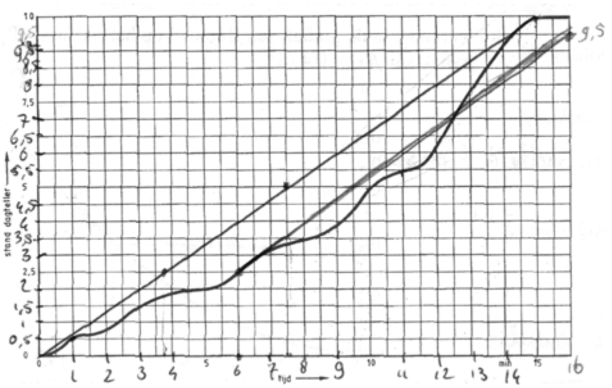


figure 6.15 Inge's drawings in the total distance travelled graph of Bommel

Next to the question about what his velocity would be if it no longer changed after six minutes, she wrote the following in her workbook:

> Yes, then he would go 7 km in 10 minutes, so that means he is travelling at 42 km/h at that point ...

The observations that students interpreted the Mr Bommel graph by correctly using average and instantaneous velocity is also illustrated by a discussion between two boys about Mr Bommel's speeding violation.

One boy calculated the average velocity over a somewhat longer interval (including a piece where he slowed down) than the other. The latter believed his answer was closer to the highest velocity. He calculated the 'velocity on a single piece'. During this discussion, they referred to characteristics of the graph ('he slows down here') and worked correctly with the intervals for displacement and time.
In the following task, a piece of the Mr Bommel graph is approximated with a graph made according to a formula. For the linear continuation in this task, Inge drew a line from a point that was somewhat too steep (fig. 6.16).

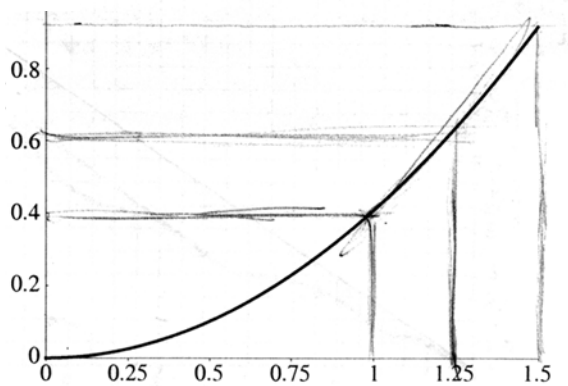


figure 6.16 Inge's linear continuation which is too steep

In her answers Inge did not write anything about the distinction between instantaneous and average velocity. She apparently did have an idea about the meaning of a linear continuation concerning the progression of the graph if the velocity no longer changed, but she did not yet have this idea in terms of tangents. We observed this with many students during the computer lesson with Slope. However, most students reasoned with lengths of intervals; with the signified for the signifiers Δs and Δt in the difference quotient, without using these signs yet.
After working on the Mr Bommel tasks, the students began using the computer program Slope. Immediately after this lesson, the teacher and observer discussed the experiences and the observer wrote a lesson report, from which – along with the video of two of the pairs (one pair at each school) – it appeared that the work of one pair was characteristic for the observations.
This pair, two girls at *school B*, appeared to struggle with the work, which was partly because it was difficult for them to drag the points in the computer program. They had also problems with the goal for which they used the computer tool. At first they wanted to obtain the slope of the red line (the guess for the linear continuation) instead of using the blue triangle (signifying the difference quotient) to get a picture of the linear continuation and then tune the red line to this value.

In the following transcription, one can see how this pair attempted to use the blue points to obtain the slope of the red and that they did not understand why the two points were on the parabola. They had already drawn the red line 'by eye' as a linear continuation of the parabola from $x = 0$ (fig. 6.17).

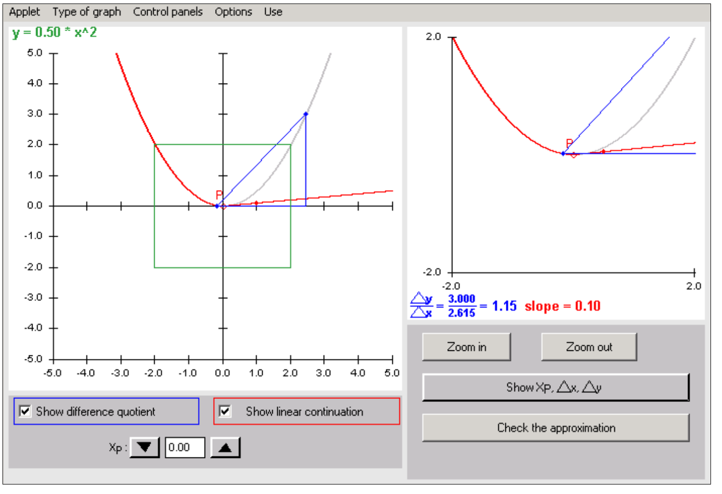


figure 6.17 Slope and the linear continuation from $x = 0$

1 Eve: So what is it that they want? Do they want you to move that blue point with that thing there? That round part, the top …

2 Eveline: What is the connection between the blue triangle and the linear continuation? One is always going parallel to this. Would that also happen if I did this? No. Hmm.

3 Eve: Can I, if you …[Takes the mouse.]

4 Eveline: But we should … look, here is something, here is a formula [She points to the division for the slope of blue triangle.]

5 Eve: That's the difference. That is the difference between that and that point, I think.

6 Eveline: Wait a minute, so we go … but look '= 1.15…' what do they mean by that?

The first thing one notices is the students' language: 'what do they want' and 'what do they mean by that'. They were trying to figure out what was expected of them. They hardly believed their own result. For them, the program was a black box for which they had to determine its purpose and operation. Their search process of trial and error was interrupted when the teacher passed. They then asked their teacher how it was supposed to be. He did not really ask them where they had got stuck, but told them that the two blue points were on the parabola and explained what they could do with them: “Put them close together and you can estimate the slope of the linear continuation”. Eveline repeated as follows:

7 Eveline: Really close together and then it indicates actually how it’s supposed to go further?

8 Teacher: Yes.

9 Eveline: So that wasn’t really difficult or anything.

10 Teacher: No, no, but it is an interesting discovery. It’s a very important discovery. You, ..., yes, now it looks kind of obvious …

11 Eveline: Yes, I could have thought that up myself if I had lived two centuries ago, I think. Only I wouldn’t really have got into it. But okay …

12 Teacher: You wouldn’t have tackled this problem.

13 Eveline: No, I don’t think it’s really a problem.

At this point the two students thought they understood how the approximation process and the blue triangle worked and they used them in the program for the games. They became interested, their language became more active, and they continued even after the bell rang:

14 Eveline: I don’t think you can do that, put them closer together. I think it’s just … see, you just can’t put it any closer together. Okay.

15 Eve: Well, it’ll just have to accept this.

16 Eveline: This approximation is good.

17 Eve: What is the approximation? 1.9? [writes it down] We should also write down what we do every time. With the other games. But let’s finish this first.

18 [...]

19 Eve: Yeah, that’s a good approximation.

20 Eveline: […] Yeah, I would actually like to have a test we could do like this.

21 Eve: Total score 25 [the maximum].

22 Eveline: But now we have to do the next game. Game, game … ummm, with all the tools?

In the next lesson at *school B*, the students referred independently to the dynamic image of the blue triangle and the program Slope. The teacher asked the students to go through the activity once again with the formula with which the graph approximated a piece of the Mr Bommel graph. The question was: could they estimate Mr Bommel’s velocity at a specific time using this formula? During this discussion the

function notation emerged for the first time with the calculation of a difference $y(x+0.001) - y(x)$ for determining the displacement Δy.
Until then, displacements were increases on an *s-t* graph, and were not associated with a difference. However, the way in which Heleen contributed to the discussion showed that she understood the notation (see next transcript). Perhaps she understood this partly due to the graphic dynamic image from working with Slope, although it is also possible that these students' previous knowledge of physics played a role.

1 Teacher: You've worked in the computer lab with the program. There is a reason for this. Open your books to exercise 46.

2 [The teacher writes $f(x) = 0.4 * x^2$ on the board.]

3 Teacher: To solve this you have to estimate a continuation at the point where $x = 1$. Does everyone have that in front of them? [The teacher drew the graph on the blackboard.]

4 Teacher: The central information is this piece of the graph from the Mr Bommel task and a formula that describes that graph. The problem is to understand what the instantaneous velocity is at $x = 1$.

5 Chris: I placed the formula on the graphing calculator and calculated the slope between an interval of 0.0001.

6 Teacher: That is probably a method that you remembered from last year. [Chris had repeated this year course.] Can you recommend something in this context?

7 Chris: Then you have to know what the difference is between the two points to extend the line, $y(x+0.001) - y(x)$ divided by 0.001.

8 Heleen: Wouldn't it be better to choose a point just before that? $y(x+0.001) - y(x-0.001)$. That seems more logical to me if you want to know the average.

9 Teacher: And why do you say that?

10 Heleen [points to the graph on the blackboard]: One point above and one below, draw a line between them and then calculate the slope. Actually just like with the computer.

11 [The teacher draws the two points in the graph, and writes on the board: $(y(x+0.001) - y(x- 0.001)) / 0.002$.]

12 Teacher: Arne, can you follow this?

13 Arne: There it is the average…

14 Teacher: The average velocity between two points is a good predictor for the instantaneous velocity. That's also what you did with the computer. That blue triangle is a good estimate. We conclude that this is a good estimate for the red line.

The teacher asked a question for an explanation by Heleen (line 9). In her answer, Heleen linked the average as the quotient of two intervals of different dimensions with this quotient as a measurement of the slope. The fact that she referred to the computer program here suggests that this activity led to this reasoning. With the computer program Slope, Heleen had access to a dynamic picture and a language with which she could discuss this approximation process. The teacher then concluded the discussion perhaps too quickly for the class by stating that this was a good method for approximating the slope of the red line. Nevertheless, this appeared to be a good time for this to happen. The students had in mind the geometric intervals on the graph and the limit process to which this notation referred. Heleen was an average student and we assumed that her remarks were representative for a large group in the class.

Towards the end of this unit, there was a task about a stone that is thrown straight upwards by a catapult. The height reached by the stone in metres can be approximated with the formula: $h(t) = 30t - 5t^2$. The students were asked to approximate instantaneous velocities and to draw a *v-t* graph of the motion of the stone. The teacher at *school A* first asked what the velocity was at $t = 0$:

1 Student: Zero.

2 Teacher: Is it zero? At $t = 0$ it shot straight up, so I think it's going very fast at $t = 0$…

3 Inge: From 0 to 1 it goes 25 upwards.

4 The teacher repeats this and continues: Yes, but that's the average velocity for that piece going from 0 to 1, but I'm asking for an instantaneous velocity …

5 Student: 30.

These three different answers led to a discussion about the instantaneous velocity at $t = 0$. All the students participated. The remark of Inge illustrated an approach which we observed more often during this lesson. Some students only calculated the vertical increase to denote a slope or change, in particular when they worked with a formula without a kinematical context. This is not surprising because the vertical increase signified the displacements in the discrete case. In these situations, reasoning in terms of the context of motion helped to involve the corresponding horizontal interval in the notions of slope and average or instantaneous change.

Then the teacher asked a student to describe the motion of the stone with the *v-t* graph. A student remarked that the velocity increased steadily while it was starting to fall (after 3 seconds), even though the graph continued to go downwards.

1 Mark: It's going slower and slower.

2 Teacher: And here [He points to the negative part.]

3 Mark: It's starting to fall.

4 Teacher: And… [He follows the graph.]

5 Mark: The velocity gets higher and higher.

6 Teacher: Inge, what is the velocity when it hits the ground?

7 Inge: 30 m/s.

8 Teacher: Yes, at $t = 6$ and then you get 30, -30 m/s.

9 Teacher: And where does it reach its highest point? How can you see that on this graph? [He points to the *v-t* graph.]

10 Inge: At 3. [The point of intersection with the horizontal time-axis.]

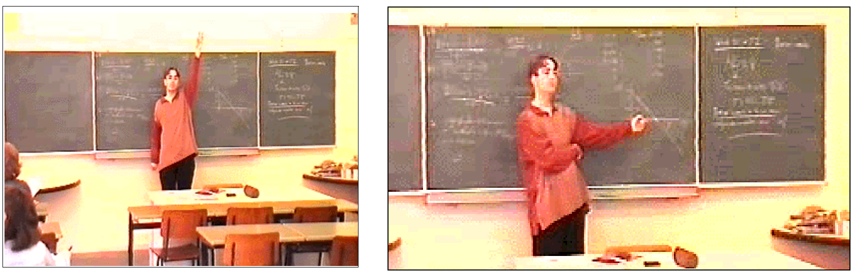


figure 6.18 The teacher discussing the trajectory of a thrown stone.

The discussion hardly addressed the technique of the difference quotient. The students talked about increases, the difference between average and instantaneous velocity, the interpretation of negative velocity, and characteristics of the *v-t* graph in terms of the motion of the stone. This illustrated how the concepts acquired their meaning in the description of the motion.

Then the teacher at *school A* asked the students to write down the most important parts of this chapter during the next 15 minutes. He would then discuss these points during the last few minutes of this last lesson. During this discussion, mathematical techniques and kinematic insights were reviewed, but the discussion was too brief and too fast to draw any conclusions about the whole class. It was striking that the central problem, understanding change, was not referred to here.

6.1.6 Final test

During the experiment, there was a great deal of attention at both schools for concept development and less attention (in comparison to the chapter from the book) for practising algorithmic skills. However, the experiment did not always proceed as planned.

Consequently, we wondered how the students would score on the written test, which they all took during a lesson, and specifically how their scores would relate to the more conceptual questions, on the one hand, and the more technical questions, on

the other. We had suggested a number of test questions to the teachers, who ultimately put their tests together independently.
The tests given at the two schools were different; this was due to the divergent time periods and because material from a previous chapter was also being tested at *school A*. A question about a data-based graph of the growth of a sunflower was included in both tests, although at *school B* the graph was accompanied with a formula. The test was taken by 33 students: 19 students at *school A* and 14 students at *school B*.
In the discussion of the test results below, we divided the questions into the following categories:

- questions in a mathematical context asking students to calculate $\Delta y/\Delta x$ on an interval with a given formula;
- questions asking students to approximate the slope at one point on the graph with a given mathematical formula (using $\Delta y/\Delta x$ or the graphing calculator);
- questions asking students to approximate/calculate average change in a context problem using the difference quotient (with a given formula or with a graph);
- questions asking students to approximate instantaneous change (velocity) using a difference quotient.

Results and illustrations
This unit begins with a brief discussion of two questions in a mathematical context. Students at *school B* were asked to calculate a difference quotient $\Delta y/\Delta x$ for a given mathematical formula. Ten students (out of 14) did this without any problem. The given formula was: $y = x^3 - x^2$. Several students became confused when determining Δy (also a difference). None of the students made this mistake systematically. Nevertheless, on these questions in a mathematical context, which focused directly on technical skills, several students at both *school A* and *school B* (7 out of 33) did not use the difference quotient to approximate a slope.
The students did well with the interpretation of the difference quotient as a measure for change in a context problem. There were also few difficulties with using units in this process. At both schools, there was a question about determining the instantaneous velocity with a given distance-time graph at a time when the velocity was reasonably constant. In their answers, most students (21 out of 33) used an interval to solve the problem; of this group, 7 used a ‘small’ interval, and 5 used a tangent. Moreover, at *school B*, this question was asked about an instant when the velocity changed continuously. In their answers, 4 out of 14 students used a tangent and also 4 used a small interval. The other students completed the problem only partially or not at all. The tangent solution of one of the students, Mette, at *school B* for the problem about the falling velocities of a parachute jumper turned out to be representative for such ‘tangent solutions’ (fig. 6.19).

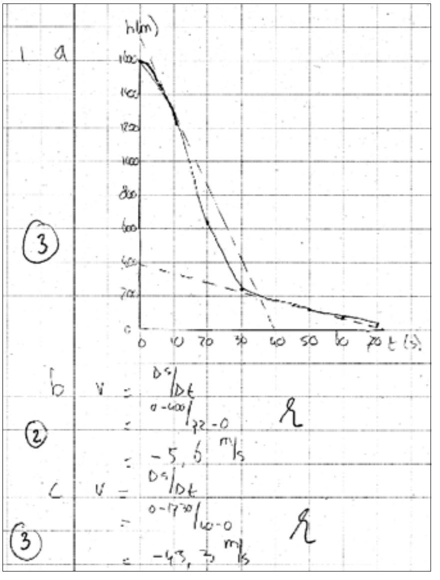


figure 6.19 Approximations of instantaneous velocities of a parachute jumper

In a different, but comparable, context problem (about the sunflower), the students were also asked to interpret the difference quotient and the slope of a graph. In their answers, many students interpreted the difference quotient properly. At *school B*, a mathematical model for growth was provided with this question.
Some students only used values they could read from the graph, while others used the formula. Chris' answer was characteristic of students who used the graph for calculating the growth per day in a certain period of 3 days by reading the corresponding values and dividing them:

Δy / Δx = 100 cm / 3 days = 33.3 cm per day.

It is striking that it was Chris who did it in this way. He had repeated the course, and during the lessons he suggested using the difference quotient and the graphing calculator several times (e.g. page 152). For the students who used the formula, Remo's answer can be used to illustrate the calculation of the difference quotient on interval [5 ; 8] (fig. 6.20).

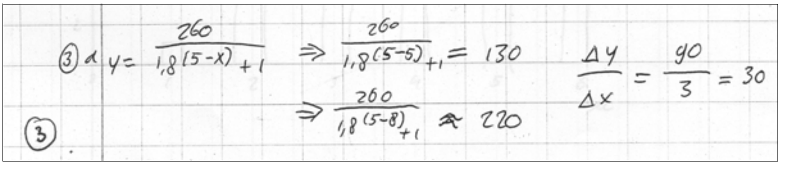


figure 6.20 Approximations of the growing speed of a sunflower

More complicated was the question at *school A* concerning a Ferris wheel for which a goniometric formula was provided that described the link between the time and the elevation of one of the cabins on the Ferris wheel. Here the students had to decide if and how they would use the difference quotient with a question for the rotation speed. Lisanne's answer was the following (fig. 6.21).

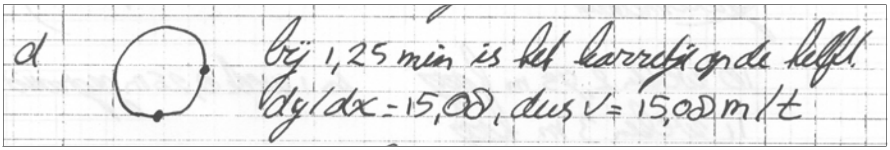


figure 6.21 Approximation of rotation speed

She probably determined *dy/dx* halfway up with the graphing calculator (using the 'calc' function), which the majority of students (11 out of 19) appeared to use. A few students (2 out of 19) seemed to think they were supposed to calculate an average velocity. Approximately one-third of the students (6 out of 19) did not complete the problem. Only one student used the rotation speed which was 'hidden' in the formula.

Finally, the test given at *school A* had a question about the difference between instantaneous and average velocity. Most of the students' answers used the distinction in the length of the time interval (period). Inge's answer is shown below; it was more detailed than the answers of the majority of the students.

> The average velocity is how fast you go, for example, during several minutes. If you, for example, go 1000 metres in five minutes, your average velocity during this period is 1000/5 = 200 m/min. But that does not mean that you are going a constant velocity of 200 m/min. For example, you might start slowly and then go faster and faster, or you might begin fast and then slow down. The instantaneous velocity is the velocity at a specific instant, for example, that you are going 100 m/min after one minute. So the instantaneous velocity is exactly how fast you are going at one instant, which therefore has nothing to do with the average. An average is your velocity taken over a certain period.

Other students referred to characteristics of the *s-t* graph (on a linear graph, instantaneous velocity = average velocity) and the accompanying calculations ($\Delta s/\Delta t$ with a very small Δt for an instantaneous velocity). For instantaneous velocity, one student referred to the reading on the speedometer.

We were satisfied with these results both conceptually and technically. It was striking that few differences could be ascertained between *school A* and *school B* in the area where kinematic knowledge played a role. However, with these test questions it was difficult to determine the effect of aspects such as the dynamic conceptual model of Slope. Did the dynamic model of Slope actually take root with the students? It were especially the numerical questions about approximating the slope at a point on a graph on which the students did not do very well, while applications in a context problem were satisfactory. This was true especially for the students at *school B*. Perhaps too little attention was ultimately paid to the approximation process.

6.1.7 Interview with three students

Following the experiment, we had the opportunity to talk with three students from *school A* about their experiences. We particularly wanted to know what they thought about this approach, especially the extensive attention for the building of concepts. All three students stated that they felt that the entire chapter had taken a very long time. Two of the three felt there was a lack of a clear structure in the student materials with explanatory blocks. They would like to see a clear explanation of how certain procedures work.

> Inge: In the maths book you have these pink blocks and then you know that is important, that it's something you must know. But in this chapter, that wasn't really clear. I thought the book was easier, but I'm also used to it.

The fact that there was not much explanation in the written material was seen as a problem by the students. They had to figure it out for themselves. But one of the students, Suzanne, actually appreciated this aspect:

> Suzanne: I didn't think there was such a big difference. In this chapter there were more drawings, more explanations. The book is more just lists of sums with an explanation block that suddenly appears. I think I understood it better due to the other approach.

The students were enthusiastic about the computer lesson with Slope. They did not refer to the difficult beginning that we had observed, while Suzanne did state that she was not so skilled with a computer. Inge appreciated the dynamic presentation in Slope more than a static picture.

> Inge: Yes, because then you understand the picture immediately. You see it move instead of just having a drawing.

This response fits in nicely with what we intended for the didactic value of such a computer lesson in strengthening a dynamic conceptual model. The students were divided on the value of the historical context. Inge felt she had no need to see Galileo struggle with this material.

> Inge: If it ends up being wrong, what use is it? If you immediately present the correct way, then I can simply accept it.

We were struck by Inge's desire to quickly accept and use an idea. This desire was also expressed, but less explicitly, by the third student, Bjorn. Inge liked to see a new concept and have its relevance explained so that she can immediately start working with it. This may reflect the attitude of the whole class. Was changing such an attitude and the importance of their contributions to open-ended activities sufficiently addressed in the class?

The observations of the class discussions showed that during these ten lessons, the teacher summarised the situation regarding the overarching issue every four lessons. Probably, not all students experienced the lessons as if they themselves contributed to the invention of mathematical tools and physical notions. During this experiment, there was insufficient time to pay a lot of attention to the students' contributions and to discuss them with the whole class.

6.2 Reflection on the first experiment

This reflection primarily concerns optimisation, refining and shifting the accents of the instructional sequence and the research questions, based upon our findings of the first teaching experiment.

