



A Designer Speaks: Malcolm Swan

MARS/Shell Centre
University of Nottingham
England

Designing a Multiple Representation Learning Experience in Secondary Algebra

Abstract

This paper describes some of the research-based principles that I use when designing learning experiences to foster conceptual understanding. These principles are illustrated through the discussion of one type of experience: that of sorting multiple representations. I refer to learning experiences rather than tasks, because tasks are only one component of the design. Close attention is also paid to the role of the teacher in creating an appropriate climate for learning to take place.

After a brief excursion into my own theoretical framework, I describe the educational objectives behind my design and provide a detailed explanation of it in one topic, that of algebraic notation. This is followed with an explanation of the principles that informed the design and the evolution of the task. Finally, I briefly indicate how the design might be generalised to include other topics.

Introduction

My own designs for novel learning experiences are based on a three-way analysis of: the purposes they are intended to serve; the learning theories related to those purposes; and the empirically tested principles for design that emerge from these theories. Clearly, different learning theories apply when different learning outcomes are desired. A full analysis of these purposes, theories and principles is available in [Swan \(2006a\)](#).

There are perhaps five distinct purposes for learning mathematics: (i) developing fluency when recalling facts and performing skills; (ii) interpreting concepts and representations; (iii) developing strategies for investigation and problem solving; (iv) awareness of the nature and values of the educational system and (v) an appreciation of the power of mathematics in society (e.g. [Cockcroft, 1982](#); [NCTM, 1988](#); [Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999](#); [Stigler & Hiebert, 1999](#)). In the examples described below, my purpose is conceptual.

The learning theories I adopt for this purpose are derived from the social constructivists: concepts are co-created as language and symbols are appropriated and internalised ([Bakhtin, 1981](#); [Vygotsky, 1996](#)). Collaborative discussion is therefore essential. Other relevant design principles include the importance of focusing directly on significant conceptual obstacles ([Bell, 1993](#); [Wigley, 1994](#)); building on the knowledge students already have ([Black & Wiliam, 1998](#)); creating tension and cognitive conflict that may be resolved through discussion ([Bell, 1993](#); [A Bell, Swan, Onslow, Pratt, & Purdy, 1985](#); [Brousseau, 1997](#)); using tasks that are 'accessible, extendable, encourage decision-making, creativity and higher order questioning' ([Ahmed, 1987](#)); using multiple representations to create connections ([Askew, Brown, Rhodes, Johnson, &](#)

William, 1997); and using tasks that allow students to shift roles and explain and teach one another (Bell, Swan, Crust, & Shannon, 1993b).

These principles challenge many teachers' existing orientations towards mathematics, learning and teaching as outlined in Figure 1 (Swan, 2005). A 'transmission' orientation in which explanations, examples and exercises dominate must give way to a more collaborative orientation in which students work together on 'interconnected', 'challenging' tasks; tasks that confront common difficulties. This model of learning should not be confused with that of "discovery" teaching, where the teacher simply presents tasks and expects students to explore and discover the ideas for themselves. Here, the teacher's role includes: assessing students and making constructive use of prior knowledge; making the purposes of activities clear; challenging students through effective, probing questions; managing small group and whole class discussions; encouraging the discussion of alternative viewpoints; drawing out the important ideas in each lesson; and helping students to make connections between their ideas.

Of course, principles guide rather than determine a design; the quality of the outcome also involves both creative input, based on prior experience and imaginative extensions from it, and systematic development through feedback from trials. These are much more difficult to describe.

In coming to understand a concept, a student must single it out and bring it to the forefront of attention (identify); notice similarities and differences between this concept and other similar ones (discriminate); identify general properties of the concept in particular cases of it (generalise) and begins to perceive a unifying principle (synthesise) (Sierpinska, 1994). Working with teachers, I have developed five task 'types' that encourage these processes (Table 1).

Figure 1: From a transmission to a collaborative orientation (Swan, 2005)