The choices made for the first experiment concerned the operation of emergent modelling with a sequence of graphic tools, the role of computer tools in this process and a problem-posing approach. However, the results indicated that the experiment did not proceed as we intended on a number of points. For example, the computer lesson with Flash was cancelled at *school A*. As a result, several parts of the lesson material required so much effort from the students and teacher that there was hardly any time left for discussions of more overarching problems. In these situations, it would have been better to design activities concerning the development of new concepts and representations in the light of such a central issue. Moreover, it was our intention that this central issue would shape the interpretation of, and the perspective on the activities for the students. This applied especially to focusing on patterns and the transition from discrete to continuous models.

In the present section we will address the tentative and temporary character of some models for students and the resulting importance of class discussions, based on those models, to achieve consensus on the next steps. We will first discuss the focus on patterns in displacements.

6.2.1 Focus on patterns

A characteristic aspect of this experimental learning trajectory is the progression between different types of graphs. The trajectory begins with a time series and trace graphs and then introduces discrete displacement graphs and distance-travelled graphs. Our intention was that trace graphs, used with predictions about the time series of a hurricane, would focus the students' attention on patterns in displacements. Questions which ask for precise descriptions of these patterns, were used to

support the students to picture the displacements vertically next to each other. During this process and its sequel, the students' learning about the relation between velocity and distance travelled should be interwoven with the characteristics of graphs.
The results of the first teaching experiment indicated that the students did produce various approaches for predictions. However, they did not draw two-dimensional graphs to analyse the patterns. The class discussions did not focus on the patterns, but rather on the influence of the quantity of data on the precision of the predictions. Contributions from the students when drawing graphs and class discussions appeared to be essential as preparation for the work with Flash.
After this, the students worked with Flash on a number of context problems. Our idea was that students would develop concepts about the use of discrete graphs during this work. However, we were unable to systematically investigate the role of Flash in this regard. Our observations concerned only two pairs of students, but indicated that there was such a development (section 6.1.2). Moreover, it was unclear what the influence of the observer was and what the usefulness of this development was for the sequel. Due to the cancellation of the computer lesson at *school A*, there was no basis there for a follow-up discussion on the work with Flash, while at *school B* the discussion proceeded too quickly for us to ascertain how all the students were reasoning with discrete graphs.
It is therefore desirable to modify the lesson material and the instructions for the teacher in such a way that a certain amount of precision in the prediction is required; this will evoke a more precise description of the patterns and will create opportunities for the students to draw graphs. In addition, activities for class discussions after the computer activities are needed (see section 6.2.3).

6.2.2 From discrete to continuous models

The results from the third section (see page 135) indicated that the step from discrete to continuous models was a difficult leap at that time in the trajectory. During this transition, the character of the horizontal axis also changed from a discrete dimension of the 'number of the measurement', to the continuous dimension of time. This transition took place in parallel with the transition in the interpretation of the vertical axis: from a discrete bar graph as a representation of displacements, to a bar graph of average velocities. The two transitions are of course related, but require a careful delineation to prevent the confusion that we ascertained. The lesson material can be improved in this regard. For example, there was a task about a graph with a continuous time axis, while discrete values were actually displayed on the graph.
The reason for using a continuous time axis is the depiction of average velocities on intervals. We intended that students would be motivated to depict these average velocities, because displacements in discrete graphs become very small when they are the result of measurements with a high frequency. This motivation apparently

did not emerge. Students worked individually on these tasks, and there was no time left in the planning of the experiment to focus extensive attention on this topic. More careful planning, with information to enable the teacher to respond to unintended events, is desirable here.

6.2.3 Class discussions for reflection and consensus

In the sections above, it emerged several times that extra information for the teacher was necessary to guide the learning process along the intended trajectory and to stay within the underlying idea of guided reinvention. During the experiment there were regularly discussions, with varying emphasis, about concepts and graphs. For example, students and teachers spoke in terms of velocity about patterns of displacements. Sometimes continuous lines were drawn through the peaks on the graph. In this lesson material the initiative for undertaking modelling activities was left to the students; this resulted in parts of models having a temporary and tentative character with mutual differences, of which the meaning is probably rather diffuse.

Class discussions are necessary first of all to reflect on the activities and the students' contributions to establish consensus about essential characteristics. Secondly, in class discussions one can evoke motives from the students for taking the next step in a specific direction. These motives frame the intention for and interpretation of the subsequent activities. For example, during the transition from the graph of the Mr Bommel context problem to graphs based on a formula, the teachers were unsuccessful in clarifying the need for such formulas; as a result, there was virtually no transfer of reasoning with intervals between these graphs.

To guide the process of emergent modelling, the teacher requires information about the intended development of models: what do we expect from the activities of the students and how can these activities be used to make model shifts more explicit? The teacher therefore has an important, two-part role: providing 'construction space' to the students and guiding them in the intended direction. Examples of this are the discussion concerning more precise predictions of the path of the hurricane (in section 6.2.1) and the discussion of students' graphs of the falling ball (in section 6.2.2). Since we now have a clearer picture of the role of the teacher; we will provide more information about this role, about the importance of the computer lesson and about the relation between the development of models and the central problem of understanding change.

Besides substantive arguments for supporting the teacher, there is also an organisational argument. During the experiment, the planning was hampered at both schools a number of times by unexpected events such as cancelled lessons, discussions of tests or a malfunctioning computer lab. It turned out to be difficult to deal with these unexpected events. In the sequel to the first experiment, we will describe the process in such a way that it will be sufficiently robust for the daily course of affairs in education. This description must ensure that the teacher can deal flexibly and adequately

with unexpected remarks, events and organisational changes in the spirit of guided reinvention. To this end the teacher must be able to regulate the teaching process in a meta-didactic fashion (Lijnse, 2002).
In retrospection, we see that the description of the instructional sequence in chapter 5 contained hardly any 'teaching steps'. This description was more an elaboration of steps to be followed by the students than a theory of instruction for the teachers. It turned out that we had not formulated any teacher-related observation criteria (see end of chapter 5, page 118). During the second experiment, we paid more attention to this aspect.
Finally, we noted that much attention to motives to proceed can lead to neglect strengthening confidence. For that matter, this can not be concluded directly from the test results, but was obvious from the students' moods, class atmosphere, and sometimes their inability to complete their homework. Therefore, we must not disregard a balance between supporting the invention of solution strategies, and practising specific procedures for achieving confidence.

6.3 Modifications to the instructional design

The conclusions in the reflection on the first teaching experiment led to the following questions:

– How can we better build on the students' contribution and improve the discussion of their contribution during the transition from time series to discrete graphs, and during the transition from discrete graphs to continuous models?
– What is the role of Flash during the students' reasoning with discrete graphs?
– How can we ensure that the teacher has mastered the didactic processes sufficiently to regulate this process in the spirit of guided reinvention and regarding emergent modelling and a problem-oriented approach?

As a result from the first experiment, we were better able to describe the intended learning process. In the teacher guide and during the preparatory discussions, we provided more information to the teacher about the intended course of the learning process, the organisation of the lessons and class discussions. Regarding the learning process and the class discussions, we gave possible solutions from the students and how these can be utilised.
This information should support the teacher for preparing the lessons and for teaching decisions during the lessons. As an example, the frames below show the instructions for the discussion about the activity concerning the progression of the hurricane during the first lesson.

The distances between the dots, the displacements, provide a picture of the change in velocity. Students can describe the motion and make predictions with patterns on the trace graph. But the students will also experience that it is sometimes difficult to see the patterns.

As part of the hurricane-activity, ask for the students' answers and their approaches. Most students will probably only continue the last displacement to determine when the hurricane reaches the coastline.

Students do not have to focus on calculations to determine velocities. The most important activity is measuring displacements and gaining insight into the progression of the motion. Make sure that the students focus on the following question: how does that 'going faster and faster' happen? To this end, let students propose solutions (with a graph or otherwise). A graph with the displacements placed vertically next to each other gives an impression. However, if no students come up with that idea independently, do not draw the relevant graph on the blackboard yet.

We then looked for a problem situation that would ask for a more precise comparison of patterns of displacements. This need for a more precise comparison is expected to motivate the students to place the displacements vertically next to each other or to draw a two-dimensional graph of the total distance travelled. As a problem situation, the cheetah and zebra problem is presented with only a table of experimental data (inspired by Kindt, 1979).

The fastest sprinter in the world is the cheetah. In 17 seconds the cheetah can reach a top speed of more than 110 km per hour and can maintain this speed for a distance of more than 450 metres. However, the cheetah quickly becomes tired after this, while a zebra, which can reach a top speed of 70 km per hour, can maintain a speed of 50 km per hour for more than 6 km. When can a cheetah catch up with a running zebra? The positions of a zebra and a cheetah who is starting to sprint have been measured every five seconds. The distances (in metres) between the positions are listed in the table below (table 6.1).

cheetah	76	116	133	134	132	100	55	36
zebra	95	97	96	94	95	94	98	96

table 6.1: How much head start does the zebra need to stay away from the cheetah?

We expected several students to draw trace graphs, since these graphs were part of the previous activities. To be able to reason properly about the head start, however, graphing the distance travelled vertically is preferred. We supposed that a number of students would do that. The teacher should encourage the students to use their own approach to this problem. The discussion of these approaches brings about the differences between and advantages of trace graphs and of two-dimensional discrete graphs for describing motion and doing predictions. When none of the students

comes up with a two-dimensional graph, the teacher should guide the class discussion in this direction. Consensus should be achieved about the methods for describing motion using discrete graphs. If consensus is achieved, the discussion of this activity seems a suitable preparation for working with the Flash computer program.
During the computer lesson with Flash, we expected a development in the students' reasoning methods. This is a development where understanding, language and use of mathematical and physical models will emerge and will shape the organisation of motion. In the second teaching experiment we collected and analysed more systematic data about the students' reasoning while working with Flash and Slope.
In the following lessons, we planned reflective class discussions about this assumed development in reasoning. The discussion after Flash was based on the question for a description of the motion of a bungee jumper. Students worked on the activity in small groups and wrote the solutions on an overhead projector sheet. The teacher ensured that most sheets were presented in class and used these presentations to reflect on their reasoning and on the work with Flash.
For the students, this is an activity which reviews the technical skills for drawing graphs. For the learning process, however, it is intended as a reflection on language, concepts and graphs in the light of describing motion and doing predictions. This reflection is supposed to take place in the class discussion based on the students' solutions. The following instructions for the teacher were therefore included.

> The homework was the exercise about the bungee jumper. Use this exercise to see if the students can work independently with the Flash graphs. Ask the students – for example, in groups of four – to choose the best graph (displacements or distance travelled) and to draw this on an overhead sheet. Then have several (or all) groups present their sheets in front of the class. Which one provides the best description of the motion of a bungee jumper? Hopefully, the students will use reasoning about the change in velocity and the relation between the progression of the displacements on the graph and the distance travelled. If not, refer to the work with Flash.

As preparation for the horizontal time axis, in the worksheets we ask several times for the length of the time interval of the relevant displacement. In one of the activities the teacher can address this aspect explicitly; it concerns a motion with a constant velocity that was introduced in the previous task (with measurements every 0.5 seconds) (fig. 6.22).

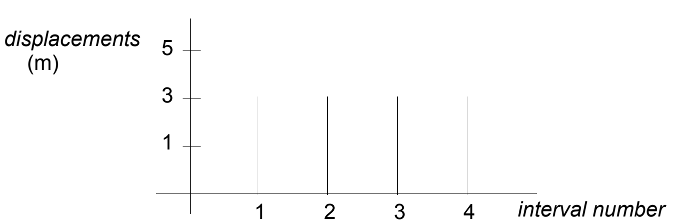


figure 6.22 A displacement graph of constant velocity

Assume that in the above exercise the positions are measured twice as often. The question is: How do the graphs of displacements and of distance travelled look in this case? We suggested the following for the discussion of these activities.

> Measuring twice as often results in twice as much data and leads to the vertical diminishing of the displacements. This makes it difficult to read the graph. If the question then arises about how this diminishing can be avoided, we expect some of the students to come up with the idea of placing average velocities on the graph instead of displacements. If this is the case, ask the question: is this the best we can do to describe and predict motion? If no students come up with an idea, then point to the displacements in the increasingly small time intervals: what do these mean for the average velocity in the accompanying time interval? Does that also become smaller?

These suggestions should create opportunities for the students to come up with the idea that presenting average velocities is a possibility for solving the problem. The interpretation of time intervals is promoted by comparing displacement graphs (for example, in fig. 6.24).

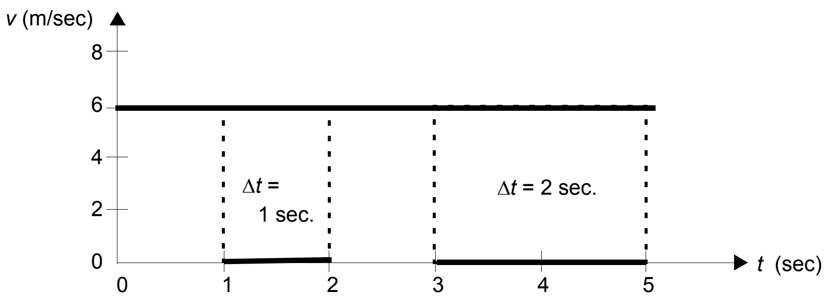


figure 6.23 Constant velocity and displacements in time intervals

The connection between displacements and the accompanying time intervals should provide for opportunities for a transition to a continuous time axis. This graph makes it simpler to talk about average velocity, and it is possible to introduce it at an early stage. As a result, we expected students to come up with the idea that depicting average velocity is an alternative when they experience the diminishing displacements with discrete graphs.

> Below you see two graphs with displacements. In both graphs, the time interval that accompanies every displacement is provided. Which of the two motions travels the greatest distance? And which reached the highest velocity?

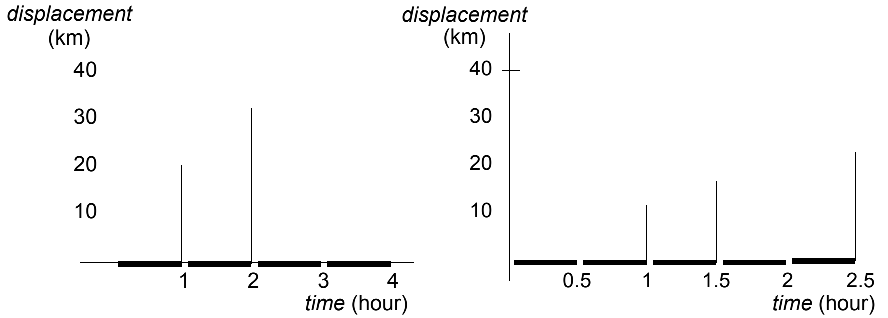


figure 6.24 Numbers on the horizontal axes are replaced by time-intervals

Preceding the work with the continuous proportionality $v \sim t$, we first go a little further with graphs of constant velocities in the assumption that students, based on the problems posed, will come up with the idea that the area accompanying a time interval is an indication of the displacement during that interval.

> Can you use this graph (see fig. 6.23) to reason that the displacement in the first interval from 1 to 2 seconds is half of the displacement during the interval from 3 to 5 seconds?

> You see another *v-t* graph below (see fig. 6.25). In the first interval of this graph, is the distance travelled more, less, or about the same as in the second interval?

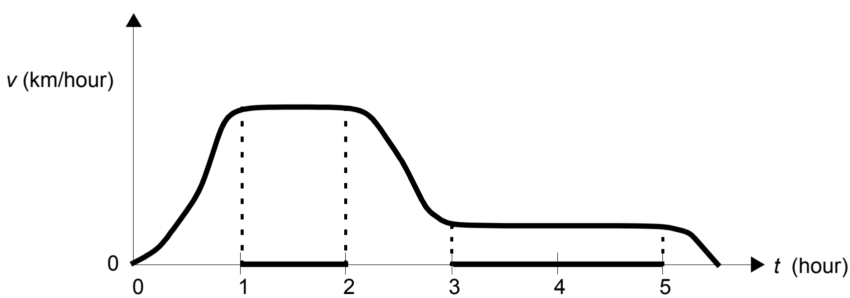


figure 6.25 Changing velocity and displacements in time intervals

With this preparation, we expected students to develop the notion that approximating a graph with piecewise constant velocities is worthwhile for determining distance travelled. A notion that would help in tackling the Galileo-activity about free fall.

6.4 Second teaching experiment

The second teaching experiment took place in a 10th grade class of 22 students at *school C*, a school in the Dutch city of Utrecht. The experiment lasted eight lessons. More information about the methodological arguments for the set up of this experiment can be found in section 4.4.

During two sessions with the teacher, we first discussed the underlying ideas of the series of lessons followed by the plans for the first two lessons. We talked about the teacher's manual that accompanied the student material and we discussed the organisation of the lessons, the most important activities, how to introduce the activities, and how the students' solutions could form a basis for the following lessons. The teacher approved of the approach we had sketched out and his planned role during the class discussions.

The description of the second teaching experiment is limited to the modifications of the instructional design as outlined in the previous section, and to the work of the students with the computer tools. In section 6.4.1 we first describe the transition from time series to discrete graphs and from discrete graphs to continuous models of motion. After this we analyse the role of the computer tool Flash in greater detail.

6.4.1 From time series to continuous models of motion

The question was: using the revised teaching and learning materials, to what extent will the students be able to structure phenomena of motion, together with the development of mathematical and physical tools for describing motion? This is a process that ultimately leads to gaining insight into the basic principles of calculus and kinematics. In this section, we primarily address the part of the learning trajectory from the weather context to the interpretation and use of average velocity graphs. The questions that played a role in this process are: what is the contribution of the students to the formulation of problems concerning patterns in displacements? Following the activities with Flash, how is consensus achieved about the usefulness and use of the two discrete graphs, and what is the role of the teacher in this process? Will the teacher be able to create opportunities for the students, based on their contributions, to use continuous models of motion?

Results

The introduction of the learning trajectory went as expected, and the students' contributions were more productive than in the first experiment. During the class discussion, many elements from the entire lesson series were covered briefly. The teacher consistently used the language and solutions of the students as a starting point, and it became clear to them *how* and *why* they were studying weather situations in this context.

The activity about the cheetah and the zebra was intended to support the students'

thinking in the need for drawing two-dimensional graphs. On their worksheets we saw a diversity of solutions. However, only two students had actually drawn two-dimensional graphs of the distance travelled. The other students solved the problem by reasoning with the table.
During the class discussion about the attempts with the table, the option of using the graph was suggested at exactly the right moment by one of the students. The students initially disagreed about the amount of head start the zebra could have. All students seemed to have developed a need for a clearer answer. At that moment, on the teacher's initiative, one of the two students draw his two-dimensional graph on the board. The fact that this graph indeed offered a solution to the diversity of previous answers was clear for many students. We concluded this from the number of students who contributed in the accompanying class discussion, about the horizontal and vertical shifts of the graph. The teacher completed this discussion with a reference to the global problem of describing motion for predictions.
The aim of this activity, i.e. to evoke the need and to discuss the usefulness of two-dimensional graphs, was achieved. As a result, the students had a means available for organising and thinking about the subsequent problem situations. During the activity about seeing a pattern in the displacements of the falling ball on the stroboscopic photo, we again saw the intended diversity of solutions, including the two types of graphs from Flash. The teacher got the students to show their solutions and problematised the differences between the students' solutions. During this discussion, the class gave the impression that they reached a consensus about the usefulness of the two discrete graphs as tools for describing motion and doing predictions. This was a useful preparation for the computer lesson with the computer program Flash. Using Flash, they were then able to interpret these graphs and immediately began investigating the motions captured on the stroboscopic pictures.
During the computer lesson, the students increasingly reasoned about the characteristics of, and relation between, the two graphs and their meaning for the specific situations. A few students took a little longer to work this out. This group continued for some time to relate the characteristics of the graphs to the displacements in the stroboscopic situation. After the computer lesson, the teacher first discussed their experiences. He used overhead sheets with screen shots for this part of the discussion. Many students participated actively in the discussion. The only deficiency was that the relation between the graph of the displacements and the slope of the graph of the distance travelled was still hardly emphasised.
The students then worked in small groups on the reflection-activity about the bungee jumper. Every group was given overhead sheets on which to draw graphs of the displacements and of the distance travelled by the bungee jumper. After 15 minutes, a representative from each group presented their sheet. The teacher again stated that they should briefly explain the graphs and after this the class could ask questions. During the first presentation, the teacher asked the student to use characteristics of

the motion while talking about the graphs, and made occasional references to experiences with Flash. In the following presentations and discussions, the students took over this role. Based on this activity, a class consensus was achieved about the usefulness and the interpretation of these two discrete graphs.

The changing character of the horizontal axis was discussed in the next lesson: from measurement number to a continuous time axis. However, the class discussion proceeded with difficulty and almost no students participated actively. One of the reasons was that most students gave the impression that they had not done their homework. However, our analysis of the students' worksheets showed that most of the students did work on these tasks. In addition, the class discussion was disturbed because one student contributed an answer that the teacher did not understand immediately (if measurements were made twice as frequently, her distance-travelled graph became approximately three times as steep). The teacher stated that he would look at the graph with the student later on. However, she continued to interrupt the teacher. The teacher was then unable to guide the students' reasoning about the horizontal axis. He asked students the questions that were compatible with his reasoning, a reasoning that he carefully attempted to establish. In retrospection, he admitted that he felt somewhat uncertain about the intention of the tasks. This uncertainty, combined with a 'difficult' student, led to a discussion that was not based on the solutions presented by the other students.

On the basis of this class discussion, we cannot say if the students understood the need for the horizontal time axis. During one of the final activities from this section, a majority of students made the correct remark that the graph with average velocities remains equally 'high' and that the graph of the displacements 'diminishes' if you make measurements more frequently. Some students had written too little or had drawn only a single graph with their answer, or had written nothing at all in their workbook. We were surprised by a solution, where the displacements did not 'drop' because she modified the vertical scale. The effect is comparable with the effect of scaling measurement values on time intervals, which took place with the graph of average velocities. Unfortunately, she did not contribute this solution and it was not discussed in the class.

In the following activities with the *v-t* graphs provided, the teacher was successful in supporting the students' reasoning about approximations with piecewise constant velocities. These velocities and the accompanying time intervals were used by the students to approximate the distance travelled. However, large parts of the graphs proceeded too constantly (see fig. 6.25) to make this approximation process explicit. In this task, it was too easy to calculate with the values that could be read directly from the graph, but the students estimated and reasoned well with constant velocities and the accompanying time intervals.

In the next three lessons, the students first worked on the problem about the speeding violation by Mr Bommel and then on a lesson with the computer program Slope. It again turned out to be difficult to make the transition from reasoning with the graph about Mr Bommel to reasoning with a graph of a formula. In the contextual activity concerning Mr Bommel, the students used intervals and divisions to estimate average velocities. Most students drew linear continuations to answer the question about the progression of the graph if the velocity no longer changed after a specific time. However, with comparable questions concerning the graph of the formula, we did not observe such answers. In retrospection, we see that there were possibilities to utilise the reasoning applied by students for this transition. We assumed too quickly that the students would understand the similar aspects in the context problem and the abstract situation. We see the graph and the difference quotient as a shared tool, while for most students it is primarily still a description of a situation that they first have to understand.