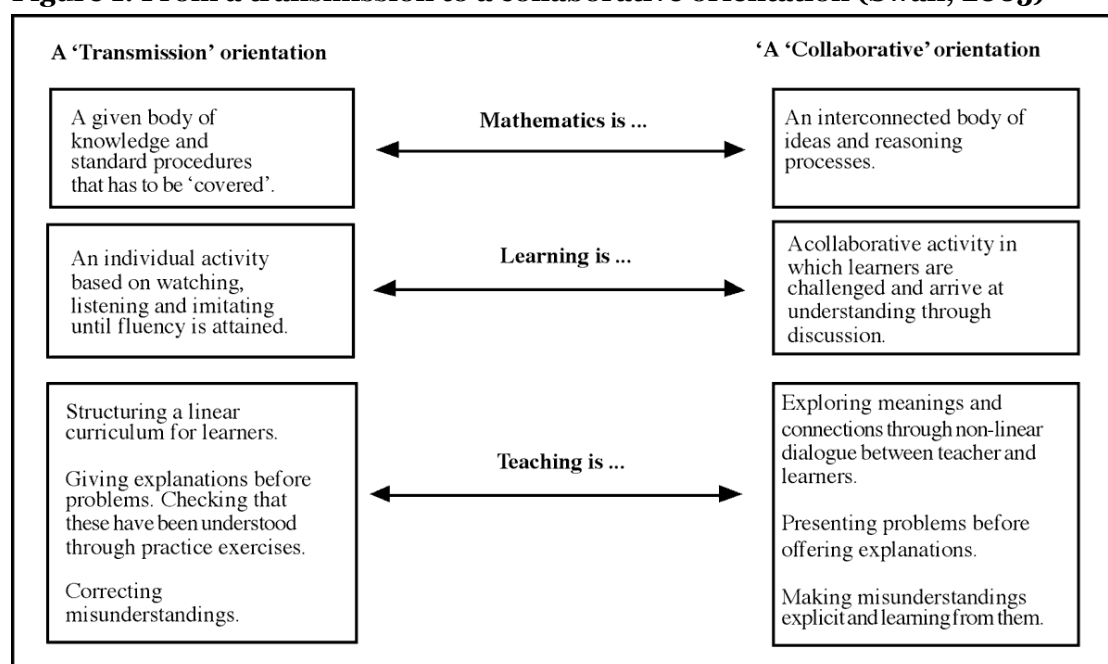


Table 1: Five task 'types' that encourage concept development

Classifying mathematical objects	Students devise their own classifications for mathematical objects, and/or apply classifications devised by others. In doing this, they learn to discriminate carefully and recognise the properties of objects. They also develop mathematical language and definitions. The objects might be anything from shapes to quadratic equations.
Interpreting multiple representations	Students work together matching cards that show different representations of the same mathematical idea. They draw links between representations and develop new mental images for concepts.
Evaluating mathematical statements	Students decide whether given statements are always, sometimes or never true. They are encouraged to develop mathematical arguments and justifications, and devise examples and counterexamples to defend their reasoning. For example, is the following statement always, sometimes or never true? If sometimes, then when? "Jim got a 15% pay rise. Jane got a 10% pay rise. So Jim's pay rise was greater than Jane's."
Creating problems	Students are asked to create problems for other students to solve. When the 'solver' becomes stuck, the problem 'creators' take on the role of teacher and explainer. In these activities, the 'doing' and 'undoing' processes of mathematics are exemplified. For example, one partner may create an equation, then the other tries to solve it.
Analysing reasoning and solutions	Students compare different methods for doing a problem, organise solutions and/ or diagnose the causes of errors in solutions. They begin to recognise that there are alternative pathways through a problem, and develop their own chains of reasoning.

The educational objectives of the design.

I now describe a specific example of one of the task types, "*Interpreting multiple representations*". In this task, students sort cards showing alternative representations of algebraic expressions into sets, so that each set has equivalent meaning. In so doing, students must discriminate between commonly confused expressions and explain their differences. This task focuses on an aspect of mathematics content that is often taught, with limited success, in a mechanistic manner: the interpretation and manipulation of algebraic expressions. In particular, it involves: translating between algebraic expressions words, tables, and geometrical areas; recognising the order of arithmetic operations implied by conventional notation; recognising equivalent expressions; using the distributive laws of multiplication and division over addition. The design is taken from a nationally distributed resource for teachers ([DfES, 2005](#)), and was developed as part of a research project with low-attaining 16-19 year old students in Further Education Colleges ([Swan, 2006a](#); [Swan & Green, 2002](#)).

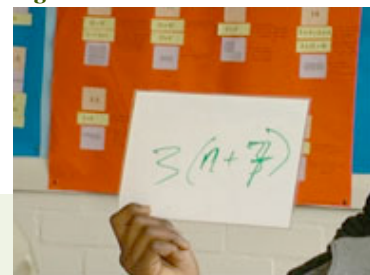
An outline of the design

The preparation for the lesson involves enlarging and cutting out sets of cards (Figure 3) so that they may be easily viewed by groups of 2 to 4 students. Before beginning the task, the teacher asks students to respond to a few algebra questions on paper, without consultation. These answers are not discussed, but are put to one side. The teacher explains that they will be reconsidered to at the end of the session. The purpose of these is to help students become aware of any changes in their own thinking. These questions are of the form:

What does $4(x+5)$ mean? Does it mean the same thing as $4x+5$? If not what is the difference?
 What does $(5x)^2$ mean? Does it mean the same thing as $5x^2$? If not what is the difference?
 What does $(x+6)^2$ mean? Does it mean the same thing as x^2+6^2 ? If not what is the difference?

Next, the teacher asks the students a few oral questions, to assess how well they can translate between words and symbols. Students present their answers using 'mini-whiteboards'. These are small, easily erasable laminated boards (Figure 2). Their advantage is that the teacher can rapidly assess many responses.

Figure 2: Mini whiteboard



Show me an algebraic expression that means:
 Add 7 to n , then multiply your answer by 3;
 Multiply n by 3, then add 7;
 Add 2 to n , then divide your answer by 4;
 Multiply n by n , then multiply your answer by 4;
 Multiply n by 6, then square your answer.

Students now work in groups. The set of cards (Figure 3) are presented in three stages.

Figure 3: Cards for matching (DfES 2005)**A1 Card set A – Algebraic expressions**

E1 $\frac{n+6}{2}$	E2 $3n^2$
E3 $2n+12$	E4 $2n+6$
E5 $2(n+3)$	E6 $\frac{n}{2}+6$
E7 $(3n)^2$	E8 $(n+6)^2$
E9 $n^2+12n+36$	E10 $\frac{n}{2}+3$
E11 n^2+6	E12 n^2+6^2
E13	E14

A1 Card set B – Explanations in words

W1 Multiply n by two, then add six.	W2 Multiply n by three, then square the answer.
W3 Add six to n , then multiply by two.	W4 Add six to n , then divide by two.
W5 Add three to n , then multiply by two.	W6 Add six to n , then square the answer.
W7 Multiply n by two, then add twelve.	W8 Divide n by two, then add six.
W9 Square n , then add six.	W10 Square n , then multiply by nine.
W11	W12
W13	W14

A1 Card set C – Tables of numbers

T1					
n	1	2	3	4	
Ans	14	16	18	20	

T2					
n	1	2	3	4	
Ans			81	144	

T3					
n	1	2	3	4	
Ans		10	15	22	

T4					
n	1	2	3	4	
Ans	3		27	48	

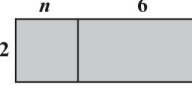
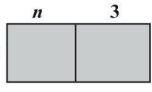
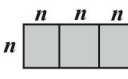
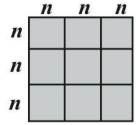

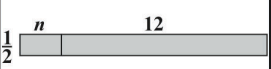
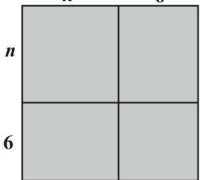
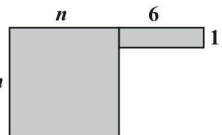
T5					
n	1	2	3	4	
Ans			81	100	

T6					
n	1	2	3	4	
Ans		10	12	14	

T7					
n	1	2	3	4	
Ans		4		5	

T8					
n	1	2	3	4	
Ans	6.5	7	7.5	8	

A1 Card set D – Areas of shapes

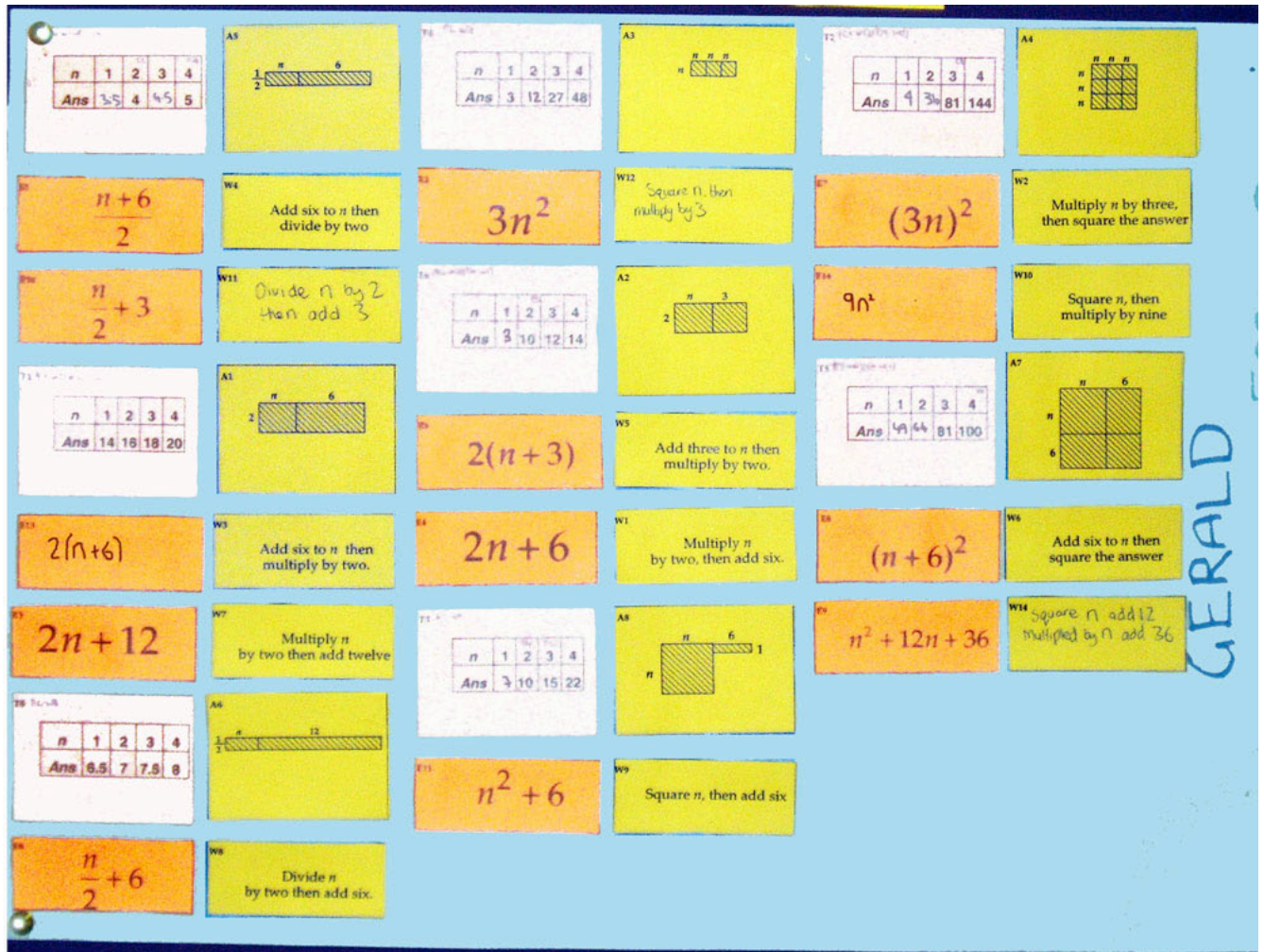
A1 	A2 
A3 	A4 
A5 	A6 
A7 	A8 

First, students take it in turns to match the *algebraic expressions* to the *written explanations*. As they do this, they must lay the cards out side-by-side on the table. This activity forces students to attend to the order of operations. (So for example, students must discriminate between "Add six to n then multiply by two" from "multiply n by two then add six"). Students use the blank cards to produce additional expressions or explanations when they cannot find ones to match. Some students may notice that some expressions are equivalent, e.g. $2(n+3)$ and $2n+6$. This is not commented on this at this stage.