During the computer lesson with Slope, the students were able to apply the method with the difference quotient to many different graphs. In the following lessons, the students developed the mathematical method for approximating local slopes and the use of the graphing calculator in this process. The mathematical question about a local slope was answered correctly by most students, although in their answers, we saw that the students were not practising any standard method. We saw a great diversity in approaches with, in most cases, an explanation that showed the students understanding of the method they were using. In addition, a continuous velocity graph was approximated by some students with piecewise constant average velocities for calculating the distance travelled. This process was not yet a standard procedure for them and showed more understanding than the method of counting boxes that was discussed in chapter 2 (page 29).

Moreover, it appeared that the teacher succeeded in discussing the contributions of students in such a way that this shaped their intentions about a possible way to proceed. In these cases, the activities of the students turned out to be productive for the problem situations that followed. The teacher was especially capable of guiding such discussions when students contributed a diversity of approaches. This diversity in reasoning and approaches was created by activities with no standard solution procedure (e.g. the activity about the cheetah and the zebra), and by computer lessons during which the students were confronted with many problem-situations and the tools in the program afforded a variety in reasoning. In the latter cases, students and teacher referred to these tools in discussions afterwards (e.g. referring to Flash when discussing the bungee jumper activity).

An important aspect of the class discussions was that the teacher did not answer the students' questions, but limited his contribution to making the students' arguments more explicit, and then posing a question about these arguments to the class. In the situations where no time was taken for such discussions (due to lack of time, lack of contributions from the students, or problems with maintaining class discipline), it appeared that the teacher tried to expose the underlying algorithm. In these cases the teacher's explanation was not retained by all the students in the class (e.g. with the bar graph approach to a continuous *v-t* graph).

Illustrations of the results

During the activities on the hurricane problem, one student suggested continuing the last displacement to make a prediction. The teacher asked the class what the student had assumed. Suzanne then stated that she had approached the problem differently. Due to this question about her assumption, the attention of the class was therefore framed by the pattern in the displacements.

1 Teacher: Try to follow my reasoning for a moment. What is her solution based on?

2 Student: Metres, using a scale, looking at the course of the hurricane.

3 Teacher: Something else… what has she assumed?

4 Student: That the conditions are not going to change.

5 Teacher: Exactly, she assumed that what happened during the last 12 hours will continue in exactly the same way. Is that probable?

6 Student: No.

7 Teacher: Okay, who did not assume that the velocity would change? Suzanne?.

8 Suzanne: I looked every 12 hours, and then the distance increased by half a centimetre.

9 Teacher: Okay, you repeatedly add 0.5. And then?

10 Suzanne: The last piece is about 3 cm [the teacher points to the 2.8] and if you continue this with 3.5, then it comes onto land at $\frac{1}{3}$. And that's 10.00 o'clock.

During the problem about the cheetah, there was a discussion about catching up with the zebra based on the numbers provided. There was a disagreement about whether or not you could determine this. Several students had added up all the numbers to determine the distance travelled. Ultimately, there was a difference of 17 metres, they therefore concluded that the cheetah could catch up with the zebra if the latter had a head start of 17 metres. Other students disagreed, saying that the cheetah was going more slowly at the end, and that the necessary head start could therefore be smaller. The teacher then asked if anyone had made a graph to describe the situation; Steven was asked to draw his graph on the blackboard (fig. 6.26).
A discussion then took place about what you could read from the graph. The teacher guided this discussion. He did not give any answers, but he made arguments more explicit and presented them back to the class.

1 Teacher: You have seen that the total distance that the zebra travels is somewhat less than that of the cheetah. Well I think that's brilliant, but what can you do with it?

2 Steven: Now you can see at what time the cheetah will catch up. Where they will cross?

3 Teacher: Is that right?

4 Student: Yes.

5 Teacher: Why?

6 Steven: Well look, if the cheetah wants to be next to the zebra, then it has to be at the same point as the zebra. Then they have run an equal distance.

7 Samir: Yes, okay. But what you're saying is that if they start running at the same point, there will be a distance between …

8 Teacher: Okay, Samir, you saw that correctly. What Steven explained is that if they begin at the same time, it will catch up with him after so many seconds [points to the graph]. But why? Then they have both run an equal distance. But is that the question? What was the question?

9 Samir: When can the cheetah still catch up with the running zebra.

10 Teacher: Okay, then we'll see if we can answer that question with this argument and then I will ask what Anna has done, because she had a completely different graph.

11 Steven: That's possible, then you have to shift the graph of the cheetah somewhat so that the upper point…

12 Teacher: Come up here and show us.

13 [Steven shows on the graph how the cheetah can be shifted somewhat to the right and then it just touches the graph of the zebra. The teacher helps Steven and shows it in a different colour of chalk; at first Steven wanted to erase the old graph.]

14 Teacher: Okay, and what is the answer then?

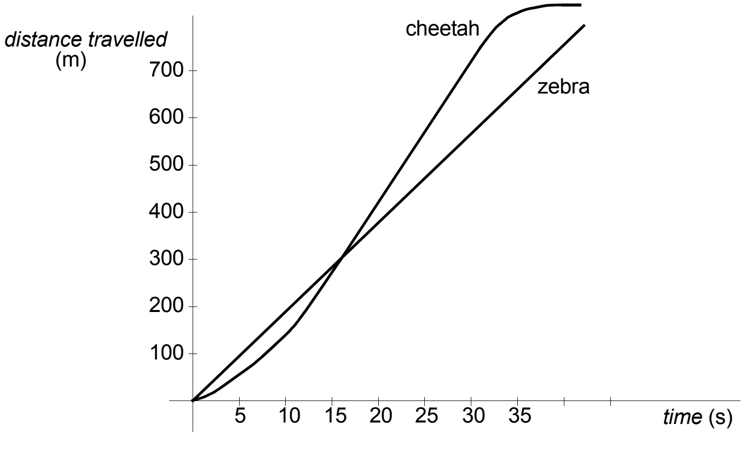


figure 6.26 Steven's graph of cheetah and zebra

15 Joost: That if he begins so many seconds later…

16 Teacher: Seconds?

17 Samir: You can shift the other upwards…

The teacher then drew Samir's suggestion on the graph and discussed the differences between the two transformations of the graph with the class. Finally he asked about Anna's graph. She had a very different graph that she drew on the blackboard (fig. 6.27). She explained that she only placed the values from the table on a graph, and that she could not go any further with this graph.

figure 6.27 Anna's displacement graph of cheetah and zebra

Other students then tried to relate the graphs to each other. The teacher finally summarised that Anna's graph provided a picture of the change in velocity, but that you could not directly see the moment the cheetah would catch up with the zebra from this graph. The following lesson began with a discussion of the students' solutions that accompanied the task about the stroboscopic photo of the falling ball. Several of their graphs are included below with various dimensions on the horizontal and vertical axes (fig. 6.28).

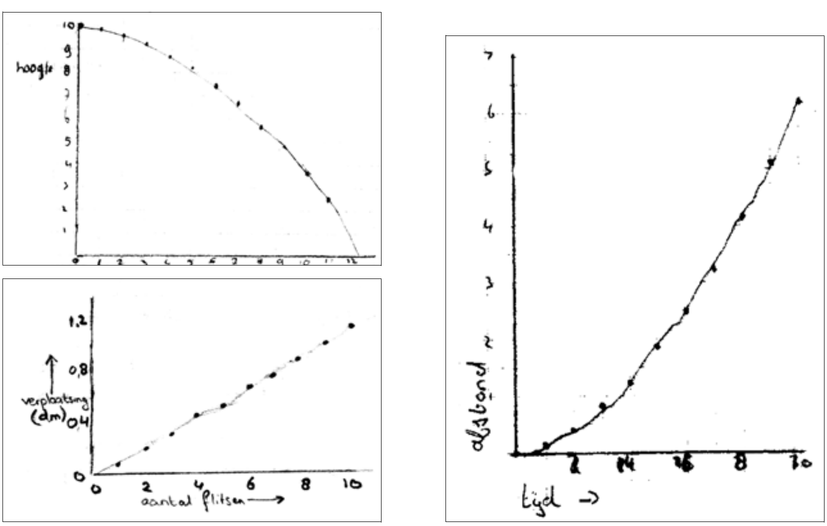


figure 6.28 Students' graphs of a falling ball

The advantages and disadvantages of the various graphs were discussed and they ended with two types of graphs for motion in general: a graph with displacements in sequential time intervals and a graph with the total distance travelled. The students then worked on the computer lesson with the program Flash (see the following section for a description of the computer lesson). After the computer lesson, there was a discussion about their experiences with Flash. The teacher first addressed the distinction between the two discrete graphs.

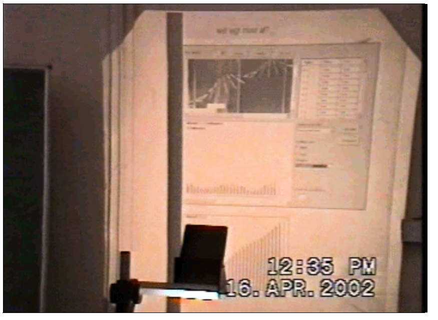


figure 6.29 Class discussion after the computer lesson with Flash

1 Teacher: First the graph with the displacements. What does it signify if two of the bars are of equal length? Here is a moment when the black and grey bars are equally long. Suzanne, what does that mean?

2 Suzanne: That they have gone the same distance.

3 Teacher: Does anyone else think it could mean something different?

4 [Nobody responded.]

5 Teacher to Suzanne: When have they gone an equal distance? What do you mean?

6 Suzanne: At the same moment.

7 Teacher: And what do you mean by the same moment?

8 Suzanne: Between the pictures.

9 Teacher: Between the …, between the picture that you have clicked onto, there and the following picture. Between these two pictures. Okay. Does that mean that they also travelled the same distance in total?

10 Suzanne: No, that's in the graph underneath isn't it?

11 Teacher: Okay, that was the following question. Explain how you can see here that they have travelled the same distance. If the bars are equally long here, what does that mean Suzanne?

12 Suzanne: Then they have travelled the same distance in total.

During this discussion, the connection with and limitations of discrete graphs were addressed, but neither of these aspects were discussed completely. The students subsequently worked on the activity about the bungee jumper. After 15 minutes, the teacher invited the first representative of a group to make a presentation.

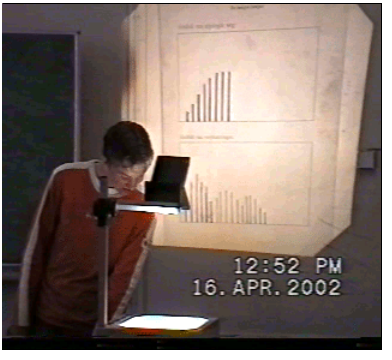


figure 6.30 Student demonstrating his graphs of the bungee jumper

1 Teacher: Glen, you have about 30 seconds to explain the upper and lower graphs.

2 Glen: The distance travelled becomes larger and larger. You can see that on the first graph. The second graph has the displacements. Let me see, how did that go again? Oh, yeah. He first falls downward, then he's at the end of the elastic, and then he goes back up and then he goes back down again. The displacements become smaller and smaller. He goes back and forth more and more slowly.

3 Teacher: Okay, I hear two things. The displacements become smaller, and he goes slower. Now explain again about the bottom graph where you use velocity.

4 Glen: In the beginning he falls faster and faster downward…

5 Teacher: How do you see that on the bars?

6 Glen: The bars go up, they get longer and longer. Once he gets to the bottom, he goes slower and the bars become shorter. After that he goes back up and the bars become longer again.

During this presentation the first explicit connection between the discrete graphs emerged with references to the motions of the jumper. The next group, which emphasised this connection as well, also discussed the relation between the slope of the line and making a discrete graph of distance travelled and change in velocity.

1 Teacher: Okay. [looking at the class] Do any of you have any comments? [nobody responds] Any criticism? Then we'll go onto the next group. Martine, do you have something?

2 [Martine has taken the sheet out of her notebook and puts the sheet down, but Natasja goes to the board to give the explanation.]

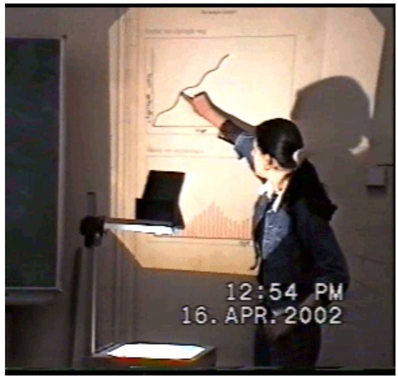


figure 6.31 Student demonstrating her graphs of the bungee jumper

3 Natasja: Well then, this is the distance travelled and that means that the bungee jumper goes down here and therefore he goes faster, because he travels a greater distance in a shorter time. And here he goes down again: And then he travels a smaller distance in the same time and then he goes back up and back down again…

4 Teacher: May I make a very small addition to interpret what you are saying, because you're all saying it very well, only what you're also saying is that you can see how fast he's going by looking at how steep that thing is. Or not?

5 Natasja: Yes.

6 Teacher: Can you explain that a bit better?

7 Natasja: Yes, if he goes down then he goes faster, and if he goes up then he goes slower.

8 Teacher: How can you see that by how steep it is?

9 Natasja: Well, because in this little piece of time [she points at an accompanying increase], he travels quite a long distance [points finger up and down along the displacement]. While in this one here it takes him longer to go the same distance [she moves her finger along a less steep part of the graph].

10 Martine: In the same time.

The use of these graphic interpretations was also apparent in the students' workbooks. Next to a graph of the distance travelled, which increased linearly at a specific time, virtually all the students noted that the velocity from that time was constant. With a question about whether two graphs (a graph of the distance travelled and a graph of displacements) could be graphs of the same motion, many students had the answer that this was impossible because from a specific point, the graph of the distance travelled became horizontal, while there were still displacements on the other graph. Apparently the horizontal slope on the distance-travelled graph was correctly associated with standing still.

However, in the exercise about the consequences of more frequent measurements the students had various solutions. The majority of students placed the extra measurements between the previous measurements; and as a result the graph did not become any wider.

For the horizontal axis, some students were already using time indications, but most just numbered the intervals. However, the discussion was dominated by one student and was delayed by the teacher's hesitation on how to proceed (on which student's reasoning and in which direction).

1 Teacher: ...know the graph. Because it's about … exercise 17. From exercise 17 you know that it was three metres per second there, no, per interval, and it seems to me you could read somewhere else that, if it was 0.5, then you could therefore go 3 metres in a half second. But that gives you 6 metres per second. Right? Then the question here is: assume that every one hundredth of a second you …

2 Student: Then you would just get 3 cm per … one hundredth of a second.

3 Teacher: So you get much less as a displacement… than this bar, because this does not have a 3 cm displacement in one hundredth of a second. Much less. So you can then calculate…

4 Student: Yes, but in total it's still correct.

5 Teacher [hesitating]: Yes…

6 Student: If you would continue to 100, then you would still get one.

It was completely unclear to us to which aspects of the graph they were referring to in their remarks. The student did make some useful remarks (lines 4 and 6), but the discussion focused primarily on drawing the requested graph together with the teacher. The teacher then discussed exercise 17 and returned to the next exercise about the consequences of measuring twice as frequently. The importance of the time duration between two interval numbers had already been discussed. Anna was the first to respond:

1 Teacher: What I now …, the step that we are now going to make, was a step that I tried to bring up a while back. Because… how have you drawn the graph? With exercise 18, how does it look to you? Anna?

2 Anna: Exercise 18?

3 Teacher: Yes.

4 Anna: Then it becomes three times as steep on my graph. Approximately. At least… that's what I think.

5 Teacher: Three times as steep… this one?

6 Anna: Yes, uh…

7 Teacher: Or this one…? No you haven't thought this all the way through yet. Think about it a bit longer. First look at how this ...

The teacher didn't know exactly what he should do with her answer and made the question more precise and asked the class again:

1 Teacher: The question is if it also became twice as wide?

2 Student 2: No.

3 Student 3: Yes, uh… that depends. If you simply go to 4 seconds then it becomes twice as wide. Yes.

4 [The teacher draws the axes of the graph with a division of the horizontal time axis.] He asks: How long do the bars become?

5 Samir: Half as long.

6 Teacher: Why?

7 Samir: Because uh, in half of the time you only go the half…

8 Teacher: Yes, you take a picture twice as often … so you would also only be able to travel half the distance.

9 Anna [softly]: I just asked him that and he said something else! Good grief.

10 Teacher [heard Anna]: We're going to go back over that in just a minute, Anna.

11 Anna: Yeah, okay [offended].

A little while later during the class discussion there was confusion about what exactly was on the graph on the blackboard:

12 Student 4: Is that one interval number, 0.25?

13 Teacher: Yes. So I can also write in 0.25, 0.50, 0.75 and 1.0. What happens in the first interval of 0.25 seconds?

14 Student 5: Then it becomes the half of the uh…

15 Teacher: And that was?

16 Student 5: ... uh, three and now it's one-and-a-half.

17 Teacher: Yes, that is one and a half…1.5. What happens during the next twenty-five hundredths of a second?

18 Student 6: ... it becomes 1.50.

19 Teacher: How much is added?

20 Student 6: 1.50.

21 Teacher: Yes, another 1.50. So where am I then?

Here there was virtually no discussion. The teacher attempted to have the students follow him (in a Socratic fashion) while he was finishing *his* graph. But it was

unclear if student 6 really meant that there was an additional 1.50. A while later it became even more difficult because the graph of distance travelled on the board still had interval numbers on the horizontal axis. Afterwards it turned out that Anna had thought: measure more frequently with the same measurement result (he still travelled 3 metres every 0.25 seconds).
A large diversity in students' solutions is apparently not a sufficient precondition for a productive class discussion. The students must attempt to relate their solutions to the teacher's intended reasoning. Perhaps it would have been more effective here to have drawn two or three of the student's graphs on the blackboard (or have had the students draw the graphs) with a discussion about their various assumptions and methods. This could have been done in exactly the same way as the teacher discussed the students' solutions for the problem about the hurricane; he could have used the students' remarks to design problems that led them in the intended direction.
During the next lesson, the students were primarily working independently with the activities concerning the transition from displacements to average velocity as a composite quantity for the vertical axis. This transition resulted from the problem of the displacements becoming smaller if you measure more frequently and from the comparison of various measurements with various time intervals. The solutions provided by most students to a task about the consequence of measuring more frequently indeed showed how the displacements became very short on most of the graphs. Martine's solution was strikingly different, however. On her graph, the displacements stayed the same length; this was because she repeatedly chose a different scale for the vertical axis. In fact, this is a choice that is symmetrical with scaling the displacements according to average velocities (fig. 6.32).

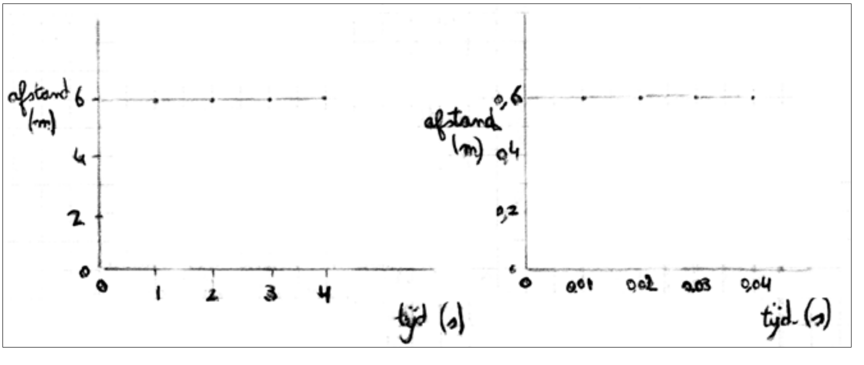


figure 6.32 Modifying the scale of the vertical axis to display small displacements

The following solution (from Martine) shows that measuring more frequently for the average velocities does not have any effect on the graph of constant acceleration (i.e. the vertical axis does not have to be modified) (fig. 6.33).

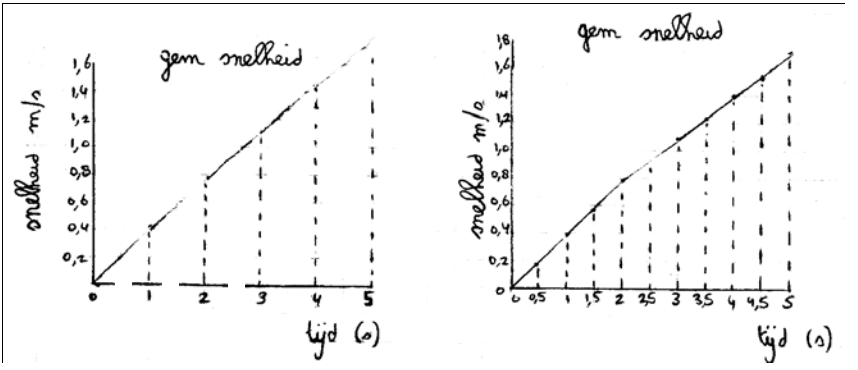


figure 6.33 The vertical axis of a velocity graph is not modified

Finally, an activity about the graph of a parachutist (page 21) showed that several students were capable of using the bar graph approach of *v-t* graphs. The question was: from what height did the parachutist jump? One-third of the students used average velocities or approximations with bar graphs (both with fixed time-intervals and with time-intervals adapted to the shape of the graph). Some interpreted the graph incorrectly or attempted unsuccessfully to use a formula. The other students answered the question only partially or not at all.
From the answer to a question about calculating the slope of a graph, it appeared that most of the students had largely mastered this mathematical concept (six students did not complete the question entirely), while no standard method was applied for determining the slope. The question was the following:

> Two functions are given: $f(x) = x^2 + 4x$ and $g(x) = 2x - 6$.
> Determine if the graph of f at the point where $x = -1$ has the same slope as the graph of g.
> Explain what you have done.

Most of the solutions of the students were correct. However, there was a great diversity in approaches. Many students either used the graphing calculator or did it by hand. We also saw mixed forms where they had checked their calculations.
With a question about ways to determine an instantaneous velocity, virtually all students provided an answer using an *s-t* graph and approximating local slope using small intervals, sometimes with a reference to graphic elements from the computer program Slope such as “make the triangle small”.

6.4.2 The role of Flash

The idea of the computer program Flash was that students could investigate many situations with the discrete graphs of distance travelled and of displacements. Interpretations of graph characteristics and the relations between the two graphs should emerge during the investigation.

Results

As a result of the previous lessons, students quickly focused on the patterns in the graphs. Flash afforded the students to develop graphical reasoning for describing motion and doing predictions. We noticed a dialectic relation between tools-in-use and mathematical sense-making. The graphical tools in Flash shaped the activities while, at the same time, the activities shaped the understanding that emerged; an understanding of graphical characteristics of discrete two-dimensional graphs as models of motion.

Unlike the previous experiment where the students did not know if the graphs depicted height of an object, we did not observe them searching for interpretations of the graphs. Nor did we hear remarks like ‘what do they want us to do?’ We conclude that the graphs in Flash were compatible with the students’ current reasoning. We analysed the work of three pairs in detail. The students in one of these pairs were weak in mathematics. We noticed that their transition of reasoning (from concrete situations to characteristics of graphs) took place more slowly than with the other two pairs. The weakest pair often resorted to using the stroboscopic photo or the time series in the stroboscopic picture to interpret the graphs.