Secondly, students are given the cards showing *tables of numbers*. This time, there are no blank cards, but there are gaps in the tables that need completing. These ensure that students continue to make connections after matching.

Very soon, students complain that there are not enough of these cards for one-to-one matching and then, more slowly, begin to notice that in some cases, different expressions correspond to the same table, such as $2(n+3)$ and $2n+6$; $(n+6)/2$ and $n/2+3$; $(3n)^2$ and $9n^2$. At this stage, they have no explanation for this, it is simply an observation. The teacher encourages students to test these equivalences using other values for n , including very large numbers, decimal numbers and fractions: "Does this always match, even when you choose $n = 250$? $n = 2.6$? $n = 1/3$?"

Thirdly, students are issued with the area cards and match these to the other cards, using the areas of these pictures to explain why different pairs of expressions might be equivalent. They do this by arranging the matching groups of cards onto posters and writing explanations alongside (Figure 4). Throughout this process, the teacher prompts students to make connections between representations: "How can you see that this area matches this expression directly?", "How can you see that these words match this table directly?"

Figure 4: Making Posters

A plenary discussion is now held in which the teacher encourages students to assess and generalise what they have discovered, using further questioning. For example:

Draw me an area that shows this expression: $2(n+3)$.
 Write me a different expression that gives the same area.
 Now draw me an area that shows this expression: $4(x-7)$.

Finally, students revisit their answers to the questions that were asked at the very beginning of the session and (i) correct their work, (ii) describe the causes of their mistakes.

Movies (from [Swan & Green, 2002](#))

Movie 1: Matching Cards



This movie shows three students in discussion with the teacher. This class is a 16-19 year-old group of low-attaining students retaking Mathematics at an FE college.

Movie 2: Extending and Generalising



This movie shows the end of a lesson, where the teacher tries to extend what students have discovered.

Design principles that informed the design

The following principles have been developed from my own research and a recent consultation with 150 mathematics educators (Swan, 2005; Swan & Lacey, 2008). These principles may be realised in the design, when it is faithfully implemented.

Teaching for conceptual understanding is more effective when we:

- **Use rich, collaborative tasks**
The tasks we use should be accessible, extendable, encourage decision-making, promote discussion, encourage creativity, encourage 'what if' and 'what if not?' questions (Ahmed, 1987). The main activity here is flexible and non-linear. Students do not need to start or finish at the same point. Students are able to choose which cards to interpret. This enables everyone to engage with the activity. The task is also flexible in that students may be encouraged to develop further cards at any level of difficulty.
- **Develop mathematical language through communicative activities**
Mathematics is a language that enables us to describe and model situations, think logically, frame and sustain arguments and communicate ideas with precision. Students do not know mathematics until they can 'speak' it. Interpretations for concepts remain mere 'shadows' unless they are articulated through language (Vygotsky, 1996). We find that many students have never had much opportunity to articulate their understanding publicly. The main task here focuses on the communicative aspects of mathematics by developing oral and written mathematical language.
- **Build on the knowledge learners already have**
This means developing formative assessment techniques so that we may adapt our teaching to accommodate learning needs (Black & Wiliam, 1998). This lesson does not follow the traditional pattern for explanation followed by exercise. Instead, the teacher asks expose and assesses existing ways of thinking and reasoning before explaining. During the main activity the teacher listens to the discussions before joining in, and when she does so, she attempts to prompt students to articulate their thinking and reasoning. Teacher explanation follows this discussion, it does not pre-empt it.
- **Confront difficulties rather than seeks to avoid or pre-empt them**
Effective teaching challenges learners and has high expectations of them (Bell, 1993; Wigley, 1994). It does not seek to 'smooth the path' but creates realistic obstacles to be overcome. Confidence, persistence and learning are not attained through repeating successes, but by struggling with difficulties. We have built conceptual obstacles into this design. Expressions that are often confused, such as $3n^2$ and $(3n)^2$ are deliberately included to provoke discussion.
- **Expose and discuss common misconceptions and other surprising phenomena**
Learning activities should expose current thinking, create 'tensions' by confronting learners with inconsistencies and surprises, and allow opportunities for resolution through discussion (Bell, 1993; Bell et al., 1985; Brousseau, 1997). The activity encourages misconceptions and alternative interpretations to surface so that they may be discussed. Conflicts originate both internally, within the individual and externally, from an individual's interpretation of another person's alternative viewpoint.
- **Use higher-order questions**
Questioning is more effective when it promotes explanation, application and synthesis

rather than mere recall ([Watson & Mason, 1998](#)). Throughout the activity, teachers are encouraged to prompt students to reflect and explain through the use of open prompts that begin "Explain why..."; "Show me an example of.."; "How do you know that...?"

- **Make appropriate use of whole class interactive teaching, individual work and cooperative small group work.**

Collaborative group work is more effective after learners have been given an opportunity for individual reflection. Activities are more effective when they encourage critical, constructive discussion, rather than argumentation or uncritical acceptance. Shared goals and group accountability are important. Teachers are advised to gradually establish 'ground rules' for discussion among students and then behave in ways that encourage dialogic and exploratory talk ([Alexander, 2006](#); [Mercer, 1995, 2000](#)) ([Table 2](#)).