Illustrations of the results

One of the stronger pairs of students quickly used the graphs to describe motions in terms of fast, faster and going constantly faster. The concepts were related to characteristics of graphs. The following transcript illustrates their method at the beginning of the computer lesson with the photograph of the falling ball and the question about what the [Continue] button does.

1 Ellen: Press Continue one more time.

2 Marloes: No, you can’t do that because it stays the same… Wait, if we do this… no, then it continues in a linear line.

3 Ellen: Yes, it just goes further like it was. It doesn’t take account of the fact that the ball…

4 Marloes: Goes faster.

5 Ellen: Yes, it goes faster.

6 Marloes: Does it continue the last distance or something?

7 Ellen: Yes, it doesn’t take account of the fact that the ball goes faster and faster.

While working on the task about the rotating stick one student in this pair remarked that the motion of the middle of the stick is shown by a displacement graph with the same height, while the motion of the end produces a curving graph. They linked these graph characteristics to characteristics of the motion of the stick.

1 Ellen: What is the difference between the two motions?

2 Marloes: The red one goes in loops.

3 Ellen: Ummm.

4 Marloes: The red one goes faster and faster and then slower and then faster again and then slower again, the blue one goes almost straight.

5 Ellen: Oh, yeah, okay.

6 Marloes: So…

7 Ellen: Then you have to say that …

8 Marloes: So the middle goes almost straight.

9 Ellen: No, the middle is going with constant velocity.

With the weakest pair, this process went much more slowly. Their language about the actual motion and the abstract characteristics of the graph were intermingled at the beginning. Occasionally they appeared to understand the difference and the connection between the two graphs, but this hardly seemed to take root with Mary (one of the weakest pair).
In the following transcript, they were working with the falling ball and were trying to determine what happened by looking at the graph of displacements.

1 Mary: Ingrid, you say the ball is falling faster and faster right?

2 Ingrid: Yes, that's right isn't it?

3 Mary: Go to that thing. To the graph.

4 Ingrid: Look, there is more and more distance between the two balls, you know, from that picture.

5 Mary: No, but look, if you say the ball is falling faster and faster…

6 Ingrid: Yes, what then?

7 Mary: You can see that from the distance, can't you, how much that continues to increase, but this is always the same. [She refers to the increase in the displacements, which remains constant.]

8 Ingrid: But it keeps falling faster, otherwise it would be a straight line. Then you have this [she gestures to show a horizontal line].

9 Mary: But wait a minute, where does that graph come from? Is that the graph of velocity or distance?

10 Ingrid: Of the distance between two balls.

11 Mary: Well then, but you say that it's falling faster and faster. But that couldn't be so, because the same amount is added each time. Not more in a specific time or something.

12 Ingrid: If you do it this way, this [she seems to be clicking regular distances], then it goes, then it continues to fall at the same rate… yeah I can't explain it. It's just the way it is.

13 Mary: If that's the way it is, then it would just stay at one point?

14 Ingrid: No, this is the distance between two flashes, you know, so between 3 and 4, so it's falling faster and faster, because the flashes happen in the same time… and the balls are further and further apart.

15 Mary: Or wait a minute, it is the distance between two balls. Sorry. No I saw it wrong.

The description of graph characteristics and of actual motion are interwoven in their language. Due to Flash, they could, however, try something out to test their expectations about the graph (a constant velocity results in constant displacements). In the following fragment, they are comparing the middle and end of the rotating stick.

1 Mary: Well, the blue stays more the same, more constant, or not. And the red, it's wavy. [Blue is the middle, red is the end.]

2 Ingrid: The red goes very fast and then it goes very slowly again.

3 Observer: And can you also see that on the other graph?

4 Ingrid: Yes, that the red suddenly goes very fast, and then it goes, like here, then it goes about the same, and then suddenly it goes very fast again. While the blue goes up at the same rate.

5 Mary: Yeah, blue goes more steadily.

6 Mary [Reads the following question] …explain how you arrive at your answer...

7 Ingrid: Well, the end travels a longer distance, because the graph is higher.

In the above mentioned fragment we can see how they repeatedly returned to the meaning of the photograph. Fifteen minutes later, they tried to use the graph to determine if the cheetah catches up with the zebra. The observer again approached them. Mary once again forgot what the graphs meant. She suddenly wondered why the two graphs were different. 'They are about the same motion aren't they?' The pair determined the meanings of the graphs by using the time series in the stroboscopic photograph.

1 [They look at the distance-travelled graph.] Mary: He catches up…

2 Ingrid: So he doesn't quite catch up.

3 Observer: Why not?

4 Ingrid: Because the red doesn't rise above the blue.

5 Observer: Do you understand that too?

6 Mary: Yeah, I get it now. I just didn't understand for a minute what kind of graph it was but now I see it.

7 Observer: And if you look at the other graph, the displacements, then he catches up with him earlier or not?

8 Mary: Yeah, then the red does come out above the blue… doesn't it?

9 Ingrid: Yes, but that means that it in a specific … between two pieces how much distance it travels. Not that it catches up.

10 Mary: You know, what I don't understand, what is the difference between this graph and the other one. They are always the same, aren't they?

11 Ingrid: No, look, this [displacements graph] is the distance between two dots. And this [distance travelled graph] is the distance between the first dot and the … for example, this, the fourth dot. That's the difference.

12 Mary: Oh, yes.

Their language about the graphs did not stabilise during the lesson. Mary remained very close to the visual phenomena ('the red does come above the blue') while Ingrid talked about displacements using various formulations ('between two pieces' and in a following sentence 'between two dots'). Probably she meant the same thing, but this made communication more difficult and probably did not strengthen their understanding. Due to the sequence of inscriptions of time series to two-dimensional graphs, it was still possible for the two students to talk about events and to trace their origin. Especially for these students, the class discussions afterwards and the activity about the bungee jumper were essential in gaining understanding and confidence in using the specific language and concepts that accompany the discrete and graphic models of motion.

6.5 Summary

In this final section of chapter 6, we summarize the main results of the teaching experiments. In chapter 7, we discuss these results and the consequences for an instruction theory on the teaching and learning of calculus and kinematics. In addition, we will focus in chapter 7 on the more general themes, such as emergent modelling and the problem-oriented approach, as included in the research questions.

Our experiences in both experiments indicated that a context with time series, like the hurricane-context, is suitable to make the students aware of the importance of being able to make predictions. The students became aware that displaying change of position is a possible way to proceed for predicting motion. Moreover, we succeeded in creating opportunities for the students to start reasoning with two-dimensional discrete graphs.

To a reasonable extent we succeeded with an computer tool in supporting the students to invent specific characteristics of these discrete graphs and connections between graphs of displacements and total distance travelled. However, for some students, the exact difference between a discrete graph of displacements and one of distance travelled was probably not completely clear even after the computer lesson. We offered the teacher a framework of reference to discuss the computer activities. This framework, together with a follow-up activity for the students was necessary to achieve classroom consensus about the meaning and use of the graphs and the accompanying language.

During the subsequent activities the students experienced the limitations of the discrete graphs and the distinction between instantaneous and average velocity. We succeeded in providing the students with a motive to look for other ways to display change. Unfortunately, the students' solutions were scarcely discussed in the class and as a result we did not get a good picture of whether the graphs of average velocities were a sensible way to proceed for the students. In this case, we had underestimated the difficulty of the intended transition to continuous models, and we could not provide the teacher with the information required in order to teach in a problem-posing fashion. The next chapter contains suggestions for this transition to continuous models and continuous motion graphs.

We succeeded in creating opportunities for the students to interpret velocity-time graphs by using bar-graph approximations of piecewise constant average velocities. The context of Galileo's assumption on free fall appeared useful for approximating the distance travelled of an accelerating motion. We are of the opinion that the transition from discrete measurement values to working with graphs of average velocity is a suitable one. However, the association with area under the velocity-time graph did not emerge in the students' work in either of the experiments.

We did also succeed in preparing the students to extent their kinematical reasoning to continuous distance-time graphs and to provide them with motives to start reasoning with vertical and horizontal intervals in these graphs for determining velocity. The students did draw tangent-like continuations for approximating the continuation of the graph if the velocity did not change after a specific point. We did not sufficiently succeed in the transition from a continuous distance-time graph to reasoning with mathematical formulas and their graphs. The next chapter discusses ways to improve this transition.

The graphic and dynamic picture provided by a computer tool and the language developed while working with this program appeared to be useful for the following lessons. They supported class discussions and the development of the mathematical and physical concepts.

Concerning the central issue of the instructional sequence – describing and predicting change – it was found that reflection on the corresponding modelling process can be improved.

7 Conclusions and discussion

This research project aimed at making an empirically grounded contribution to a local instruction theory for the teaching and learning of calculus and kinematics. The main research question was:

How, and to what extent, can the teaching and learning of the principles of calculus and kinematics be integrated in a guided reinvention course on modelling motion using computer tools?

This course should prevent the conceptual problems that students have with these topics, as outlined in chapter 2. For the design of the course we used the design heuristics associated with emergent modelling and problem posing, and computer tools to support students' inventions. The approach was further inspired by a study of the historical development of calculus and kinematics, and by theories on symbolising. Consequently, on a more general level, the aim of this research was to contribute to the understanding of these design heuristics, of the role of computer tools in education, and of guided reinvention as a paradigm for the teaching and learning of mathematics and physics.
In chapter 3, the main question was split into two research questions. The answers to the first question, concerning the intended local instruction theory, are described in section 7.1, while the second research question, concerning the choices made for designing the teaching materials, is answered in section 7.2. In section 7.3 we reflect on guided reinvention, the integration of mathematics and physics courses, symbolising, and the distinction between expressive and explorative modelling. We conclude this chapter with recommendations for future research and educational practices in section 7.4.
The answers to the research questions were, in most cases, empirically confirmed by our teaching experiments. In some cases our reasoning is hypothetical, based upon experiences we gained during the research. As a result, the conjectured instruction theory described in the next section has been partly confirmed by the teaching experiments, and is partly a reconstruction empirically supported by the teaching experiments.

7.1 Question 1: the emergence of a local instruction theory

We aimed to implement a process of teaching and learning which could be characterised as guided reinvention. This implies that the students' understanding of the intended notions should be rooted in, and developed from, their understanding of everyday phenomena. The guidance consisted of giving the students activities for which they had no standard procedures, that motivated them to expose and explore their own solution procedures, and to compare these to a general problem which needed to be solved. Solving the general problem led them to organising and modelling activities for describing and predicting motion. These activities shaped a trajectory for the learning of calculus and kinematics. The first research question concerns the conjectured local instruction theory:

1 *How can students develop the basic principles of calculus and kinematics in a process of teaching and learning that can be characterised as guided reinvention?*

In this section we draw conclusions on *how* students can acquire the notions aimed at in a dynamic growth process in which symbols and meanings develop together. This process is related to the means of support for achieving it: the instructional sequence with the student activities, the guidelines for the teacher and the tools provided.

From displacements between successive positions to trace graphs as representations of changing position with changing velocity

The first step in the trajectory was to foster students' thinking about change of position as a measurable quantity signifying motion, because change of position provides the imagery needed to make trace graphs. For this step we presented students with a context and a problem that led them to reason with displacements.
We chose a weather context in which we used the successive positions of a hurricane. Questions for extrapolations of the hurricane's trajectory guided the students to look at and analyse displacements between successive positions. This enabled the students to take displacements at equal time intervals as a basic notion for describing the hurricane's motion (section 6.2.1). This experience was in line with observations by Boyd & Rubin (1996), who found that displacements appeared to be a basic structuring element for describing motion and making predictions.
We conclude that these activities are a good starting point for the learning of calculus and kinematics, because they focus the students on graphical characteristics of descriptions of motion. Making predictions in such a context requires the students to co-ordinate patterns in successive positions.

From trace graphs to two-dimensional graphs

The next step was to describe motion using trace graphs. One-dimensional trace graphs were introduced after discussing the use of displacements in making predictions. These trace graphs signified successive displacements at equal time intervals, and required them to display the patterns in the displacements. Co-ordinating these patterns encouraged the students to invent two-dimensional graphs, with successive displacements drawn next to each other, or graphs with the displacements added up and displaying how the total distance travelled developed. We found that some students used only the last displacement in making their predictions on the course of the hurricane, while others took the pattern of increasing displacements into account (see section 6.1.1). The differences between constant velocity and increasing patterns were used by the teacher to pose the problem of displaying patterns in trace graphs.

In the next activity (the cheetah chase) students were free to develop their own strategies for comparing the motion of the two animals. Some students pictured the displacements with two-dimensional discrete graphs. We did not see the variety in students' inventions found by DiSessa et al. (1991), probably due to the students' previous coursework and the purpose of this activity. The teacher was able to discuss the students' reasoning with the whole class (section 6.4.1) and the reasoning with the graphs and their characteristics acquired a central position. All the students seemed familiar with the signifying domain, in this case the displacements in trace graphs, and with the problem for which the two-dimensional graphs were a solution (how to picture and reason with displacements and their patterns).

We conclude that predicting change with trace graphs and an invention-oriented activity supports the students in the transition to two-dimensional discrete graphs. The suggestions in the teacher guide for discussing patterns in displacements with the whole class offers the teacher opportunities for class discussions and ways to proceed in line with the envisioned learning trajectory.

Finding relations between graphical characteristics and properties of motion

After introducing discrete two-dimensional graphs, investigating various situations led the students to start using these graphs as a model for reasoning about mathematical and kinematical notions and relationships. The imagery underlying the emerging notion of velocity and changes in velocity was the length of, and change in, displacements in successive time intervals. This notion became related to patterns in two-dimensional discrete graphs of displacements and distances travelled. Constant velocity was displayed with a constant-displacement graph and a linear distance-travelled graph. Moreover, students came to understand the mathematical relation between these two graphs as taking sums and differences, a relation where the average displacement played a central role. We expected this imagery to prevent the students' potential misinterpretations discussed in chapter 2 (page 25).

A consequence of the students' freedom to develop their reasoning was that time had to be allotted to discussing and learning from each other's findings. The results of their investigations were discussed to reach a class consensus on the emerging notions. It was helpful for the teacher to guide and focus such a discussion using an activity in which the students could present their reasoning with the graphical characteristics. During these investigations and class discussion, the quantities of velocity and distance travelled became related through the discrete graphs, although velocity was still directly related to identifiable displacements in corresponding time intervals and was not yet being used as a compound quantity.

However, letting students investigate many situations is time-consuming. Therefore we used the computer tool Flash for supporting their reasoning on various motion situations represented by stroboscopic pictures. In just one lesson, the tool afforded all the students to relate specific kinds of motion with characteristics of graphs of displacements and distances travelled. Some students returned occasionally to the stroboscopic picture, while others started to reason with only the two graphs (section 6.4.2). They realised that clicking on successive positions in the picture in Flash was equivalent to measuring displacements. Gradually, in subsequent activities, graphical reasoning replaced the use of identifiable displacements. We observed that reasoning with graphs started to signify a way of describing motion and making predictions.

The differences between the students during the computer activities required a teacher-guided discussion about the graphical characteristics in relation to describing motion and making predictions. This discussion was needed to reach a consensus about the results and the way to proceed. An activity in which students had to describe and present a specific motion (of a bungee jumper) supported this discussion. The presentations and the accompanying discussion guided their thinking towards using graphical characteristics for reasoning about motion (section 6.4.1). We conclude that such a discussion is useful and such an activity supports this discussion.

In all the teaching experiments, some students invented a compensation strategy for finding and using the average displacement which was related to Oresme's middle speed theorem (page 87). Discrete constant displacements signified constant velocity, and this velocity was the average velocity when displacements in a displacement-graph above and below the constant displacement outweighed each other, and added up to the same total distance travelled. The accessibility of such a graphically supported compensation strategy was also identified by Bakker (2004) in a study on statistics education.

The following activities focused on change in small time-intervals. The students noted that instantaneous velocity could not yet be determined, because it was still related to an identifiable time interval and they did not have enough measurements in the discrete case. The notion of instantaneous velocity could have been problematised then, by using the students' knowledge of and experience with values from a speedometer in a car, motorbike or bike. Their knowledge of the characteristics of the discrete graphs provided a framework of reference for the imagery in the continuous case, such as the interpretation of linear continuations and points of intersection.

From discrete graphs to continuous models of motion

At this stage we planned to make the transition from discrete to continuous models of motion. We were not able to collect evidence for students' contributions on the need for this transition (section 6.4.1). However, bar graph approximations of continuous velocity-time graphs signified displacements in corresponding time intervals for the students. They worked with these bar approximations and their reasoning – based upon displacements – helped them to understand and manipulate continuous velocity-time graphs and the difference between instantaneous and average velocities. An example was their flexibility in dealing with the middle speed theorem (section 6.1.3).

We did not see students using incorrect dimensions for quantities like cm^2 for distance travelled via the calculation of the area under a velocity-time graph (page 21). Constant average velocity in a velocity-time graph signified a horizontal line for the students, where velocities below and above the line outweighed each other and added up to the same total distance travelled.

The transition to continuous models and their graphs can be improved by a more explicit discussion of the relation between proportional reasoning (e.g. Galileo's hypothesis for free fall) and data-based reasoning. The historical interpretation of instantaneous velocity (70 km/h means that if you would maintain this velocity for one hour, you would cover a distance of 70 km) can also be used for such a focus on proportionalities. Both the mathematics of models of proportionalities (tables, graphs and formulas), and the differences with discrete graphs of measurements can be exploited to support students in understanding the nature of continuous models and finding ways of reasoning with them. This seems to be needed for evoking the feeling that the use of continuous models is a promising way to proceed.

The transfer of mathematical notions from mathematical models to applications – and vice versa – might be encouraged with an explicit focus on, and use of, analogy reasoning. Kaper & Goedhart (2003) described using analogies of structural features in the field of chemistry education. They found that analogy reasoning could be used productively for students to construct an analogy between the structural features of two domains. Analogies between reasoning about total distance travelled according to a continuous model of constantly accelerating velocity and reasoning with displacement graphs and graphs of average velocities might be used to offer opportunities for the students to construct these similarities themselves.

Constructing the difference quotient as a measure for average velocity and the difference between average and instantaneous velocity

The students' use and understanding of intervals in continuous graphs was built upon their reasoning with discrete graphs. Linearity in continuous distance-travelled graphs signified linearity in the discrete distance-travelled graphs with constant dis-

placements (as vertical increments) in equal and successive time intervals (displayed horizontally). As a consequence, linearity in continuous distance-travelled graphs also became related with constant velocity through the discrete graphs.
The reasoning with lengths and changing lengths of intervals in the discrete graphs, and their understanding of average velocity as a compound quantity, permitted the students' use of difference quotients as a measure for average change. As a result, *parts* of these continuous distance-time graphs signified a *vertical displacement* together with a corresponding *horizontal time interval*. The quotient of these two dimensions resulted in the average velocity over the actual time interval. Students could be supported at this stage in the learning process by being presented with a context together with a continuous distance-time graph and posing questions related to the motion (e.g. Mr Bommel's driving activity, see page 145).
In addition, the approximation of instantaneous velocity built upon linear continuations of a distance-time graph (a continuation with *the* velocity at that very moment). Drawings which signified this reasoning could be evoked by an open-ended problem, where a continuous distance-time graph was presented with a problem on the velocity at a moment when the velocity was changing (e.g. exceeding the speed limit). The students did not yet have the mathematical tools for solving these problems at their disposal. They drew tangent-like *straight* lines to denote the continuation of the graph if the velocity was not changing from a certain moment. Most students answered that the precise value of this velocity could not be determined and only a few invented an approximation process by themselves (see page 148). The teacher could then have used the variations in their reasoning and drawings to discuss the approximation process and to build upon their inventions.
We conclude that context problems with continuous graphs and questions for instantaneous change supported students in inventing ways to reason with these graphs. These inventions provided for the imagery for the difference quotient as a means for calculating average change and approximating instantaneous change.

The difference quotient in graphs of mathematical formulas

The last step in our trajectory concerned the transition to reasoning with graphs of mathematical formulas. We conjectured that approximations of situational distance-time graphs with graphs of mathematical formulas could be used for this transition. A computer program (Slope) was used to present students with various investigations and to afford them the use of chords for approximating instantaneous change. However, during the teaching experiments, their reasoning with the continuous distance-time graphs was not discussed.
As a consequence, students seemed to be familiar with the signifying domain (graphs and their intervals), but most of them did not have a clue how to use the graphical tools or the related $\Delta y/\Delta x$ notations together with a formula $y = f(x)$ in the

computer program for approximating instantaneous change. The computer tools were not compatible with the students' reasoning and the activities were not as productive as the first computer lesson with the Flash program, in the sense that the students did not develop the use and characteristics of the difference quotient during their investigations. However, we noted that after this computer lesson their knowledge of the approximation process and the dynamic visualisations in the software helped them in reasoning with the difference quotient.

We conclude that in the transition of students' reasoning with data-based distance-time graphs to graphs of mathematical formulas, more attention is needed for approximation processes in data-based graphs. A discussion of ways to approximate instantaneous velocity can offer opportunities for explaining the limitations of these approximations. During this discussion, when all the students understand the approximation process and the need for values over small intervals, the use of mathematical formulas as models of data-based graphs, or parts of them, can come to the fore. In this way consensus can be reached on the similarities between the graphical reasoning in both cases, and in the understanding of the notational extensions as a result of using a formula.

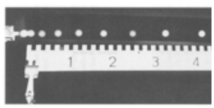
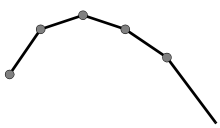
Finally, we conclude that the students' activities with a computer tool like Slope are helpful in supporting the teaching and learning of the approximation process with the difference quotient. It offers a dynamic graphical visualisation that can be referred to during following lessons by simulating the dynamics with gestures and talking about 'the blue triangle'. This also fosters reasoning about the approximation process in situations where the teacher and students do not have a computer available.

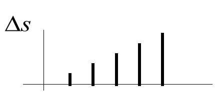
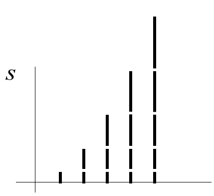
Summary

The relation between the inscriptions, the intended activities and conceptual development of the students can be summarised in a table. Gravemeijer et al. used such a table for describing a learning trajectory on measurement and flexible arithmetic (Gravemeijer et al., 2003).

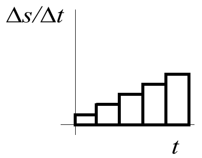
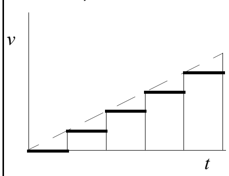
To emphasise the 'tool' character of the inscriptions, they referred with the label 'tool' to these physical representations. The 'imagery' column refers to the history that frames students' perception. By providing this column we make it plausible that students understand the tool and how the tool derives its meaning. The activities and discussions that address specific concepts should result in motives to proceed. In our approach we focused especially on these motives, and we have therefore added them to the table.

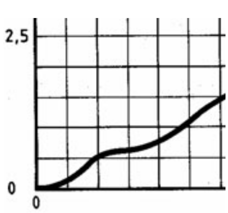
The table summarises (*i*) how students are expected to act and reason with the tools, (*ii*) how an activity relates to preceding activities, and (*iii*) the conceptual development aimed at by that activity.

<table>
<tr><th>tool</th><th>imagery</th><th>activity</th><th>concepts</th></tr>
<tr><td rowspan="2">time series
(e.g. satellite photos & stroboscopic pictures)
</td><td rowspan="2">real world representations signify real world situations</td><td>predicting motion
(e.g. weather)</td><td>displacements in equal time intervals as an aid for describing and predicting change</td></tr>
<tr><td colspan="2">should result in a feeling that the ability to predict motion with discrete data is an important issue</td></tr>
<tr><td rowspan="2">trace graph of successive locations
</td><td rowspan="2">signifies a series of successive displacements in equal time intervals</td><td>compare, look for patterns in displacements and make predictions by extrapolating these patterns</td><td>displacements as a measure of speed, of changing positions, but difficult to extrapolate</td></tr>
<tr><td colspan="2">resulting in a willingness to find other ways to display displacements for viewing and extrapolating patterns in them</td></tr>
<tr><td rowspan="2">displacements in tables and graphs.

focus on reasoning with the graphs in Flash

</td><td rowspan="2">signifies patterns in displacements of trace graphs
(and cumulative)</td><td>compare patterns and use graphs for reasoning and making predictions about motion (also at certain moments: interpolate graphs)

refine your measurements for a better prediction: displacements decrease</td><td>displacements depicting patterns in motion;
constant displacements = constant velocity = linear (discrete) graph of distances travelled;
precise predictions of instantaneous velocity cannot be made with discrete information;
a displacement represents a constant velocity on a time interval.</td></tr>
<tr><td colspan="2">should result in the need to know more about the relation between sums and differences, and in the need to know how to determine and to depict velocity</td></tr>
</table>

<table>
<tr><th>tool</th><th>imagery</th><th>activity</th><th>concepts</th></tr>
<tr><td rowspan="2">graphs of average velocities

reasoning and calculations with intervals Δs and Δt</td><td rowspan="2">average velocity during a time interval signifies a displacement in that time interval (shift from length to area as a representation for displacement)</td><td>with this type of graph students depict change of velocity and patterns remain visible with as many measurements as you have</td><td>average velocity as a compound quantity; graph of successive average velocities with a continuous horizontal time-axis; where the areas of the bars signify corresponding displacements</td></tr>
<tr><td colspan="2">this should result in an interest in knowing what to do when you want to test a hypothetical continuous model</td></tr>
<tr><td rowspan="2">bar graphs of piecewise constant velocities (and lines of summit)
</td><td rowspan="2">each bar signifies a displacement

the continuous graph signifies a constantly changing velocity at any instant as a (hypothetical) model of a motion that depicts instantaneous velocities</td><td>try to describe free fall;
check Galileo's hypothesis with discrete approximations of the continuous model;
approximate continuous distance- time graphs with displacements</td><td>a velocity-time graph can be approximated with piecewise constant velocities; where areas of bars represent displacements;
insight into the link between area, velocity and distance covered</td></tr>
<tr><td colspan="2">should result in asking whether this approximation method can be applied in all situations and in ways for displaying constantly varying distance travelled</td></tr>
<tr><td>a continuous graph of distance travelled
</td><td>signifies a constant change of position and refers to the cumulative displacement graph in Flash</td><td>reasoning with graphs about average and instantaneous velocity
(e.g. in the context of breaking the speed limit)</td><td>straight lines in distance graph: height = Δs, width = Δt, slope ($\Delta s/\Delta t$) = measure for average velocity on time interval</td></tr>
<tr><td></td><td></td><td colspan="2">results in asking how to be more precise about instantaneous velocity and how you can improve approximations when you have a graph and a (hypothetical) formula at hand</td></tr>
</table>

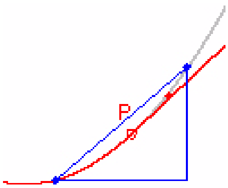
tool	imagery	activity	concepts
computer program Slope with linear continuation 	signify what happened when velocity no longer changes from a certain moment (building upon experiences with Flash and definitions of velocity using potential displacements)	find instantaneous velocities from distance-time graphs	difference quotient as a measure of the slope of a line and an aid for finding slopes of tangents

table 7.1 A summary of the local instruction theory

The teaching experiments for this trajectory were limited to eight to ten lessons, which was not long enough for all the discussions we had planned. Parts of the trajectory (especially the discrete graphs) can be dealt with in an earlier grade (e.g. see Yackel et al., 2003). This will then be recapitulated in grade 10, but more rapidly than during our experiment. In addition, the students will continue lessons with calculus and kinematics after this sequence. Notions like acceleration, frame of reference, limit concept and derivative function should follow these basic principles in pre-university courses physics and mathematics.

Concluding remarks

We reached a better understanding of the teaching and learning of the basic principles of calculus and kinematics; an understanding which is reflected by the presented contribution to an instruction theory. Nevertheless, this contribution is based upon two teaching experiments in three schools. More experiments are needed to establish a robust theory which will hold up in different educational situations.

In this reinvention process we did not see wrong interpretations of velocity-time graphs, nor did we see wrong interpretations of points of intersection in velocity or distance-time graphs. However, we must note that we did not investigate the students' reasoning in various situations. The way students reasoned and developed the notion of difference quotient and their use of motion graphs supported our conjecture that this learning trajectory led to an understanding that prevented the conceptual problems described in chapter 2.

In the instructional design, the concepts were developed from students' common sense reasoning about displacements in trace graphs. The learning of these concepts built upon their daily life reasoning and the imagery for new concepts was rooted in their experiences and provided the basis for the following steps in the teaching sequence. The lessons were more successful when we designed these steps well.

This led to a framework of graphical and kinematical relations in which slopes and areas of graphs came to the fore as mathematical entities for describing and predicting motion.

Finally, we note that our focus on teaching materials as a means of support for creating teaching and learning processes runs the risk of neglecting other aspects required for realising a guided reinvention process. For instance, it should be possible and the students should experience it as important, that they reveal their ideas and discuss each other's reasoning in pairs, small groups, and class discussions. Moreover, other unanticipated events may occur influenced by the students or by lessons before or after the mathematics lesson. Such events affect the teaching process, demanding the teacher's time and skill to manage them on a meta-didactical level, in line with the intended learning trajectory. In our analysis we focused on content-related ways for the teacher to support the students' reasoning. We note that teacher interventions aiming at encouraging interaction processes between students are also important and should not be neglected (Dekker & Elshout-Mohr, 2004).

Our intended teaching and learning processes were supported by emergent modelling, a problem-posing heuristic, and IT tools. In the next section we will reflect on how these choices benefited our teaching design.

7.2 Question 2: design heuristics and computer tools

We aimed at a process of teaching and learning in which students contributed to a dynamic growth process of emerging symbols and meanings. To achieve such a process, we explicitly chose two design heuristics, and we integrated the use of computer tools in our teaching materials. The second research question evaluated these choices and is dealt with in this section:

2 *To what extent does the course of this process empirically support the adequacy and the understanding of our choices: the role of computer tools and of the design heuristics related to emergent modelling and problem posing?*

Emergent modelling (EM) refers to a process of teaching and learning in which models emerge from informal solution strategies, and the use of the models supports the emergence of formal mathematics. The related design heuristic – *looking for models that can come to the fore as models of a situation and can emerge into models for mathematical reasoning* – played a key role in accomplishing our teaching design. This heuristic involved a historical analysis of the development of calculus and kinematics, and mental experiments about both the teaching possibilities and the students' learning.

The problem-posing heuristic (PP) involved evoking the need for new tools or concepts by problematising students' experiences or activities with respect to a general content-specific goal. For this the teacher needed to have an idea of both how the students might reason, and how this reasoning could be used for evoking productive questions and suggestions with respect to the over-arching goal.

In analysing the adequacy of the design heuristics and the use of information technology we focused on three cases: (*i*) the transition from trace graphs to two-dimensional discrete graphs, (*ii*) the transition to the use of continuous models, and (*iii*) the transition from drawing linear continuations in generated data or fictive continuous graphs to using graphs of formulas and difference quotients. These cases contained crucial steps in the instructional sequence.

We compared the success of events in the teaching experiments and tried to analyse whether the differences could be explained by the choices for our design. On a wider level, we reflect upon the students' learning process we had achieved and whether it can be characterised as emergent modelling and problem-posing.

The transition from trace graphs to two-dimensional discrete graphs

In the first teaching experiment, the students had to use two-dimensional discrete graphs in the Flash program immediately after working with trace graphs of the hurricane's motion. We noted that they needed time to discover the significance of the two-dimensional graphs provided by the software (see section 6.1.2).

The notion of emergent modelling led to designing an activity which would support students' constructions of and reasoning with two-dimensional graphs, building upon their reasoning with trace graphs. We therefore inserted an activity where students had to compare data of two different kinds of motion (the 'cheetah chase' see page 163). We also provided the teacher with the students' possible reasoning, and showed how this could be used for guiding them towards two-dimensional graphing by focusing their discussion on patterns in displacements.

The notion of problem-posing led us to add information for the teacher to our materials for guiding classroom discussion so that students could reach a position where they understood that using two-dimensional graphs was a sensible way to proceed to the main goal: how to describe motion for making predictions (see section 6.3).

In the second teaching experiment, the inserted activity elicited students' reasoning on the relations between motion and graphical descriptions, precisely as anticipated. The classroom discussion based upon students' contributions provided the imagery needed for the Flash activities (see section 6.4.1). We observed that the teacher was able to regulate the discussion in such a way that students understood that displaying and investigating patterns in displacements was a sensible way to proceed for describing and predicting motion. The students' suggestion of using two-dimensional graphs was welcomed by the teacher as a valuable way of reasoning, and we concluded from their contributions and questions while discussing these graphs, that it was accepted by all the students.

During the Flash activities in the second teaching experiment, all the students used the two-dimensional graphs meaningfully in their reasoning about the motion situations. A reasoning which started with clicking in the stroboscopic photograph and describing change in successive displacements with words *and* gestures. The focus of their words and gestures changed from these individual displacements to patterns in and between the two-dimensional graphs for reasoning about motion. We conclude that the design heuristics were helpful, both in analysing the experiences with the first experiment and in improving the teaching materials.

The transition from discrete graphs to the use of continuous models

The transition from reasoning with discrete motion graphs to using continuous models was prepared in situations with data of successive positions. Students were supposed to understand the change of perspective in using continuous models and, for instance, note the difference between continuous lines in discrete graphs and in graphs of continuous (hypothetical) models. We did not succeed in making these aspects of the transition evident during the first teaching experiment. We considered that this transition was related to the change in reasoning with velocity as a proportion between identifiable values, to viewing velocity as a compound quantity which has an instantaneous interpretation.

We were not sure whether the students' reasoning during the computer activities with discrete graphs resulted in the construction of the intended notions. Emergent modelling brings about a gradual process in which ideas emerge during the students' activities, where students have different histories, use different words, and reason at different speeds. Especially after a lesson spent completely on computer activities, a classroom activity was needed to see where students had got to, to let them explain what they had learned, and to justify this knowledge with respect to the global question. Moreover, the teacher should use this wider question to create consensus about the notions learned to frame their intentions for the way to proceed. We therefore added a graph construction activity after the computer lesson with Flash (the bungee jumper activity).

This construction activity resulted in group work and a classroom discussion which addressed precisely the notions we aimed at. Students showed their discrete graphs, exposed their reasoning, and had lively discussions about each other's contributions, while the teacher was able to use their contributions to reflect on the computer activities and guide them to the intended graphical relations (see page 174).

In addition to the possibilities, the constraints of discrete graphs should also emerge in the next activities. One of these constraints yielded the interpretation of the horizontal axis, from numbers of measurements in discrete graphs to time in continuous graphs. We were not clear about this difference in the instructional sequence. This prevented the students focusing on the interpretation of the horizontal axis.

In the second teaching sequence, we added information for the teacher about ways to problematise instantaneous velocity (see section 6.3). This improvement in the teaching design was inspired by the problem-posing approach.
The new activities on the constraints of discrete graphs and the notion of instantaneous velocity did not take place as we had planned. This was partly due to the teacher's problems in managing the classroom discussion during this particular lesson (see section 6.4.1). The students hardly had time to express their thinking, although we saw possibilities afterwards in their written material.
We conclude that a reinvention process for this transition from discrete graphs to continuous models needs more time than just one or two lessons. The design heuristics helped us to analyse and improve our teaching sequence. Activities with clear class co-operation and discussion, like the graph construction activity, are important for making the crucial transitions in the dynamic growth process. We cannot conclude that we succeeded in designing a teaching process in which the whole class understood the need for continuous graphs of motion.

The transition from linear continuations to using graphs and formulas

In both experiments students had difficulties making the transition from the activity with a data-based continuous distance-time graph to working with graphs of formulas and linear continuations.
During the subsequent lessons, the dynamic images in the computer tool Slope appeared to serve as useful imagery underlying the approximation processes with a difference quotient and the use of the graphing calculator. Both the teacher and students used these dynamic images while discussing technical aspects of the difference quotient and the operation of the graphing calculator. However, the references we observed were confined to remarks during classroom discussions (e.g. see page 152). We felt more emphasis could have been put on the transition from the graphical tools in Slope to establishing notions like the difference quotient in relation to the use of the graphing calculator. Theories on the use of personal devices like the graphing calculator suggest a parallel development in the mathematical notions aimed at, and the use of these devices in classroom teaching processes (Drijvers, 2003; Artigue, 2002).
We conclude that the preparation of the notion of linear continuation in the discrete case and the activities with Slope, provided the students with imagery for understanding the difference quotient as a tool for approximating instantaneous change. However, during the activities with Slope in both our teaching experiments, students posed questions like 'what do they want us to do?', which was precisely the kind of question we had tried to avoid (see section on interpretative framework in chapter 4). Our analysis indicated that this kind of question was posed because we had not supported the students sufficiently in building their reasoning about graphs of for-

mulas upon their reasoning with data-based graphs. In this transition we did not succeed in achieving a teaching and learning process that could be characterised as a problem-posing approach. A content-specific motive has to be found for making the transition from linear continuations to using graphs and formulas, as in the continuous model for free fall drawn from Galileo's hypothesis.

Conclusions

Before considering each part of the second research question concerning our choices, we will quote the related questions from table 3.1 (see page 65).

2 EM: Does the previously planned sequence of graphical tools fit students' thinking and foster advanced reasoning by a shift from model-of to model-for?

On a general level, we conclude that we achieved an emergent modelling process, especially in the shift from modelling motion to reasoning with two-dimensional discrete graphs. This shift involves a transition *from* viewing discrete graphs as tied to identifiable displacements in stroboscopic pictures *to* viewing these graphs as entities in their own right, which support reasoning about change in velocity and the relation between constant velocity and distance travelled. These graphs remain their kinematical interpretation, but are no longer referring to identifiable displacements. The students' use of these graphs was constructed from their network of mathematical and physical knowledge, and the connections extended their ability to view motion from a mathematical and physical perspective and to understand velocity as a compound quality. This shift from graphs as referents to graphs as mathematical entities is related to the *model of* to *model for* transition described in section 3.5.1. We conclude that this design heuristic fosters the emergence of students' understanding of the intended mathematical and kinematical relationships.

The dialectic relation between the development of graphs and of conceptualising motion implies that the teacher has to consider both ways when dealing with teaching decisions. We have already emphasised the importance of awareness in the parallel development of the language used by the students. When the teacher gives students construction space in modelling activities, they tentatively invent inscriptions and use an 'impure' mathematical language for expressing their ideas. For instance, the way they drew continuous lines in discrete graphs and talked about these lines was not correct from a mathematical perspective, but it fitted the students' – productive – reasoning, signifying patterns in discrete graphs and their characteristics.

In an emergent modelling approach the teacher has to be aware of the possible ways in which inscriptions *as well as* the related language emerges in order to offer appropriate guidance in class discussions building upon students' contributions.

2 PP: Are students aware of a global problem that is being solved, and do the local problem situations provide the students with content-specific motives to proceed in the intended direction?

Our second design heuristic is based upon achieving a problem-posing approach. This heuristic appeared to be valuable in improving the teaching materials, especially in designing means to support the teacher in class discussions. The students' activities were especially productive when the teacher arranged the discussions of their contributions in such a way that *they* posed the problems that had to be solved. In doing so the students exposed their understanding of content-specific aspects of the problem and were motivated to solve them. Those cases in which the students were less motivated and tried to guess what was expected from them, were when they had not previously contributed sufficiently to the way to proceed. However, when teachers had to act under time pressure, we saw that these reflections were the first to be dropped.

The use of a wider question appeared helpful in getting students to explicate what they learned at a reflective level, for justifying the following steps, and for providing content-related motives to proceed; motives which frame the students' intentions during the subsequent activities.

We underestimated the difficulties for implementing the problem-posing approach in our instructional sequence. The guiding theme *how to describe change for predictions?* was not always functional for the students. The course of teaching and learning in our experiments can only be characterised as problem posing to a limited extent.

The problem-posing heuristic exceeds the design of student activities and guidelines for a series of lessons. It demands a change of attitude to act in an everyday school situation on a meta-didactical level. Teachers have to be aware of the danger that they can easily get students to look at a problem from a specific point of view, while at the same time the students still have to invent the knowledge that shapes that point of view. For investigating the students' thinking and for guiding their perception and intentions, the problem-posing design heuristic proved to be valuable.

2 IT: Do the representations in the computer tools fit prior reasoning and how do they afford advanced reasoning and sense-making?

We investigated the didactical use of two types of computer tools. The first tool was developed to enable students to investigate many cases and to afford the construction of a framework of graphical relationships for reasoning about discrete motion situations. This construction built upon students' previous knowledge and expressions, instead of on discoveries from guess-and-check strategies. The students' investigations and constructions were productive as a result of the compatibility of the tools with the students' reasoning. Students developed graphical reasonings through words and gestures with the tools provided. Consequently, we advocate students working in pairs with the software. Moreover, we saw a wide range in the quality of students' reasoning with the tools.

This use of computer tools seems to correspond with helping students to develop relationships by linking different representations. However, we prefer to speak of creating opportunities for the students to construct a variety of mental ideas by investigating many cases.

The second type of computer tools was to visualise an interactive manifestation of the model. This dynamic image provided something concrete to talk about in class discussions and signified the approximation process with a difference quotient. We conclude that computer activities can create a shared dynamic imagery of a specific notion which can be referred to afterwards. Such a dynamic image functions as a generic organiser (Tall, 1996), a generic example that 'embodies' a general property or process.

Finally we conclude that activities with computer tools need careful preparation beforehand, and need reflection afterwards. In the second teaching experiment, we added an activity before the Flash lesson to support the students' inventions of, and reasoning with graphs. The variety in solutions enabled the teacher to discuss graphical models of motion and to reach consensus about the way to proceed. As a result, the tools in the software were compatible with the students' current reasoning. Their understanding of the graphs provided by the tool, was an important condition for their meaningful and flexible reasoning with the tool.

An intended model shift – associated with emergent modelling – did take place at a different pace and to a different extent during the computer activities. Consequently, the small groups' activity afterwards, together with the students' presentations, class discussion and teacher's guidance promoted the whole-class understanding of the acquired notions. As an aside we note that the fact that computer rooms have to be reserved in advance, rather limits the possibilities for sufficient preparation of the computer activities.

7.3 Discussion

The *a priori* points of departure were: to achieve a learning process of guided reinvention, integrate calculus and kinematics, and support the students in developing symbolising with a series of inscriptions. In this section we reflect on these points.

7.3.1 *Guided reinvention*

By guided reinvention we refer to a learning process in which students experience something as if they had invented it themselves. There are two observation criteria related to such a process: (*i*) students' meaningful perception of and reasoning about the problem situations provided, especially those situations for which they did not yet have a standard solution procedure (*invention*), and (*ii*) the possibilities for the teacher and subsequent activities to use students' reasoning in line with the intended trajectory (*guidance*). In addition, we paid special attention to offering possibilities for students to pose the questions that have to be solved along the intended trajectory by adopting design heuristics related to a problem-posing approach (*supporting their awareness of a reinvention process*).

We concluded that we succeeded in most cases in creating opportunities for students to make inventions which contributed to the intended learning processes. We used contextual problems that varied from day-to-day life (e.g. predicting the track of a hurricane with satellite photographs) to more scientific situations (e.g. approximating maximum slope); situations for which they did not yet have standard solution procedures. Students invented inscriptions and reasoning which could be used by the teacher for supporting the development of (*i*) the notions of distance travelled and velocity, and (*ii*) the mathematical tools for describing change and making predictions. The students' reasoning appeared particularly meaningful and productive when it referred to the main question on predicting change.

The teaching experiments showed that it was difficult to build upon students' contributions in making the transition from data-based graphs to continuous models. In these experiments we did not succeed in providing the students with problems where it made sense for them to extend their reasoning from data-based graphs to using formulas. However, as a result of the experiments we were able to propose ways for improving this transition.

In relation with the use of computer tools we note that Flash enabled students to invent the possibility of reasoning with graphical relations for describing and predicting motion. The use of Slope could hardly be characterised as guided reinvention (see also section 7.3.4), but the students' experiences with Slope provided for a dynamic imagery, which they used to understand and trace meaning while working with the difference quotient.

The historical development of calculus and kinematics provided us with indications for the use of emerging models and of the related conceptual development. However, these indications proved limited for planning and dealing with the development of students' thinking and language. We had underestimated the importance of having clues about this development in order to offer the teacher the information needed for guiding class discussions.

We note that our aim is to describe a trajectory which supports the teaching and learning of calculus and kinematics in a class situation. This does not imply that individual learning processes follow precisely this trajectory. However, for guided reinvention to work, teachers should have an image of the trajectory and their role (when to guide and when to give students freedom for invention?).

Finally, during the last few lessons of both teaching experiments, students and teachers focused on the instrumental use of the difference quotient and graphing calcula-

tor with mathematical formulas and their graphs. This was not surprising, because the experiments took place during mathematics classes and replaced studying chapters from mathematics textbooks. The teachers wanted to be sure that students had mastered the mathematics required. Such a focus on algorithms has the danger of disconnecting them from their roots.
In summary we note that we partly succeeded in realising a process of teaching and learning that can be characterised as guided reinvention.