Table 2: Ground rules for students and the role of the teacher during a discussion

Ground rules for students	The role of the teacher.
<ul style="list-style-type: none"> • Give everyone a chance to speak. • Listen without interrupting. • Check that everyone else listens. • Try to understand what is said. • Build on what others have said. • Challenge what is said. • Demand good explanations. • Treat opinions with respect. • Share responsibility. • Try to reach agreement. 	<ul style="list-style-type: none"> • Make the purpose of the discussion clear. • Keep reinforcing the 'ground rules'. • Listen before intervening. • Join in, don't interrupt. • Don't judge or praise - this discourages contributions. • Ask students to describe, explain and interpret. • Do not do the thinking for the students. • Don't feel you need to resolve everything before leaving a group or before the end of the lesson.

- **Encourage reasoning rather than 'answer getting'**

Often, learners are more concerned with what they have 'done' than with what they have learned. It is better to aim for depth than for superficial 'coverage'. Here, students are told that comprehension is more important than completion. The teacher's role is to prompt deeper reasoning by asking students to explain, extend and generalise: "Why is it true that $2(n+3)=2n+6$? Can you produce an area diagram for $3(n-4)$?"

- **Create connections between topics both within and beyond mathematics**

Learners often find it difficult to generalise and transfer their learning to other topics and contexts. Related concepts remain unconnected. Effective teachers build bridges between ideas ([Askew, Brown, Rhodes, Johnson, & Wiliam, 1997](#)). As new cards are introduced, the possibility for building multiple connections between different representations is exploited.

- **Recognise both what has been learned and also how it has been learned**

What is to be learned cannot always be stated prior to the learning experience. After a learning event, however, it is important to reflect on the learning that has taken place, making this as explicit and memorable as possible. The end of the session serves several purposes. It allows students to share their findings through the public display of their work. As they show their posters, this gives status to their own ideas. It encourages students to extend and generalise their ideas by making small changes to the examples and then explicitly formulating rules for the equivalence of algebraic expressions. The teacher thus recognises and values the contributions of students, and then extends and institutionalises them ([Brousseau, 1997](#)).

The process of the design

In this section, I describe the process of designing the activity. The design presented here evolved from earlier versions, and no doubt will continue to evolve. In this case the final version (Figure 3) is similar to the first version (Figure 5) in many respects. The cards were originally devised to show combinations of symbols n , 2, 3, 6, together with pairs of operations add, multiply, divide and square. For each card, e.g. $2(n+3)$, an equivalent expression was devised, $2n+6$, so that equivalent expressions could be deduced. In addition, a distracter, $2n+3$, was introduced. Cards with the operations in the reverse order were introduced, so e.g. n^2+6 was accompanied by $(n+6)^2$. The area cards provided a spatial representation of these expressions. It was felt that students could review their knowledge of area while engaging with the task and use the areas to see why two algebra expressions might be equivalent.

The original version of the activity was in booklet form with advice to the teacher to make cards, *if possible*. When this was tried with teachers many did not take up this suggestion and they used the resource as though it were a textbook exercise. Students worked on the activity individually, in the order presented. The students wrote cryptic answers into exercise books like: 1-I; 2-F and so on. Towards the end of the session some teachers went through the answers using this notation and there was little discussion of the underlying concepts.

The original version suggested that the teacher should suggest that students substitute a number for n ($n=4$) to help them search for equivalent expressions. On reflection this did not encourage students to think of the letters as representing variables. Thus tables were introduced.

The original version contained no blank cards. This led students to match the final few expressions using elimination strategies. On reflection it appeared much more profitable to encourage students to devise cards of their own. In addition, some spaces were left on the tables cards, partly to encourage students to engage in substitution and also so that matching would be less easy to spot. Interestingly, the tables now make it possible to use new strategies for matching. For example, students who recognised square numbers in the table could search for a square area. Students were also surprised to discover that several expressions could correspond to the same table.

Comparing the original design to the final one it will also be noted that:

- Of the 12 algebra expressions in the original set, I have removed two simpler ones ($9n^2$; $2(n+6)$) and introduced two more complex ones ($n^2+12n+36$, n^2+6^2) along with some blank cards.
- Of the written explanation cards, I have removed two: "Square n then multiply by 3" and "Divide n by 2 then add three". The diagrams are identical and the tables are all new.

The purpose of these changes was to challenge pupils in the following ways:

- They must use the written explanations to construct the two easier expressions cards ($9n^2$; $2(n+6)$); and they already have two similar ones the teacher may refer to as a model (if this becomes necessary): ($3n^2$); $2(n+3)$
- They have to use the expressions cards to construct two easier written explanation cards: "Square n then multiply by three" and "Divide n by two then add three"; and they already have two similar ones to use as a model: "Square n then multiply by nine" and "Divide n by two then add six"

- They also have to construct two more complex written explanation cards: $(n^2 + 12n + 36)$, $n^2 + 6^2$

The introduction of the two more complex expression cards also gives pupils the opportunity to distinguish $(n+6)^2$ from $n^2 + 6^2$ (an important realisation) and, as the little poster shows in [figure 5a](#), they may also draw a diagram to show the difference. Secondly, they have a chance to look at the equivalence of $(n+6)^2$ and $(n^2 + 12n + 36)$. So, in short they are presented with the opportunity to deduce that $(n+6)^2$ is equivalent to $(n^2 + 12n + 36)$, not $n^2 + 6^2$, and give an explanation.