7.3.2 *Integrating science and mathematics*

The integration of physics and mathematics in the teaching experiments was confined to the integration of kinematical notions in an instructional sequence for mathematics lessons. Ideally, this instructional sequence is acted out in both physics and mathematics lessons.
Our research indicated that students' conceptual problems in applying mathematical notions in other topics can be prevented by integrating the learning and teaching in an application. We have shown how the history of the intertwined development of calculus and kinematics provided ways of using emergent modelling in a teaching sequence. In our approach, the learning of calculus and kinematics was rooted in grasping and organising motion. The activities for making predictions and describing motion graphically helped the students to develop a notion of velocity as a compound quantity, which supported the notion of proportionality that underlies the understanding of the difference quotient as a measure for change.
However, both in the transition from discrete graphs to reasoning with continuous models of motion, and in the transition from a continuous distance-travelled graph to reasoning with graphs of formulas, students had difficulty in understanding and using structural similarities. Their reasoning mainly remained in the context of modelling motion, while the students were also supposed to use mathematical knowledge of proportionalities and of graphs and formulas. For experts these similarities are evident, but students still had to construct them. A conceptual analysis of ways of reasoning is needed for the instructional designer to offer opportunities for students to understand which knowledge to use, and to construct the intended notions. Such an analysis asks for content knowledge of these notions and of the students' current mathematical *and* physical reasoning. We advocate more emphasis in mathematics education for proportional reasoning with graphs and formulas in different disciplines.
Other possibilities for an integrated approach lie, for instance, in the field of discrete and continuous dynamic modelling. Such modelling activities for investigating the dynamics of various situations (population growth, cooling down, consumer-production dynamics) could be the source for developing both mathematical models *and* knowledge of their applications (e.g. Michelsen, 1998; Prins et al., 2003).

We do not believe that all mathematical topics can be developed through an integrated approach. Some topics are essentially the result of a process of organisation within or between mathematical systems or structures. In fact, the trajectory in this research should also be followed both by a series of lessons where the mathematics of change is developed as a generalising principle for many applications, and by elaborating the topic within a mathematical context (e.g. difference equations and derivatives, proving, the limit concept). In these subsequent lessons, motion and the related compound dimensions could still be used to trace meaning or to support new mathematical inventions. In addition, the kinematical notions addressed in this trajectory should be elaborated in a series of physics lessons to integrate and extrapolate the same notions in the relation with force and acceleration.
Finally, the integrated approach contributes to a trend within education to develop skills (general and topic-specific) through problem-oriented case studies. The experiences we gained in our research raised two points of concern. First, the teacher needs didactical knowledge of both disciplines for dealing with class discussions and guiding students' tentative ideas. Second, the period of integrated lessons should be alternated with topic-oriented lessons to develop relations with the topic's systematics: relations that (*i*) support its understanding, (*ii*) create possibilities to trace meaning, and (*iii*) provide opportunities for new inventions.

7.3.3 *Symbolising*

Mathematics often originates from observing and organising phenomena. However, the mathematical concepts (or constructions) that describe patterns and structures also exist independently of these phenomena. We develop and communicate these concepts with inscriptions. A knowledge of symbolising is necessary to understand how these concepts and inscriptions were invented, and to be sure that teacher and students communicate the same ideas. We concluded that, especially for mathematics education, we need to apply *that* knowledge to prevent students from acquiring an instrumental use of graphs or algorithms without understanding the concepts represented by those graphs and algorithms.
Assumptions about the benefit of emergent modelling, problem posing and the use of computer tools for education are related to semiotic notions about how students perceive and symbolise problem situations. These ideas aimed at a dynamic growth process from students' intuitions to the intended learning goals. In the design of the teaching sequence, we paid specific attention to the progressive development of inscriptions, imagery, activities in context and the parallel development of mathematical and kinematical concepts (see table 7.1). Each notion and inscription, developed within a context, provided the imagery needed for the next step in the conjectured local instruction theory. These choices proved to be useful in designing and analysing the teaching and learning processes.

The students' learning processes could have been analysed from a semiotic perspective. Semiotic frameworks for such analyses are Peirce's notion of diagrammatic reasoning (e.g. Bakker, 2004) and the more linear notion of chains of signification (e.g. Cobb, 1999; Gravemeijer & Stephan, 2002b). However, we analysed primarily the mathematical activity observed in classroom discussions and the students' written materials, and this resulted in empirically supported reasoning for the teaching and learning processes in the classroom. Future research is necessary to investigate and analyse the symbolising processes of individual students.

In the classroom learning processes, we noticed that students reasoned and wrote about mathematical and kinematical notions with a tentative language and inscriptions which were not as precise as the notions aimed at. Goldin (2003) described three main stages in the development of representational systems: (*i*) an inventive and semiotic stage, (*ii*) structural development and establishment of relationships, and (*iii*) an autonomous stage. In the autonomous stage the system can function flexibly in new contexts. During the inventive stage, students tentatively used inscriptions and language to communicate their developing ideas. We noted that this sometimes created differences in thinking and use of language between the students and teacher (e.g. discrete graphs and interpretations of a continuous line of summit). We saw that the teacher played an important role in this communication, which is also supported by Bauersfeld (1995) and Van den Boer (2003). The bungee jumper activity, where students had to present results of their group work (see section 6.3), is an example of providing students with the means to communicate about each other's findings and the teacher with means to guide the classroom discussion after a computer lesson.

We also considered the dynamic and interactive tools in the computer program Slope. We studied how the dynamics of the chord in a graph in Slope were referred to in the lessons thereafter, and found that these dynamics appeared to form a strong imagery supporting the understanding of approximation processes with a difference quotient. Bakker (2004) addressed a similar dynamic feature of computer tools in the context of a semiotic analysis.

Finally, nowadays much instrumental manipulation can be done by hand-held technological devices (like graphing calculators and computer algebra). The transition from using didactical computer tools in concept development to using such devices to solve problems demands an understanding of the way in which these devices become meaningful instruments (e.g. see Drijvers, 2003).

It is of increasing importance that students learn to recognise mathematical or physical structures in phenomena, and are able to translate these structures into symbolisations so that they can be dealt with scientifically. Acquiring these skills is precisely what the semiotic perspective in this research project aimed to foster.

7.3.4 *Computer tools: discovery versus invention*

In chapter 3 we described alternatives for the traditional transmission approach. We distinguished between the approaches which have much in common with ideas underlying discovery learning and those which stress the importance of using students' own inventions. This distinction was inspired by Doerr's (1997) description of exploratory modelling compared to expressive modelling or model building.

In discovery learning, the final model is introduced and linked with known phenomena in computer simulations or with other technological devices. The understanding of the final model is established through the students' exploration of the linkages. We suspected that such an understanding might be the result of trial and error strategies, without construction of the underlying concepts and without being sure that the students' understanding was rooted in related intuitions and experiences.

In invention-oriented approaches, the final model is the result of modelling activities in which students' inventions played a central role. The related inscriptions and language are progressively developed through these activities and tool-use supports students' inventions. Our approach achieved such a process in the use of Flash. We found similar results with the use of dynamic representations in computer tools for the learning and teaching of algebra (Boon, 2004; Doorman, 2004).

We noticed that the tools we provided for the students were not always experienced by *all* the students as tools for expressing their reasoning. Rather, for some students, the activities with the Flash and Slope software could be characterised as guided discovery. During the computer activities with Flash, these students were mainly engaged in trying to understand the link between the discrete graphs and their manipulations in the stroboscopic picture. However, their history and the imagery prepared by the preceding activities made it possible for them to trace meaning. In most cases, the students were able to do this by themselves, but in some cases, the teacher or an observer was asked for an explanation. In the latter cases, the students' previous experiences provided ways to guide them from those to understanding the intended notions.

It is difficult – maybe even impossible – to design learning processes for classroom situations in which all students experience their learning process as invention. However, a learning trajectory which supports invention and which makes it possible to trace meaning, provides the teacher with possibilities for guiding the students' reasoning. This can be realised when the tools in the software are part of an emergent modelling process and are – as much as possible – compatible with the students' current reasoning.

7.4 Recommendations

This final section contains recommendations concerning instruction theories, educational practices and the integration of science topics in secondary education.

Instruction theories and design research

This research contributes to a local instruction theory on the teaching and learning of calculus and kinematics. The trajectory starts with constructions of discrete two-dimensional graphs from time series. Returning to the origin of what is displayed in such graphs appeared to be useful in supporting students in their reasoning with intervals. We emphasize that a return to where mathematical inscriptions and concepts originate should take place more often in education, since students risk dealing with mathematical notions algorithmically and without true understanding. Algorithms should be practised, but not until the students are able to reinvent them by themselves (Freudenthal, 1973). We would add that our experience showed that they do not automatically then have the ability to reinvent the learned notions by themselves at a later stage. Students should be regularly asked to trace explicitly the meaning of mathematical notions and related language and inscriptions.

Our reasoning underlying the conjectured theory was mainly based upon the students' reasoning observed in class discussions and on students' written materials. Futher research is necessary to investigate and analyse whether the symbolising processes made by individual students really fit our proposed sequence. Moreover, further research on computer tool-use and the relation between invention-oriented and discovery learning should reveal how a teacher can ensure that students really understand the acquired skills, and how the underlying notions can be traced when students no longer understand the standard procedures.

The design research approach resulted in an *empirically based* contribution to a local instruction theory. However, experimenting in a settled school program limited the possibilities for our research and the teaching experiments were confined to a series of mathematics lessons. Moreover, it was sometimes difficult for the teachers to guide *all* the students during the lessons of 50 minutes in the intended way. We recommend future experiments where mathematics and physics teachers implement this instructional sequence in their lessons. In addition, these experiments are needed for the constitution of a *robust* local instruction theory for the learning and teaching of calculus and kinematics.

The local instruction theory, including the instructional activities, offered teachers a framework of reference for planning their lessons and their practical teaching (Gravemeijer, 2004a). Further research is needed to investigate which description of the instructional sequence, together with the underlying theory, can indeed be used as a means of support for teachers and for other parties who influence the course of affairs in education (de Lange et al., 2001).

Educational practices

We did not succeed in relating the students' work to a leading theme in all our lessons. Sometimes local problems in one lesson demanded all the time and attention available. In addition, the current trends in Dutch education for more autonomous learning by the students and more computer controlled assessment do not support the conditions needed for guided reinvention. Class discussions appeared to be essential for students to reach a consensus about their tentatively developed inscriptions and the related language.

In the trend towards autonomous learning, we recommend that the importance of open-ended problems, class discussions and assessment of modelling skills should also be taken into account. Time for concept development according to guided reinvention learning processes could be gained from the current way of dealing with an elaborate collection of differential and integral techniques for a large variety of functions. The relation between the notions of calculus in discrete situations and in situations where mathematical formulas are at hand, is more important for students who are being prepared for science studies than a thorough knowledge of integral and differential techniques. In relation to this, we refer to the current availability of technological tools, hand-held and on the internet, which can give quick and accurate answers when you are able to type in the correct questions.

Students developed concepts and instrumental competencies through open-ended activities. These modelling activities should be valued and discussed seriously by the class. Current assessment practices appeared to focus both teacher and students on algorithmic skills. More emphasis should be put on assessment which addresses modelling competencies through open-ended investigations (Goldin, 2003; de Haan & Wijers, 2000; van den Heuvel-Panhuizen, 1996; de Lange, 1987, 1999).

Integration of science topics

Finally, the integration of mathematics and science topics is an important issue for education. In current scientific research many breakthroughs happen on the border of different topics. However, integration of mathematics and science in secondary education should take into account that didactical problems within the various topics have more consequences than a superficial consideration of the respective curricula might suggest. Integration has more implications than just the tuning of standards, and needs schools, teachers and researchers to invest in understanding each other's didactical problems and cultures.

This study has provided some insight into the constraints and possibilities for the integration of physics and mathematics. We recommend further research on this integration for the understanding of the teaching and learning of science and mathematics as closely related disciplines, and for implementing real changes in the way these topics are covered in schools.

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Samenvatting

Introductie

Het onderzoeksprogramma Wiskunde leren met ICT richtte zich op de vraag naar empirische evidentie voor de manier waarop computergebruik kan bijdragen aan het leren van wiskunde. Eén van de domeinen binnen dit programma onderzocht het leren van de beginselen van differentiaalrekening en kinematica. Dit proefschrift doet verslag van dat onderzoek. De overige domeinen waren statistiekonderwijs (Bakker, 2004), het leren van algebra (Drijvers, 2003), niveauverhoging bij kansrekening (Pijls et al., 2003) en het gebruik van grafische rekenmachines in het beroepsonderwijs (Hoek & Seegers, in press).

De keuze voor differentiaalrekening en kinematica is het gevolg van vele publicaties over de problemen van leerlingen met deze onderwerpen. Het blijkt dat zij hun schoolkennis meestal niet combineren met hun dagelijkse redeneringen over verandering en beweging. Bij het onderwijzen van deze onderwerpen hebben grafieken een centrale rol, maar deze zijn voor leerlingen niet voldoende transparant.

De introductie van grafische rekenmachines in het voorbereidend wetenschappelijk onderwijs en van computeralgebra in het wetenschappelijk vervolgonderwijs bieden hier wellicht een oplossing. Leerlingen hoeven niet meer uitvoerig de arbeidsintensieve algoritmen te leren om verschillende functies te differentiëren en te integreren. Daardoor kan meer aandacht worden geschonken aan het leren van de onderliggende concepten.

In het wis- en natuurkundeonderwijs zien we bovendien steeds meer het gebruik van computersimulaties. Deze worden dan meestal gestuurd door een natuurwetenschappelijk model in een geïdealiseerde omgeving. Leerlingen worden geacht om de regels van het model te ontdekken tijdens het werken ermee. Die werkwijze in het onderwijs wordt ook wel *discovery learning* genoemd (de Jong & Joolingen, 1998a). Deze aanpak kan worden geplaatst naast een benadering waarbij computers worden gebruikt om leerlingen te ondersteunen in het zelf ontwikkelen van modellen (Doerr, 1997). Het ondersteunen en begeleiden van leerlingen bij het ontwikkelen van wiskundige modellen is ook precies wat wordt beoogd met realistisch wiskundeonderwijs. Een onderwijsbenadering waarin modelleeractiviteiten van leerlingen centraal staan, sluit aan bij ideeën over hoe mensen symboliseren. Voor het geven van betekenis aan symbolen en het leren gebruiken ervan blijkt kennis over de onderwerpen waarnaar verwezen wordt, en het doel van het gebruik, essentieel.

We veronderstellen dat de problemen van leerlingen met het interpreteren van snelheid-tijd en afstand-tijd grafieken hun oorzaak hebben in onvoldoende kennis over snelheid als samengestelde grootheid en over de samenhang tussen snelheid en afgelegde weg. Leerlingen worden geacht allerlei samenhang in die grafieken te zien, maar het ontbreekt ze aan de daartoe noodzakelijke domeinkennis.

We richten ons daarom in dit onderzoek op een onderwijsbenadering waarbij symbolen, betekenissen en vakspecifieke doelen zich in wisselwerking met elkaar ontwikkelen (zie Meira, 1995). Deze onderwijsbenadering en de veronderstelling dat computers kunnen worden gebruikt om leerlingen te ondersteunen bij het ontwikkelen van modellen, vormen de uitgangspunten voor het onderzoek naar het leren en onderwijzen van differentiaalrekening en kinematica.

Leren en onderwijzen van differentiaalrekening en kinematica

Voor het onderzoek naar een alternatieve benadering van differentiaarekening en kinematica is eerst vakdidactische literatuur bestudeerd. Daaruit blijkt dat leerlingen snelheid niet zien als een relatie tussen tijd en afgelegde weg, maar als eigenschap van een bewegend object die wordt gerelateerd aan verplaatsingen en inhalen (Piaget, 1970; Thompson, 1994a).
Uit onderzoeken met studenten volgt, dat zij nog steeds niet spontaan met snelheid redeneren als een samengestelde grootheid die een relatie beschrijft tussen een bewegend object en een referentiekader (Saltiel & Malgrange 1980). In hun redeneringen lopen beschrijvingen en oorzakelijke verbanden vaak door elkaar. Hierin herkennen we een onderscheid tussen straatbeelden van natuurkundige fenomenen, die gebaseerd zijn op alledaagse ervaringen, en logisch consistente schoolbeelden die een geïdealiseerde realititeit beschrijven (Genderen, 1989). Kennelijk slaagt het onderwijs er niet in deze twee beelden te verbinden.
Bij differentiaalrekening en kinematica is het gebruikelijk om de begrippen op te bouwen aan de hand van continue grafieken. Raaklijnen en oppervlaktes spelen een belangrijke rol in de verklaring van een maat voor verandering. Het blijkt echter dat leerlingen zulke grafieken niet altijd correct interpreteren. McDermott e.a. (1987) en Clement (1985) hebben uitgebreide studies gedaan naar problemen van leerlingen met snelheid-tijd en afstand-tijd grafieken. Een bekend verschijnsel is dat leerlingen die grafieken als een beschrijving van de werkelijke situatie zien. Dit wordt niet alleen veroorzaakt door de vorm van de grafiek, maar ook door de taal waarmee we over grafieken praten (Goddijn, 1978; Dekker, 1991; Berg, 1994).
Een ander probleem is dat vaak te snel overgegaan wordt naar de formules die bij het onderwerp een rol spelen (Machold, 1992; Barnes, 1995; Kindt, 1995). In de schoolboeken staat het oefenen met die formules centraal. Hierdoor zijn leerlingen gespitst op het *hoe* in plaats van op het *waarom*. De berekeningen passen in een systematiek rond tijd, snelheid en afgelegde weg die voor leerlingen nauwelijks wordt opgebouwd vanuit hun perceptie van die grootheden.
De conclusie die we uit deze literatuurstudie trekken is dat de overgang van dagelijks taalgebruik en intuitieve noties naar formele begrippen en grafieken bij deze onderwerpen te groot is. Dit heeft tot gevolg dat leerlingen onvoldoende inzicht ontwikkelen in de wis- en natuurkunde van het modelleren van beweging.
In recente pogingen om inzicht te krijgen in een mogelijk leertraject zijn twee bena-

deringen te herkennen. Beide hebben het modelleren van beweging als centraal thema, maar kiezen een verschillende invalshoek.
In de eerste benadering wordt de oplossing gezocht in het creëren van een verbinding tussen de wetenschappelijke kennis en de alledaagse ervaringen. Deze verbinding wordt vormgegeven in computerprogramma's waarbij de wetenschappelijke beschrijvingswijzen gekoppeld zijn aan simulaties met fenomenen rond beweging. Zo kunnen leerlingen hun alledaagse kennis over beweging benutten om die wetenschappelijke beschrijvingswijzen te *exploreren*. Een voorbeeld hiervan is de software die ontwikkeld is in het Simcalc-project (Kaput & Schorr, in press).
Het zien van een relatie tussen fenomenen en het ontstaan van continue grafieken biedt echter niet de garantie dat leerlingen alle elementen van die grafieken goed interpreteren. De activiteiten voor de leerlingen zijn gericht op het exploreren van een computermodel dat gebaseerd is op een formeel systeem. Dit model is consistent met het expert-beeld van een theoretisch systeem en het is maar de vraag of leerlingen die consistentie overzien (Gilbert, 1998).
Deze benadering is te contrasteren met een tweede benadering waarbij de informele kennis van leerlingen als beginpunt wordt gekozen. Met betrekking tot computergebruik geven Doerr (1997) en Gilbert (1998) een alternatief door de computer in te zetten als gereedschap voor het *construeren* van modellen door leerlingen. Een voorbeeld van een aanzet tot zo'n werkwijze is te vinden in het onderzoek van Boyd en Rubin (1996).
Uit de analyse van deze twee benaderingen wordt geconcludeerd, dat voor de beoogde aanpak meer inzicht nodig is in de mogelijkheden om leerlingen te ondersteunen bij de ontwikkeling van inzicht in grafieken en het modelleren van beweging. We raadplegen hiertoe literatuur over perceptie en symboliseren.

Theoretisch kader

Om een beter beeld te krijgen van de oorzaak van de didactische problemen bij differentiaalrekening en kinematica beschouwen we de relatie tussen perceptie, interpretatie en kennisontwikkeling. Inzichten uit de semiotiek wijzen op de belangrijke rol die voorkennis en verwachting hebben bij perceptie en interpretatie (Cunningham, 1992; Jarvilehto, 1999). Bovendien blijken taal- kennisontwikkeling verweven, en als gevolg daarvan is communicatie met anderen, waarbij je pogingen doet om nieuwe begrippen te verwoorden, hierbij een centraal aspect (Bartsch, 1998).
Een probleem voor het onderwijs is dus hoe je leerlingen kunt voorbereiden op het interpreteren van een situatie zoals bedoeld is. Bij het gebruik van structuurmaterialen verwijzen specifieke kenmerken van die materialen naar kenmerken in de situatie. De vraag is dan of leerlingen voor een correcte interpretatie niet eigenlijk al de situatie georganiseerd moeten hebben? Cobb e.a. (1992) formuleren deze zogenaamde 'learning paradox' als volgt:

> In other words, the assumption that students will inevitably construct the correct internal representation from the materials presented implies that their learning is triggered by the mathematical relationships they are to construct before they have constructed them. (Cobb, Yakel & Wood, 1992, p. 5)

De oplossing is dat deze paradox wordt vermeden door symbolen en begrippen geleidelijk en gelijktijdig te ontwikkelen. Dit leidt tot een leerproces dat te vergelijken is met wat Meira (1995) noemt: het creëren van een geleidelijk en dialectisch proces van symbool- en betekenisontwikkeling.

We vermoeden dat een probleem, zoals omschreven in deze paradox, ook speelt bij de presentatie van continue grafieken tijdens het onderwijzen van differentiaalrekening en kinematica. Een leerproces, waarin leerlingen zelf bijdragen aan de ontwikkeling van afstand-tijd en snelheid-tijd grafieken, zal het waarschijnlijk voorkomen dat er een kloof ontstaat tussen de wetenschappelijke kennis en hun ervaringen.

Zo'n leerproces kan ook worden gekarakteriseerd als *geleid heruitvinden*. Met 'uitvinden' wordt hier het karakter van het leerproces bedoeld. De activiteiten van de leerlingen zijn belangrijker dan de uitvinding als zodanig. De activiteiten moeten ervoor zorgen dat leerlingen hun verkregen kennis zien als uitbreiding van hun eigen kennis; een uitbreiding waarbij een inbreng van ze verwacht wordt en waarvoor ze zelf medeverantwoordelijk zijn (Freudenthal, 1991). Zo ervaren leerlingen het onderwijs alsof ze het geleerde zelf uitvinden.

Verscheidene ontwerpheuristieken zijn ontwikkeld voor het stimuleren van productieve uitvindingen van leerlingen en het realiseren van geschikte begeleiding in het onderwijs (zie bijvoorbeeld Treffers, 1987). Twee heuristieken in het bijzonder zijn van nut in het licht van de beschouwing over perceptie en symboliseren: het gebruik van emergente modellen (Gravemeijer, 1994, 2004a) en de probleemstellende benadering (Klaassen, 1995; Vollebregt, 1998).

Het gebruik van *emergente modellen* richt zich op de ontwikkeling van modellen bij het organiseren van fenomenen vanuit een wiskundig perspectief. Tijdens dit organiseren vindt een verschuiving plaats, waarbij eerst proberende beschrijvingen van leerlingen een model *van* een specifieke situatie leveren, terwijl die later uitgroeien tot een model *voor* meer wiskundige redeneringen (Streefland, 1985). Computerprogramma's kunnen leerlingen helpen bij het uitdrukken van hun ideeën (Cobb, 1999; van Streun, 2000).