In conclusion, when designing the cards, I had to carefully consider:

- The affordances and limitations of each representation;
- How the cards would be used to expose common misconceptions;
- The order in which the representations must be presented to maximise conflict and discussion;
- The criteria that students will tend to use to match cards, and how I might minimise superficial matching;
- The degree to which sets of cards can open up the possibility for generalisation.

Figure 5a: Making sense of expressions

This page invites students to interpret the meanings of algebraic expressions. They then search for expressions which are equivalent, by substituting $n = 4$ into each one.

You will need to emphasize that expressions are only equivalent if they give the same answer for *every* possible substitution.

If possible, introduce this as a card sorting activity. (A photocopiable sheet is included on the next page.)

Card sorting frees the students from having to write and enables them to pay more attention to the meanings of expressions.

When they have finished arranging and rearranging the cards, they may then like to make a permanent copy by pasting their arrangement onto a large sheet of paper.

As students sort the expressions, listen carefully and note their difficulties.

On this page, the letter n stands for any number.

1. Discuss the meaning of each of algebra expression 1 - 12. Match the expressions with the explanations A-L.
2. Substitute $n = 4$ into each expression. What answers do you get? Which expressions give the same answer? Do they give the same answer for all values of n ? Can you explain why?



1	$\frac{n+6}{2}$	7	$3n^2$
2	$9n^2$	8	$2n+6$
3	$2(n+3)$	9	$\frac{n}{2}+6$
4	$(3n)^2$	10	$(n+6)^2$
5	$2(n+6)$	11	$\frac{n}{2}+3$
6	n^2+6	12	$2n+12$

A	Multiply n by two, then add six.	G	Square n , then multiply by three
B	Add six to n then multiply by two.	H	Multiply n by three, then square the answer
C	Add three to n then multiply by two.	I	Add six to n then divide by two
D	Multiply n by two then add twelve	J	Divide n by two then add three
E	Square n , then add six	K	Add six to n then square the answer
F	Square n , then multiply by nine	L	Divide n by two then add six.

Figure 5b: Check - making sense of expressions

This page looks again at the same expressions, but this time links them to areas. This should help students to begin to see *why* different expressions are equivalent.

Ask students to match the expressions on the cards with the shapes.

Encourage those that finish to invent additional expressions for each shape.


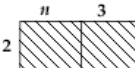
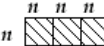
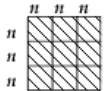


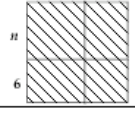

Leave time for students to make notes on the important things they feel they have learned. When they have done this, ask some to read aloud what they have written and invite questions and comments from other members of the class.

Write down expressions for the areas of each shape

Write each expression in more than one way.

Compare what you have written with the expressions on the previous page.

Does this help you to see why different expressions can give the same answer?

	Expressions which give the shaded area: $2(n + 6)$ $2n + 12$
	
	
	
	
	
	
	

Go back to the previous page and check your answers using these diagrams. Make notes on all you have learned.

**Concluding remarks**

The appendix to this paper displays two further card sets that I have devised to illustrate how this task-type may be used across the mathematics curriculum. Although such complex activities pose considerable management challenges for teachers, they have been extremely popular with both teachers and students and have strongly featured in three recent projects in which I have been involved (DfES, 2005; NCEE, 2006; NRDC, 2006). The evidence from these suggests the above task types have been popular with students, have improved learning outcomes and have also had an impact on the beliefs and practices of many mathematics teachers (Swain & Swan, 2007; Swan, 2006a, 2006b, 2007).

The good thing about this was, instead of like working out of your textbook, you had to use your brain before you could go anywhere else with it. You had to actually sit down and think about it. And when you did think about it you had someone else to help you along with you if you couldn't figure it out for yourself, so if they understood it and you didn't they would help you out with it. If you were doing it out of a textbook you wouldn't get that help. After I did it I found that I used a lot of brain power, but I felt dead clever. Do you know that when you have actually done something and you actually put all your effort into something.. it makes you feel dead clever. I've told all my friends that I have actually done a bit of work in maths. 'Cause I never thought I was any good at maths, but I was alright with that. (Lauren, a 16-year-old low-attaining student).

Acknowledgements

This research was funded at various stages by The Esmée Fairbairn Charitable Trust, The Learning and Skills Development Agency, and the Department for Education and Skills.

References

- Ahmed, A. (1987). *Better Mathematics: A Curriculum Development Study*. London: HMSO.
- Alexander, R. (2006). *Towards Dialogic Teaching: Rethinking Classroom Talk* (3 ed.). Thirsk: Dialogos.
- Askew, M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective Teachers of Numeracy, Final Report*. London: Kings College.
- Bakhtin, M. M. (1981). *The dialogic imagination: Four essays by M.M. Bakhtin* (C. Emerson & M. Holquist, Trans.): University of Texas Press.
- Bell. (1993). Principles for the Design of Teaching. *Educational Studies in Mathematics*, 24(1), 5-34.
- Bell, A., Swan, M., Crust, R., & Shannon, A. (1993b). *Awareness of Learning, Reflection and Transfer in School Mathematics; Report of ESRC Project R000-23-2329*: Shell Centre for Mathematical Education, University of Nottingham.
- Bell, A., Swan, M., Onslow, B., Pratt, K., & Purdy, D. (1985). *Diagnostic Teaching for Long Term Learning. Report of ESRC Project HR8491/1*: Shell Centre for Mathematical Education, University of Nottingham.
- Black, P., & Wiliam, D. (1998). *Inside the black box : raising standards through classroom assessment*. London: King's College London School of Education 1998.
- Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics* (N. Balacheff, M. Cooper, R. Sutherland & V. Warfield, Trans. Vol. 19). Dordrecht: Kluwer.
- Cockcroft, W. H. (1982). *Mathematics Counts*. London: HMSO.
- DfES. (2005). *Improving Learning in Mathematics*. London: Standards Unit, Teaching and Learning Division.
- Mercer, N. (1995). *The guided construction of knowledge*. Clevedon, Philadelphia, Adelaide.
- Mercer, N. (2000). *Words and Minds*. London: Routledge.
- NCEE. (2006). *America's Choice: Mathematics Navigator*. Washington, US.
- NCTM. (1988). NCTM Curriculum and Evaluation Standards for School Mathematics: Responses from the Research Community. *Journal for Research in Mathematics Education*, 19, 338-344.
- NRDC. (2006). *Maths4Life: Thinking through Mathematics*. London: DfES.
- Sierpinska, A. (1994). *Understanding in Mathematics*, . London: Falmer.
- Stigler, J. W., Gonzales, P., Kawanaka, T., Knoll, S., & Serrano, A. (1999). *The TIMSS Videotape*

Classroom Study: Methods and Findings from an Exploratory Research Project on Eighth-Grade Mathematics Instruction in Germany, Japan, and the United States (NCES 1999-074). Washington, DC: National Center for Education Statistics.

Stigler, J. W., & Hiebert, J. (1999). *The Teaching Gap* (2 ed.). New York: The Free Press.

Swain, J., & Swan, M. (2007). *Thinking Through Mathematics research report*. London: NRDC.

Swan, M. (2005). *Improving Learning in Mathematics: Challenges and Strategies*. Sheffield: Teaching and Learning Division, Department for Education and Skills Standards Unit.

Swan, M. (2006a). *Collaborative Learning in Mathematics: A Challenge to our Beliefs and Practices*. London: National Institute for Advanced and Continuing Education (NIACE); National Research and Development Centre for Adult Literacy and Numeracy (NRDC).

Swan, M. (2006b). Learning GCSE mathematics through discussion: What are the effects on students? *Journal of Further and Higher Education*, 30(3), 229-241.

Swan, M. (2007). The impact of task-based professional development on teachers' practices and beliefs: A design research study. *Journal of Mathematics Teacher Education*, 10(4-6), 217-237.

Swan, M., & Green, M. (2002). *Learning Mathematics through Discussion and Reflection*. London: Learning and Skills Development Agency.

Swan, M., & Lacey, P. (2008). *Mathematics Matters - An Executive Summary*: National Centre for Excellence in Teaching Mathematics.

Vygotsky, L. (1996). *Thought and Language* (A. Kouzlin, Trans. 9th ed.). Cambridge: Massachusetts Institute of Technology Press.

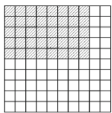
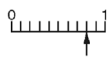
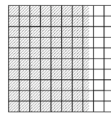
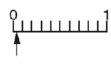
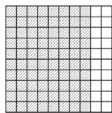
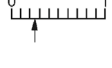
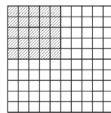
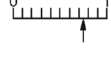
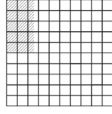
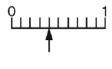
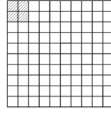
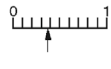
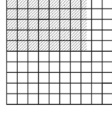
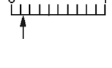


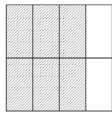

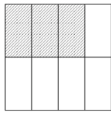
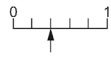
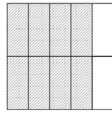

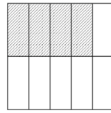
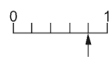
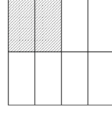
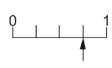
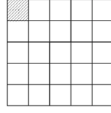
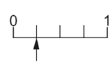
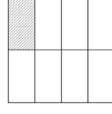



Watson, A., & Mason, J. (1998). *Questions and prompts for mathematical thinking*. Derby: Association of Teachers of Mathematics.

Wigley, A. (1994). Models for Mathematics Teaching. In A. Bloomfield & T. Harries (Eds.), *Teaching and Learning Mathematics* (pp. 22-25). Derby: Association of Teachers of Mathematics.

Appendices: Two further examples of multiple representation tasks

Example 1: Visualising and ordering decimals and fractions (DfES, 2005)

In this example, students are first asked to order a set of decimals into order of size. This exposes many misconceptions (e.g., size is related to the number of digits). They are then given area and/or number line diagrams to help interpret the numbers. Finally groups of cards are arranged in order of size. The whole process is repeated with the fractions and, over several sessions the six sets are combined.