De *probleemstellende benadering* richt zich op het verschaffen van inhoudelijke motieven aan leerlingen bij hun modelleeractiviteiten. Centraal staat daarbij een overstijgend kernprobleem. Dat probleem zorgt ervoor dat het voor leerlingen duidelijk is hoe de beantwoording van de opgaven uit de leergang je verder kunnen helpen. Het beantwoorden van een opgave roept bij leerlingen, denkend aan het kernprobleem, vervolgvragen op die zouden moeten worden opgelost. Zo zullen ook de leerlingen ervaren wat de logica is in de opgavenreeks.

Dit brengt ons tot een splitsing van de hoofdvraag in twee onderzoeksvragen:

1. *Hoe kunnen leerlingen de beginselen van differentiaalrekening en kinematica leren volgens een leerproces dat gekarakteriseerd kan worden als begeleid heruitvinden?*
2. *In hoeverre ondersteunt het verloop van dit leerproces de adequaatheid van en het inzicht in onze keuzen ten aanzien van emergent modelleren, de probleemstellende benadering en het gebruik van computerprogramma's?*

Methodologie

Het doel van het onderzoek is om een beoogde onderwijsbenadering te realiseren en verwachtingen over het leerproces van de leerlingen te analyseren. Een uitwerking van deze benadering moet dan eerst ontwikkeld worden. *Ontwikkelingsonderzoek* is een methodologie die zich richt op het realiseren van innovatieve leeromgevingen voor het onderzoeken van onderwijs- en leerprocessen. Dit onderzoek richt zich in het algemeen op de ontwikkeling van empirisch ondersteunde, vakdidactische onderwijstheorieen (Gravemeijer, 1994; Lijnse, 1995).

Ontwikkelingsonderzoek kent globaal drie fasen: het eerste ontwerp, het onderwijsexperiment en de analyse. De eerste fase start met het beschrijven van een gewenste aanpak in een hypothetisch leertraject (Simon, 1995) of scenario (Klaassen, 1995). Deze beschrijving is een lokale uitwerking van onze veronderstellingen over hoe het onderwijs-leerproces te ondersteunen is, hoe we verwachten dat het zal verlopen en waarom we dat verwachten. In ons geval is dit een eerste uitwerking van de beoogde lokale onderwijstheorie over het leren en onderwijzen van differentiaalrekening en kinematica.

Zoals hiervoor beschreven is het onderzoek gestart met een literatuuronderzoek naar problemen met, en benaderingen van deze onderwerpen. Parallel aan dit literatuuronderzoek is een pilotexperiment uitgevoerd om enkele leerlingactiviteiten te onderzoeken, en om te analyseren wat de mogelijkheden zijn voor de onderwijsexperimenten binnen de huidige schoolorganisatie. Vanwege het onderwerp is gekozen voor experimenten in 4 vwo (Natuur-profielen).

Na de analyse van het pilotexperiment is een hypothetisch leertraject geformuleerd en uitgewerkt voor een eerste versie van het lesmateriaal. Die versie is in het eerste onderwijsexperiment uitgeprobeerd in twee 4 vwo klassen op verschillende scholen. Het lesmateriaal en de verwachtingen zijn vooraf met de docenten doorgesproken. Van alle lessen zijn geluidsopnamen gemaakt. Video-opnamen zijn gemaakt van klassikale discussies en van een tweetal leerlingen tijdens de computeractiviteiten. Na afloop van de lessenserie is het leerlingenwerk ingenomen en zijn de proefwerkuitwerkingen gekopieerd. Met deze data is geprobeerd om het leerproces van de leerlingen te reconstrueren en om veronderstellingen te toetsen.

Een tweede onderwijsexperiment bleek nodig voor het optimaliseren van het lesmateriaal, voor een betere uitwerking van de overgang van discrete naar continue grafieken, en om meer systematisch data te verzamelen over de bijdrage van de computeractiviteiten aan het leerproces van de leerlingen. Dit tweede experiment vond op één school plaats en data werd op een vergelijkbare manier verzameld als bij het voorgaande experiment. Het voornaamste verschil is dat nu opnamen gemaakt zijn van drie tweetallen tijdens de computeractiviteiten.
Het interpretatieve kader voor het analyseren van de data werd voornamelijk bepaald door keuzen ten aanzien van het lesmateriaal en de relatie met het begrijpen en verbeteren van de leerprocessen.
Met de beschikbare data zijn we gekomen tot een empirisch onderbouwde bijdrage aan een lokale instructietheorie voor het leren en onderwijzen van differentiaalrekening en kinematica.

De leergang

Wiskunde is de discipline waarmee we de wereld om ons heen structureren. Dat maakt het noodzakelijk voor ontwerpers van lesmateriaal zich te verplaatsen in leerlingen en zich hun perceptie van probleemsituaties voor te stellen. Daarbij kan het helpen om de historische ontwikkeling van de onderwerpen te analyseren. Welke problemen waren de aanleiding en hoe werden ze aangepakt door mensen die de standaardoplossingsmethoden nog moesten ontwikkelen?
De eerste ideeën over bewegingsleer komen we tegen bij Aristoteles (circa 350 v. C). Hij probeerde verschillende soorten materie te karakteriseren op grond van hun eigenschappen, en veronderstelde dat objecten naar hun natuurlijk plaats vallen met een constante snelheid die evenredig is met het gewicht van het object.
Heel lang blijven de ideeën van Aristoteles gangbaar en onaangetast. Het bestuderen van bewegingen en veranderingen wordt op een aantal plaatsen in Europa pas weer onderwerp van studie in de dertiende en veertiende eeuw. Men bestudeert in die tijd situaties waarbij een eigenschap bezig is te veranderen. Rond 1360 geeft Oresme aan deze discussie een – voor ons – belangrijke bijdrage, namelijk die van de grafische voorstelling. Het ging hem hierbij niet zozeer om *wat* er precies gebeurt, maar *hoe* je dat wat er gebeurt, kunt beschrijven.
Oresme paste deze techniek ook toe op – geïdealiseerde – bewegingen. De bijzondere denkstap die hij hierbij maakte, was dat snelheid een eigenschap van objecten is die afhangt van de tijd. Dankzij zijn keuze worden de meetkundige figuren discrete varianten van de ons bekende snelheid-tijd grafieken.
Oresme, en later vooral ook Galileï (1564-1642) bij empirisch vastgestelde bewegingen, geven betekenis aan het werken met grafieken zonder dat momentane snelheid gedefinieerd is als differentiaalquotiënt. Dijksterhuis merkt hierover op:

> Het is een situatie, die zich in de geschiedenis van de wiskunde herhaaldelijk heeft voorgedaan: mathematische begrippen worden vaak - men kan bijna wel zeggen: in den regel - reeds lang intuïtief gehanteerd, voordat men ze met volkomen scherpte kan omschrijven, en fundamentele stellingen worden vaak intuïtief ingezien voordat men ze strikt kan bewijzen. (Dijksterhuis, 1980, p. 218)

Vanaf dan ontwikkelt zich de differentiaalrekening via Newton en Leibniz (beiden 17^{e} eeuw) en via Cauchy (19^{e} eeuw) tot de uitwerking die de basis vormt voor het huidige onderwijs.
Deze geschiedenis leert ons dat de aanzetten voor de differentiaalrekening voortkomen uit het modelleren van beweging. Er werd in eerste instantie niet gekeken naar grafieken van functies, maar naar meetkundige vormen die direct betekenis hebben in de context. De grafieken komen naar voren als modellen van relaties tussen afstand, tijd en snelheid van bewegende objecten. Geleidelijk aan worden grafieken een opzichzelfstaand object: de grafiek representeert de samenhang tussen twee variabelen. Pas in een later stadium gaan grafieken functioneren als model voor wiskundig redeneren over het differentiëren en het integreren van functies.
Op basis van de geschiedenis en van ervaringen in een pilot-experiment besluiten we dat het thema *greep krijgen op verandering* in de lessenserie centraal moet staan en dat we leerlingen kunnen ondersteunen met een serie grafieken: van discrete context-nabije grafieken tot de continue snelheid-tijd en afstand-tijd grafieken.
Dit thema is uiteindelijk geconcretiseerd binnen de context van een naderende orkaan. De vraag voor de leerlingen is: Hoe kun je voorspellen wanneer de orkaan het land zal treffen? Die situatie wordt gepresenteerd aan het begin van de leergang en gedurende de lessenserie wordt daar een aantal keren op teruggeblikt: Wat kunnen we nu beter? Hoe zouden we nog preciezer kunnen zijn? In deze aanpak sluiten we niet aan bij intuïties van leerlingen over snelheid en gemiddelde snelheid als samengestelde grootheid, maar bij intuïties over verplaatsingen als maat voor een veranderende snelheid.
Bovendien zijn bij deze opzet computerprogramma’s ingezet. Leerlingen werken onder andere met het programma Flits waarin bewegingen zijn vastgelegd met stroboscopische foto’s. Leerlingen kunnen met dit programma redeneren over die bewegingen zonder belemmerd te worden door de tijdrovende activiteit van het meten zelf. Hierdoor kunnen ze sneller ingaan op kenmerken van bewegingen in relatie met grafieken.

Twee onderwijsexperimenten

De veronderstellingen over de achtereenvolgende denkstappen van de leerlingen en hoe die te ondersteunen zijn uitgewerkt in een serie van tien lessen voor 4 vwo. Deze lessenserie is uitgeprobeerd op twee scholen.

Uit de analyse van dit eerste experiment concluderen we dat het deels is gelukt om de leerlingen de beginselen van de differentiaalrekening en kinematica te leren in een proces dat kan worden gekarakteriseerd als geleid heruitvinden. Op een aantal punten verliep het echter niet zoals bedoeld. Het bleek voor de docenten niet altijd duidelijk hoe de klassendiscussies zouden moeten leiden tot probleemstellingen voor het vervolg. Dit deed zich met name voor bij het problematiseren van patronen in verplaatsingen, en bij het bespreken van de overgang van grafieken van metingen naar grafieken met een continue tijd-as. Dit had onder andere tot gevolg dat leerlingen tijdens de computerpractica niet direct in de gaten hadden hoe ze de grafieken van de software konden gebruiken en voor welke problemen die grafieken een oplossing boden. Bovendien ging op één van de twee scholen het eerste computerpraktikum met Flits niet door vanwege technische problemen.
We concludeerden dat een tweede experiment nodig was. Voor dat tweede experiment werd het lesmateriaal herzien. We voegden meer open opgaven toe om leerlingen de gelegenheid te geven greep te krijgen op de centrale probleemstelling, en opdat een verwachte variëteit aan leerlinguitwerkingen kon worden gebruikt voor klassendiscussies. Dankzij het eerste experiment konden we de gewenste leerroute bovendien beter articuleren en verwerken in lesbeschrijvingen in de docentenhandleiding. Tot slot zorgden we ervoor dat we meer data konden verzamelen over de computeractiviteiten van de leerlingen om die te kunnen analyseren. Het tweede onderwijsexperiment vond plaats op één school, wederom in 4 vwo.
Deze wijzigingen hadden inderdaad tot gevolg dat de verschillende bijdragen van leerlingen aanleiding waren voor productieve klassendiscussies. In de meeste gevallen kon de leerkracht deze discussies zo begeleiden, dat er consensus leek te ontstaan over het geleerde en dat de richting voor het vervolg voor leerlingen duidelijk was. Op deze wijze konden leerlingen beter voorbereid aan de computeractiviteiten beginnen en hoefden ze minder tijd te besteden aan het achterhalen van de betekenis van de grafieken en van het doel van de activiteiten.

Bevindingen

Het doel van dit onderzoek is een bijdrage te leveren aan een theorie over het leren en onderwijzen van differentiaalrekening en kinematica. De kern hiervan is het modelleren van beweging voor het doen van voorspellingen. Het blijkt dat dit kan worden geïntroduceerd in de context van een orkaan die een kust nadert. In die context ligt het voor de hand om positieverandering vast te leggen en te gebruiken voor voorspellingen. Het voorspellen leidt tot een onderzoek van patronen in verplaatsingen. Dit is een motivatie voor het construeren van twee-dimensionale grafieken van verplaatsingen en van afgelegde weg.
Door met een computerprogramma veel situaties te onderzoeken – waarbij de mogelijkheid blijft om betekenissen van die grafieken te traceren – ontwikkelen leerlingen inzicht in grafische kenmerken van die grafieken, zoals ‘helling’ en de interpre-

tatie van ‘snijpunten’. Kenmerken die voorbereiden op redeneringen met continue snelheid-tijd en afstand-tijd grafieken.
Voor het motiveren van de overgang van discreet naar continu maken we gebruik van een beperking van discrete grafieken en van een onderzoek naar een door Galileo verondersteld continu model voor vrije val. Dit is aanleiding voor een strokenbenadering met stukjes constante gemiddelde snelheid. Voor leerlingen blijkt deze overgang moeilijk en de docent heeft een belangrijke rol om de probleemstelling en de gevolgen van een bewering als ‘valsnelheid is evenredig met de valtijd’ met leerlingen te bespreken.
De volgende stap naar het differentiequotiënt en momentane snelheid vindt nog steeds plaats in het thema ‘greep krijgen op verandering’. We hebben gekozen voor een context met een continue afstand-tijd grafiek van een object dat voortbeweegt met veranderende snelheid. Het probleem voor de leerlingen is te achterhalen hoe de grafiek zou verlopen als de snelheid op een zeker moment niet meer zou veranderen. In deze leergang gebeurt dat in de context van een vermeende snelheidsovertreding. Deze probleemstelling blijkt leerlingen te helpen bij het formuleren van redeneringen die bruikbaar zijn voor het vervolg. Een deel van de grafiek kan vervolgens worden benaderd met een functie om meer precieze berekeningen mogelijk te maken. Voor de docent is het dan de taak om die diversiteit aan redeneringen te verbinden met meer wiskundige redeneringen die gebruikmaken van een functie en haar grafiek.
Als leerlingen een beeld hebben van de mogelijkheden om met een differentiequotiënt momentane verandering te benaderen, kan een computerprogramma helpen om de dynamiek van dat proces grafisch te verankeren. Bovendien helpt dat grafische beeld ook bij latere discussies als de computer niet beschikbaar is. Een hele computerles werken met het programma bleek voldoende om ook achteraf te kunnen verwijzen naar grafisch-dynamische beelden van het programma die het benaderingsproces ondersteunen.

Conclusies en aanbevelingen

Leerlingen ontwikkelden zo hun kennis over snelheid als samengestelde grootheid, de samenhang met verplaatsingen en afgelegde weg en het differentiequotiënt als maat voor verandering. Deze kennis is ondersteund door een serie van grafieken die het voor leerlingen mogelijk maakt betekenissen te construeren en te traceren. Hierdoor zijn de uiteindelijke begrippen geworteld in hun redeneringen over beweging en voorspellingen bij veranderingsprocessen.
Vervolgexperimenten moeten ervoor zorgen dat deze bijdrage aan een instructietheorie uitgroeit tot een robuuste theorie voor het leren en onderwijzen van differentiaalrekening en kinematica die geldt in verschillende onderwijssituaties.
De keuze voor emergent modelleren heeft ertoe geleid dat leerlingen met het lesmateriaal symbolen en betekenissen ontwikkelen in een dialectisch proces. Het blijkt

dat de docent niet alleen kennis moet hebben van de beoogde ontwikkeling van grafieken, maar ook van de bijbehorende taalontwikkeling van leerlingen.
De probleemstellende benadering bleek waardevol voor het geven van betekenis aan en het creeëren van inhoudelijke motieven voor de activiteiten. Leerlingen ervaren dan de achtereenvolgende activiteiten als een samenhangend geheel. De overstijgende problematiek is behulpzaam bij het reflecteren op de stand van zaken en om leerlingen te betrekken bij het denken over de volgende problemen die zouden moeten worden opgelost om verder te komen met het beschrijven van veranderingsprocessen. We zijn echter niet overal geslaagd in deze benadering.
De dynamiek van de computerprogramma's en de mogelijkheid om veel situaties te onderzoeken bieden leerlingen de gelegenheid om zelf ideeën te ontwikkelen. Computergebruik in de klassenpraktijk heeft echter als risico dat leerlingen te oppervlakkig en te snel door de activiteiten heengaan, zelfs als ze de bedoeling niet volledig begrijpen. Een goede voorbereiding in het lesmateriaal en klassendiscussies onder leiding van de docent moeten zorgen voor afstemming van de mogelijkheden van de programma's met de redeneringen van de leerlingen. Activiteiten achteraf zijn nodig voor reflectie op en een klassikale consensus over het geleerde.
In deze benadering van geleid heruitvinden is het ons opgevallen dat leerlingen de mogelijkheid hebben om betekenissen te traceren, maar dat dit niet vanzelf gaat. We bevelen aan dat met name in het wiskundeonderwijs regelmatiger aandacht wordt besteed aan de oorsprong van wiskundige begrippen, omdat die snel een eigen leven kunnen gaan leiden in beoogde algoritmen.
Onze interpretaties en redeneringen zijn vooral gebaseerd op analyses van klassengesprekken en leerlingenwerk. Vervolgonderzoek is nodig om te onderzoeken of de symboliseringen die we met deze lessenserie ondersteunen inderdaad passen bij de individuele leerprocessen van diverse leerlingen.
Nader onderzoek moet inzicht geven in het gebruik van computerprogramma's als gereedschap bij het leren van wiskunde. Het onderscheid tussen geleid exploreren en geleid construeren lijkt in de klassenpraktijk minder groot dan de theoretische uitgangspunten doen vermoeden. Het is ons namelijk niet gelukt om de computer zo in te zetten, dat tijdens de activiteiten alle leerlingen het geleerde ervaren als eigen uitvindingen.
In het huidige wetenschappelijke onderzoek liggen de belangrijkste doorbraken op grensgebieden van verschillende disciplines. Bovendien komen bij modelleeractiviteiten meestal aspecten van verschillende disciplines aan de orde. Tot slot bevelen we daarom aan dat meer onderzoek zich richt op het onderwijs in de samenhang tussen de bètavakken.

Appendix: The pilot experiment

In this appendix we describe the learning processes of four students we followed during the pilot experiment. This description gives an impression of the practices and possibilities of these students within this experiment.
The students used their mathematics booklet (Bos et al. 1998). This booklet contains a chapter on the basic principles of calculus in ten lessons. For our pilot experiment we planned to use the time needed for this chapter, and this booklet together with alternative materials.
The first alternative activity concerned an orientation on modelling motion. This activity should evoke both the need for drawing graphs, and initial reasonings about the relation between distance travelled and a changing velocity. With this activity we could also gain insight in the level of these students' reasoning. We choose a series of photographs by Muybridge (inspired by an idea of Speiser, 1994). This photographer investigated motion of animals and humans with such series. Motion in front of a grid was recorded with photographs in a fixed frequency. We used a serie of a cat (see fig. 1) that starts running the questions are posed on his walking velocity, running velocity and how this changes (Muybridge, 1985).

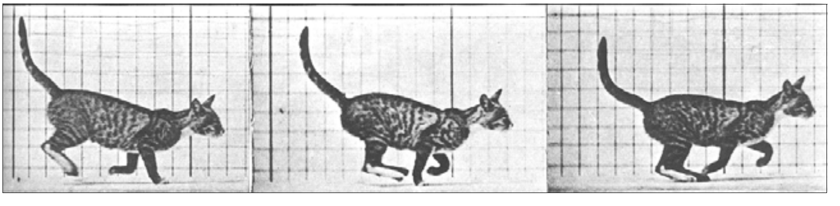


figure 1 A series of photographs by Muybridge: A Catwalk

The topic of sums and differences is inspired by Leibniz' work, and is also based upon an instructional sequence concerning the basic principles of calculus (Kindt 1997). Properties of, and the relations between series, their sums and their increments are investigated by the students in the context of mathematical formulas. In this process, the students are supposed to develop notions that can be used for problems later on. Moreover, it is a first introduction to the mathematical relationship between sums, summation symbols, increments and difference symbols. The notion of limit is only used in an intuitive way at the end of the unit.
The relationship between sums and differences is explored by the students with various graphical representations (fig. 2). From the picture and the graph, students can become challenged to proof that the sum of successive odd numbers is a square, and the difference between two successive squares is an odd number.

The relationship between these sums and differences is represented with symbols: $\Delta\, n^2 = 2k{+}1$ and $\Sigma\ 2k{+}1 = n^2$.
Students are expected to use graphing calculators when dealing with these investigations. From graphs and tables they can deduce that $\Delta 3^n = 2 \cdot 3^k$ and, with the theorem they can prove that $\Sigma 3^k = 1/2 \cdot (3^n - 3^0)$.

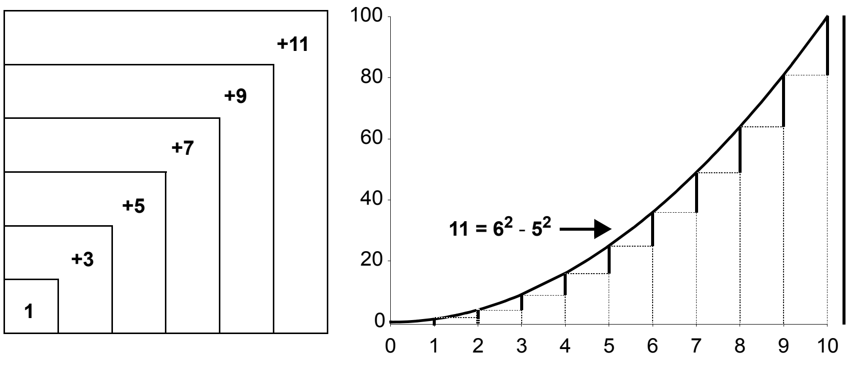


figure 2 The relation between squares and the sum of successive odd numbers

The second alternative activity on modelling motion concerned the transition from reasonings about velocity with continuous time-distance graphs to the mathematical notion of a difference quotient. This activity, about a comic-strip (see page 115) is inspired by instructional sequences that fitted this line of thinking (Kindt, 1996, de Lange, 1987). Students have to determine velocities from distance-time graphs. The intervals in these continuous graphs, which are necessary for the difference quotient, will probably derive their meaning from the preceding discrete work.
We expected that students were able to interpret the graph in terms of a slope that is related with - change of - velocity, and that they were able to measure displacements in time-intervals that can be used to calculate average velocities over the time-interval (especially when the graph is straight over the corresponding time-interval). From there the step is made to a fictive motion according a formula: $s = t^3$. Questions deal with average velocities and whether the velocity at $t = 5$ sec can be determined (or approximated) and what the graph would look like when the velocity shouldn't change from that moment.
In this context students might come up with a relation between average velocity and instantaneous velocity and the use of a difference quotient for approximating the slope 'in a point' of a graph.
We describe our observations of Suzanne, Lennert, Loes and Jonas respectively.

Suzanne

Suzanne measured distances to describe the cat's motion. She placed the distances between the cat and a fixed starting point in the table. From the ninth frame she also wrote increases below the cells of her table (see fig. 3). Probably it became difficult to determine the distances to the starting point, and she measured displacements between two frames for calculating the new distances to the starting point.

sec	0	0,03	0,06	0,09	0,12	0,15	0,18	0,21
cm	0	2	3	4	5	6	6	7

sec	0,24	0,27	0,30	0,33	0,36	0,39	0,42
cm	8	13	17	25	32	40	46
	5	5	8	7	8	6	

figure 3 Suzanne's table for the catwalk

In the subsequent activities she did not manage to determine increases with formulas, and neither to use the graphing calculator for this. In her written materials were a few blank pages, but she continued in the contextual situations where she had to reason with data. However, halfway the chapter she hardly answered any of the questions on the situation of Mr Bommel. The next task dealt with a fictitious motion according to a place-time formula $s = t^3$. In the questions for average velocities in time-intervals with a length of 1 second, she wrote correct calculations with $\Delta t = 1$. How you can use such a calculation for decreasing Δt to determine instantaneous velocity seemed not clear to her. She calculated the average velocity from $t = 0$ and drew a linear continuation according to this velocity in her graph fig. 4.

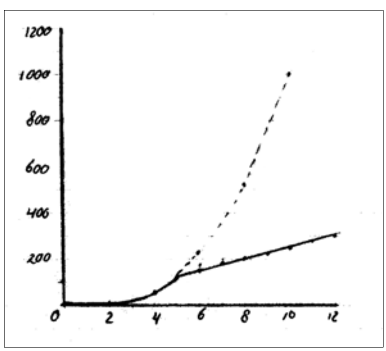


figure 4 Suzanne's graph

The activities on Mr Bommel were important to understand a approximation procedure. It is not surprising that gets stuck in the subsequent problems. She seems to understand how to calculate with a difference quotient, but isn't possible to use it for approximating local change. An example of either not understanding the idea of local change, or of being able to treat an algorithm in a specific situation but not understanding for what kind of problems the algorithm is a solution nor being able to deal with it flexible. In the first case, the question didn't evoke the intended associations.

The responsibility for Suzanne to deal with the course description for the students, and to plan her own activities, had consequences that can have big influences for what has to follow. She skipped the activities of Mr Bommel, and wasn't able to really understand what came next.

Suzanne showed much attention during the classroom discussions in the subsequent lessons. She posed questions and seemed eager to understand how to use the graphing calculator for solving problems on instantaneous change. In her writing we find the teacher's notations like: $(y1(5.0001) - y1(5)) / 0.0001$.

In the final assessment she showed that she could calculate instantaneous change with her graphing calculator in standard situations (even with a linear formula she used the above procedure). However, she got stuck with a contextual problem without a formula, on instantaneous velocity. Suzanne seems to master the algorithms and the use of the graphing calculator, but doesn't really understand the (graphical) meaning of these algorithms.

Lennert

Lennert started in the catwalk context with a table of displacements and total distances. In the first half of the chapter he followed the course description rigorously, in the second half he misses some problems. Just like Suzanne, it is not easy for him to keep up with the intended pace of the course description. From the notations in his activity book it appears that he worked a lot with the graphing calculator. Before this options is suggested he studies increases in formulas by using: $y2 = y1(x{+}1) - y1(x)$. His great care for his work is shown by a few summaries that he wrote down during the chapter.

In the question on the motion according to $s = t^3$ Lennert used graphical reasonings and the given graph. This is in line with the kind of reasoning that was evoked by the problem on Mr Bommel. He succeeded in drawing a linear continuation from a certain moment that approximates the instantaneous velocity at that moment.

In the subsequent activities we see how he is going to use the difference quotient for calculating average and approximating instantaneous velocity. He drew arrows in a graph to explain for himself the meaning of the horizontal and vertical intervals and their quotient.

Lennert didn't use the teacher's procedure with the graphing calculator. He determined function values with the trace-option and calculated Δy and Δx and the difference quotient with these values. In the final assessment he scored good. He seemed to work in his own pace, seemed to understand the notions aimed at and was able to make adoptions to the algorithms for serving his purpose. He hardly participated in classroom discussions and didn't copy the algorithms that were presented by the teacher as black box procedures for specific situations.

Loes

For describing the catwalk Loes draws a graph with time vertical and covered distance horizontal. The larger the cats displacements become, the less steep her graph is. When asked where you can do a good estimation of the speed, she answers: "where the graph is flat (...) with a constant speed."
For the problems with formulas she tries to use the idea of $y1(x+1) - y1(x)$ without a GR (with $y(x) = 2^x$). However, her notation is incorrect, it appears to be a multiplication of 2^x with $(x+1)$ (fig. 5).

$$2^x\,(x+1) \quad - \quad 2^x\,(x)$$
$$4\,(2+1) \quad - \quad 4\,(2) \quad = 12 - 8 = 4$$
$$8\,(3+1) \quad - \quad 8\,(3) \quad = 32 - 24 = 8$$

figure 5 Loes' incorrect understanding of calculating differences

This might be the result of something we overlooked. In algebra it is common to leave the multiplication symbol out of the expression. As a result of sentences like "the graph of *f*" students might interpret "*f*(x+1)" as *f* times (x+1). Loes did a lot of work on the problems in the Bommel context. Just like Lennert she also did the following problems with movement according to $s = t^3$, but she read function values from the graph.

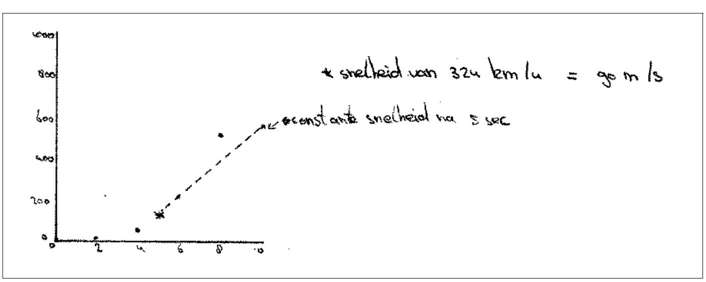


figure 6 Loes' drawing of a constant velocity from t = 5 seconds

She calculated the average distance over that time interval with the differences between these values and the corresponding time intervals. She approximated the average speed on the time interval [4, 6] for the question about the speed at moment $t = 5$, and noticed that the speed she finds is exactly in between the average speeds on [4 , 5] and [5 , 6]. She then used that average speed to draw the graph as it would be if the speed wouldn't change beyond $t = 5$ (fig. 6).

According to her you can't calculate the speed at $t = 5$ with a method based on increases. Loes emphasizes that the value which you calculate with a difference quotient is the average increase for *one x*. It seems that the number she calculates with $\Delta s/\Delta t$ is interpreted as the distance travelled in one time unit, something quite different from instantaneous velocity. It looks as if she doesn't quite know yet how a difference quotient (or even the graph) can help here for quantifying instantaneous velocities.

In the next classroom discussion the teacher doesn't quite manage to show how the graphing calculator can be used for approximating instantaneous speed. He makes a small mistake and, as a consequence, he can't answer all questions of students during the lesson. One of those questions is the same as Loes' question on approximating the instantaneous speed. The teacher wanted to illustrate the process of approximation at $x = 3$ by defining the function $y2$ in the graphing calculator as: $y2 = y1(3+x) - y1(x)$, and decreasing the value of x. He got stuck and realised his mistake just as the bell goes. In Loes' activity book the mistake is written down. This is something we saw more often. During classroom discussions the teacher had to watch time carefully. As soon as the school bell rang, the students closed their books and packed their bags, the lesson is finished.

Next, Loes made a diagnostic test from her mathematics book. Calculating averages is no problem for her. When she had to approximate instantaneous change, e.g. the slope of the graph of $f(x) = \frac{1}{2}x^4 - 3$ in (1, -3), she made errors (fig. 7).

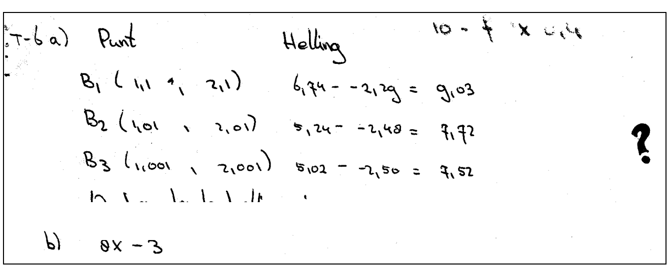


figure 7 Loes' mistake

Probably she tried to find the algorithm with a summary and an example in the book. She wrongly interpreted the numbers of the table in the book (see fig. 8). In her cal-

culation it seems as if she approximated the interval [1 , 2] and calculated with the corresponding function values.
As a result of the course of affairs during the classroom discussion and the last lessons Loes didn't grapple the notion of instantaneous change and how to approximate it.
In the final assessment she didn't give any answer on questions concerning these issues, such as the questions on determining slopes in a point of the graph (fig. 8).

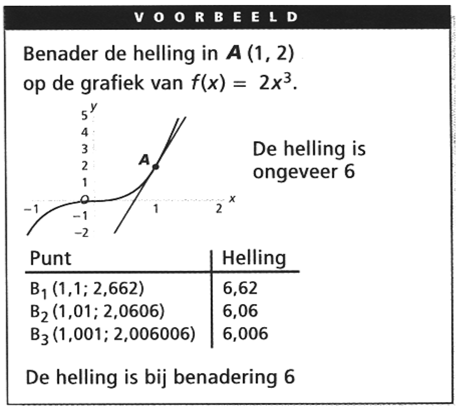


figure 8 The example from the book (Bos et al. 1998, p. 77)

Jonas

The written materials of Jonas looked a bit sloppy. In the catwalk activity Jonas drew a table with total distances and a graph of these numbers with a frame-axis horizontally. He seemed to have little problems with this activity and with the subsequent activities on sums and differences. However, he didn't prove that the difference between two successive squares is an odd number. Jonas gave short and correct answers in the Bommel activities. In the final pieces of the chapter he correctly calculated average speeds using the difference quotient. For approximating instantaneous velocities he used a method on the graphing calculator (fig. 9).

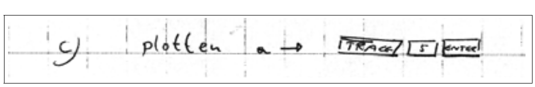


figure 9 Jonas' writing

From similar answers in subsequent questions we can conclude that he is able to use the graphing calculator. But how much did he understand? This is difficult to deduce

from his written materials. We also had hardly any data of his contributions to classroom discussions or small group work.
In the final assessment Jonas showed an excellent understanding of approximating with a difference quotient from accompanying sketches (see fig. 10).

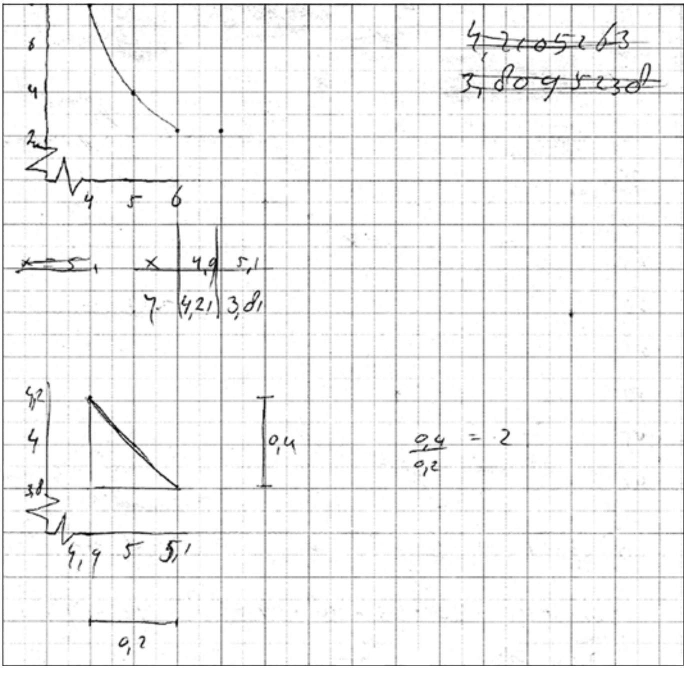


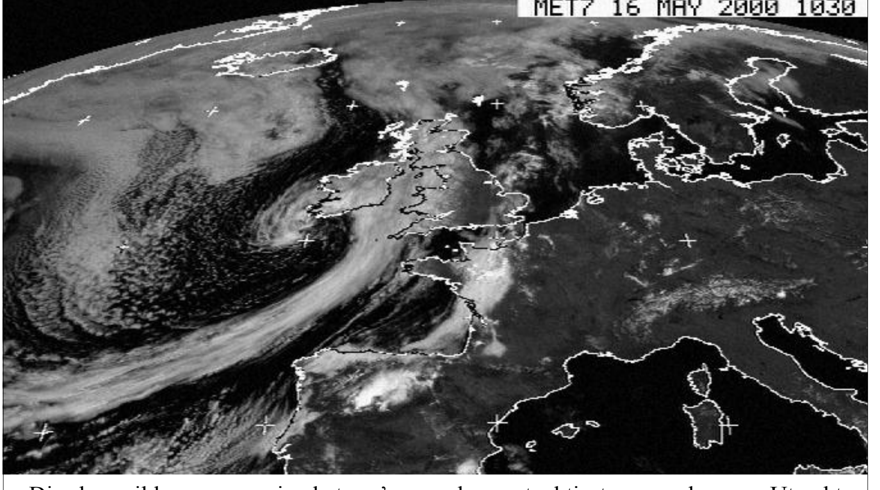
figure 10 Jonas' work on a test item

Appendix: Student activities

The student activities of the second teaching experiment are available in Dutch and can be retrieved at http://www.fi.uu.nl/~michiel.

§ 1 Niets veranderlijker dan het weer

Van alles om ons heen verandert: je kledingmaat, de gemiddelde lengte van de Nederlander, het aantal levende diersoorten, de hoogte van de zeespiegel en natuurlijk het weer van dag tot dag. Bij veranderingen is het vaak handig en soms zelfs van wezenlijk belang om verandering te kunnen voorspellen.
Bij voorspellingen van het weer wordt gebruik gemaakt van het weer van vandaag. Men begint dan met een beschrijving van het weer in Europa. Voor dergelijke beschrijvingen worden satellietfoto's gebruikt. Met satellietfoto's is bijvoorbeeld direct te zien waar het in Europa bewolkt is en hoe de wolken bewegen.
Op 16 mei 2000 om 10:30 uur zag de hemel boven Europa er zo uit:



Die dag wilde een groepje skaters 's avonds een tochtje ten noorden van Utrecht maken. Zoals je ziet is het om 10:30 uur in Nederland zonnig.
>> Heb je met deze foto voldoende zekerheid dat het die avond ook droog is? Zo ja, waarom? Zo nee, welke informatie zou je dan nog nodig hebben om met meer zekerheid te kunnen voorspellen?
>> Hieronder zie je de situatie aan het begin van die middag. De witte vlekken boven het westen van België blijken regen- en onweersbuien te zijn. Wat denk je, kan de skate-tocht nog doorgaan?

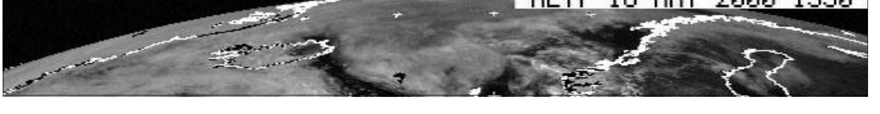
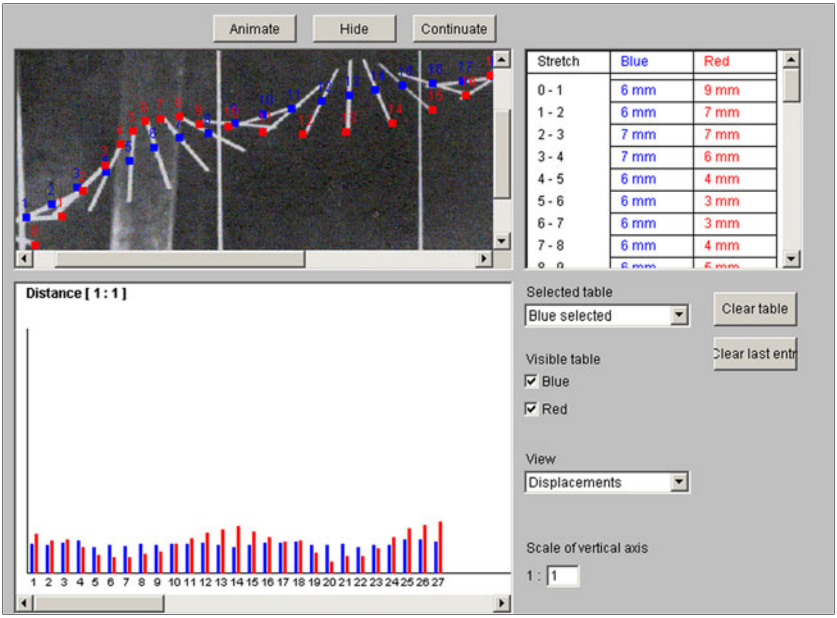
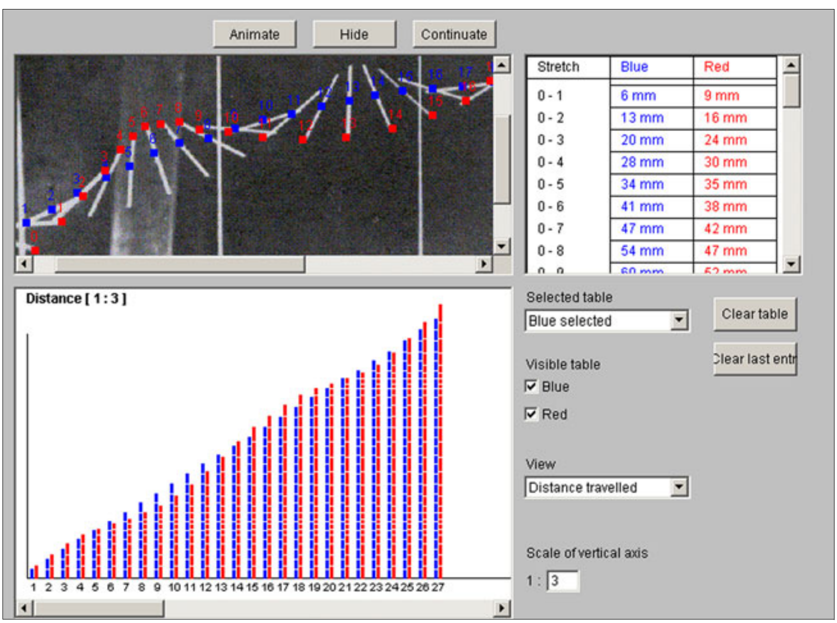


figure: A part of the first page of the student-activities

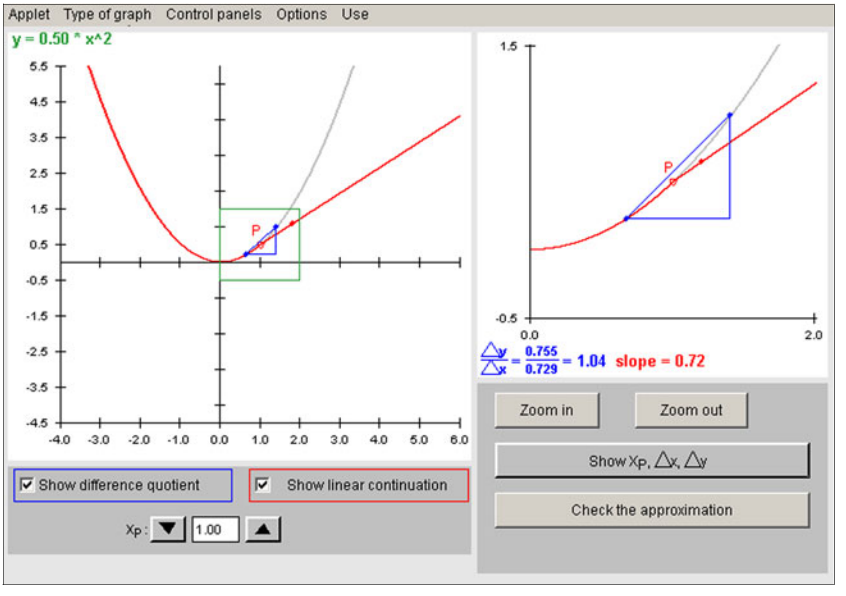
Appendix: Computer tools

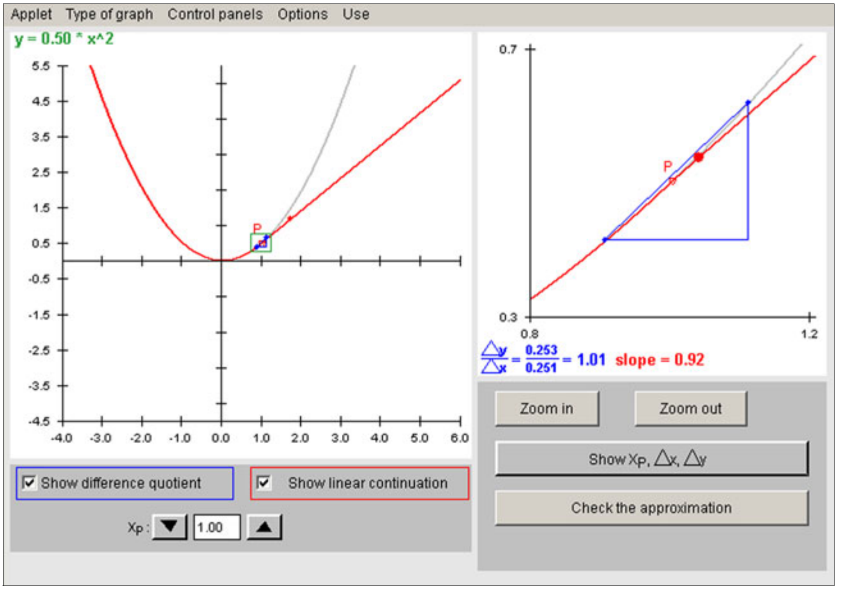
Flash and Slope are the computer tools used within the instructional sequence. The full-colour screenprints of these tools give a better impression of the tools than the other figures in this book. Two screendumps of Flash are displayed below:





The first screendump of Flash shows two graphs of displacements of a rotating stick (see page 108). The second screendump shows the graphs of distances travelled. The other computer tool is Slope. The first screendump of Slope shows the possibility of approximating change by adjusting a red continuation of the curved graph, or by using the two blue points of a chord on the graph (see page 116). The second screendump shows how an approximation can be improved by using the zoom-tool:





Curriculum Vitae

Michiel Doorman was born on 1 october 1962 in Eindhoven, the Netherlands. He completed his secondary education in 1981 at the Minkema College in Woerden. In 1988 the Utrecht University awarded him a masters in Mathematics for his thesis on the extension of a proposition in intuitionistic logic for automated theorem proving. He minored in Computer Science. From 1988 he has been working at the Freudenthal Institute. Until 1992 he was mainly devoted to software development. During the following years he has been involved in curriculum and teacher training projects, mainly concerning the role of information and communication technology in mathematics education. Since 1994, this work concentrated on upper secondary (pre-university) mathematics education in a research project on the integration of the graphing calculator, in a curriculum development project (Profi), and in a project that aimed at guiding the Biology, Chemistry, Physics and Mathematics departments in schools to cooperate. In 1998 he started his PhD research study.