Decimals	Proportions of a 100 square	Positions on a tenths scale
0.8	E1 	F1 
0.04	E2 	F2 
0.25	E3 	F3 
0.375	E4 	F4 
0.4	E5 	F5 
0.125	E6 	F6 
0.75	E7 	F7 
	E8 	F8 
Fractions	Proportions of a unit square	Positions on assorted scales
$\frac{3}{8}$	C1 	D1 
$\frac{2}{5}$	C2 	D2 
$\frac{1}{4}$	C3 	D3 
$\frac{3}{4}$	C4 	D4 
$\frac{8}{10}$	C5 	D5 
$\frac{1}{25}$	C6 	D6 
$\frac{1}{8}$	C7 	D7 
	C8 	D8 

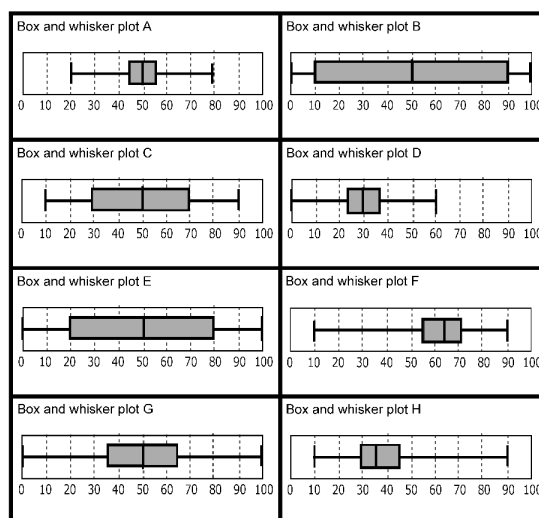
Example 2: Frequency graphs, Cumulative frequency graphs, and Box and whisker plots (DfES, 2005)

This is an example of a much more demanding task, where students match frequency graphs, cumulative frequency graphs, and box and whisker plots, for large samples.

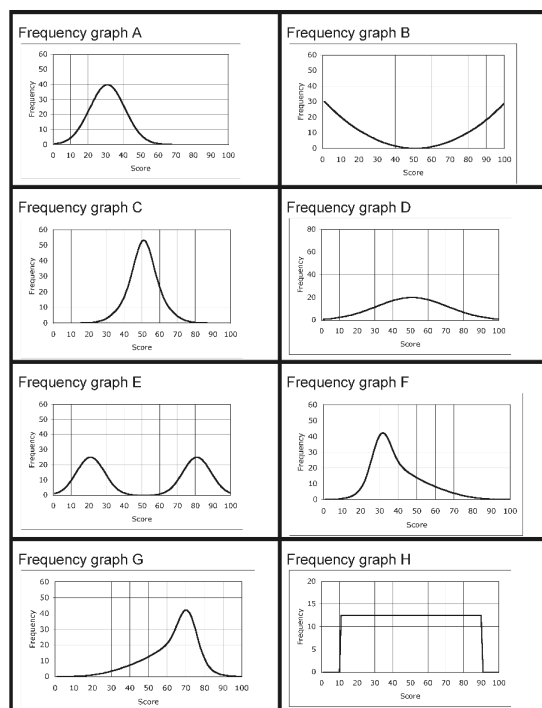
Words

This was the sort of exam where you could either do everything or you couldn't get started.	This exam did not sort out the better-performing candidates from the poorer-performing ones. They all got similar marks.
Two groups of candidates sat the exam. One group had studied the work for two years. The other group had only just begun.	This exam resulted in a huge spread of marks. Everyone could do something but nobody could do it all.
In this exam, the mean mark was greater than the modal mark.	In this exam, the mean mark was smaller than the modal mark.
In this exam, the median and the modal marks were the same. There was a very wide range of marks.	This exam was much too difficult for most candidates.

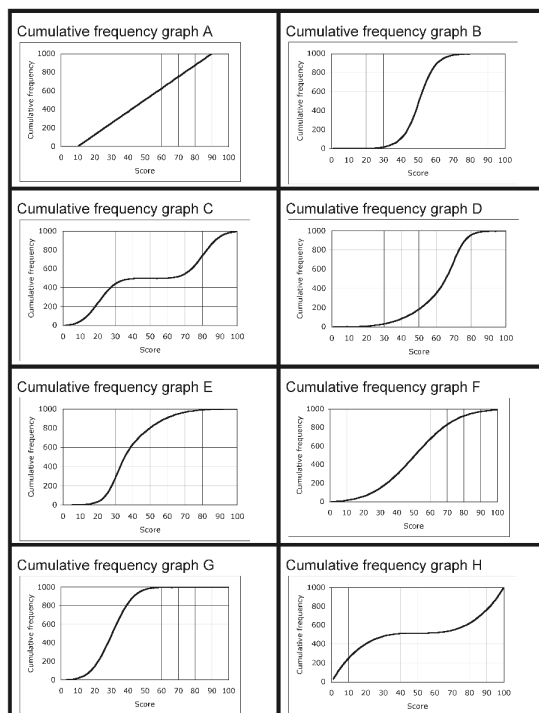
Box and whisker plots



Frequency graphs



Cumulative frequency graphs



About the author

Malcolm Swan is Associate Professor and Reader in Mathematics Education at the University of Nottingham and has been a leading designer-researcher since he joined the faculty in the Shell Centre for Mathematical Education in 1979. His interests lie in the design of teaching and assessment, particularly the design of situations which foster reflection, discussion and metacognitive activity, the design of situations in which learners are able to construct mathematical concepts, and the design of assessment methods that are balanced across learning goals – and thus have a positive backwash effect on teaching and learning. Diagnostic teaching, using ‘misconceptions’ to promote long term learning, has been an ongoing strand of this work.

He has led design teams on a sequence of internationally funded research and development projects including work for UK examination boards and the US NSF-funded *Balanced Assessment* project and the *Mathematics Assessment Resource Service* (MARS). He has designed courses and resources for the professional development of teachers, evaluating their impact on student learning and on the beliefs and practices of experienced teachers. These materials have been sent by the UK Government and the Bowland Trust to all schools and other relevant educational institutions. In 2008 he was awarded the ISDDE Prize for educational design for *The Language of Functions and Graphs*.

Swan, M. (2008) *A Designer Speaks. Educational Designer*, 1(1).
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