From Informal to Formal, Progressive Formalization an Example on "Solving Systems of Equations"

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This paper deals with the concept *progressive formalization* as one of the features of the theory of Realistic Mathematics Education. The concept is exemplified by the results of a study on the learning and teaching of solving (systems) of equations. Instructional materials were developed that supported students of about 11 years of age to use informal and intuitive strategies to solve problems. Through *guided reinvention* these strategies evolved to the more formal and standard solving methods. The study demonstrated that young students can develop a conceptual understanding of the mathematics of equations.

In the theory of Realistic Mathematics Education – known as RME – (Gravemeijer 1994, Treffers 1987), the concept of *guided reinvention* is a key feature. Students should be encouraged to reinvent the mathematics, guided by the teacher and the instructional materials. Following this approach, the students will take ownership of the mathematics and will develop a conceptual understanding. To reinvent formal and abstract mathematics, students should get the opportunity to gradually move from using informal, intuitive and concrete solving strategies through a variety of pre-formal strategies to using more formal, abstract standard solving strategies. This process is called *progressive formalization* (Freudenthal 1983, Treffers 1987) and is also a feature of RME.

The challenge for educational researchers and curriculum developers is to develop instructional sequences that support students to develop mathematical thinking, skills, and knowledge in meaningful ways embedding progressive formalization and guided reinvention. While RME is an overall theory for mathematics education, progressive formalization especially applies to algebra education. In this paper we will focus on algebra education, more specific on early algebra education and we will use data, examples, and findings from a study on the learning and teaching of solving (systems of) equations,

Progressive Formalization in the Context of Early Algebra

In many traditional curricula – among which the “classic” algebra I and algebra II courses in the USA – algebraic topics are introduced very quickly, and students get hardly time to develop conceptual understanding. There is often no connection made with existing (informal) knowledge of the students. This results in for students meaninglessly manipulating meaningless symbols. One aspect of progressive formalization is that in the learning sequence time is taken to develop the concepts and skills. So, one needs to start early. With respect to algebra, students at a young age (10 years in out study) have many informal and intuitive notions and conceptions on using symbols, organizing and describing information, using various representations to present information, and on finding generalizations for particular situations (Kaput 1995, Mason 1985). So there is much available in students’ environment and thinking world that can be used to develop an instructional sequence on an algebraic topic.

The Focus of This Paper

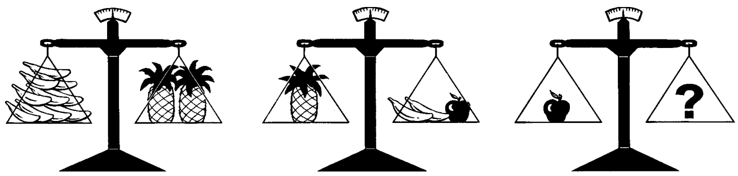
In the *Mathematics in Context* (MiC) curriculum (Wisconsin Center for Educational Research and Freudenthal Institute (Eds), 1995) that was developed for American middle school students (ages 10 to 14) we chose to start the development of algebraic skills and knowledge at an early age, and to use the four years to slowly come to a formal and abstract level. The two characteristics of RME progressive formalization and guided reinvention were two key features of the approach towards algebra in the MiC curriculum that we will elaborate on in the following.

The research that we report on in this paper concentrates on one unit of instruction of the MiC curriculum. *Comparing Quantities* is an instructional unit in the algebra strand of MiC designed for 6^{th} grade, and deals with the mathematics of solving (systems of) equations. The question we deal with in this paper is: "Does the chosen approach indeed contribute to a thorough conceptual understanding of the algebra of equations? In other words "Does a realistic approach of the beginning of the algebra of equations contributes to students development in algebraic knowledge and skills insightfully?"

Comparing Quantities, a Concrete Example of Progressive Formalization

Comparing Quantities was designed to provide students the opportunity to reinvent the mathematics of unknowns, variables and solving systems of equations. The major contexts of the unit are "shopping problems" in which students learn how to find the values of combinations of items, and to find the values of individual items when the value of combinations of items are known. The unit consists of five sections going from informal to formal mathematics.

The Unit starts with a collection of realistic situations and problems that are close to the world of students. The concept *exchanging* is introduced and students use a variety of informal symbols to present the information (words, pictures).



How many bananas are needed to make the third scale balance?

Figure 1. Compare and exchange.

In the second section the *combination chart* is introduced as a tool or strategy to find and represent values of combinations. Along the horizontal axis of the chart are the number of items of kind A (large cabinets) and on the vertical axis are the number of items B (small cabinets). The value of a combination of two kinds of items (the length of a combination of small and large cabinets) is put in the corresponding cell in the chart. A move from one cell to another means an exchange of items A for B, resulting in a change of the value of the combination. Students explore and investigate combinations and patterns in the combination charts, and use the charts to find the value of one individual item.

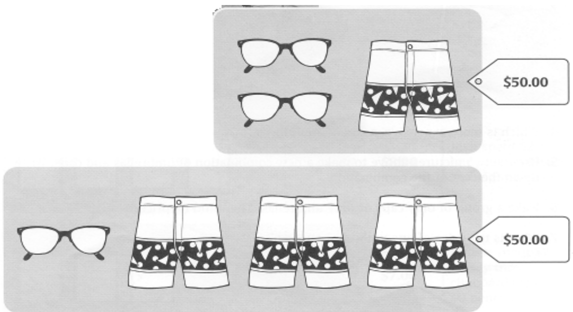
Ms. Simon wonders how she can design cabinets for a wall of 285 cm long. She decides to go to the cabinet store.
The cabinets store has a very convenient chart. Using this chart makes it easier to find out how many 60-cm and 45-cm cabinets are needed for a certain wall length.

1. Explain how Ms. Simon can find the number of cabinets she needs for a wall.

2 Can the cabinet store provide cabinets for exactly 4m/ Explain your answer?

Figure 2. Combination chart.

The third section deals with *shopping problems*, and students use exchanging, reasoning strategies, and the combination chart to solve these problems.



Without knowing the prices of a pair of glasses or a pair of shorts, which one is more expensive?
Explain how you know

Figure 3. Shopping problems, presented in pictures.

In the fourth section, the *notebook notation* is introduced, and students use this strategy to solve problems that deal with combinations of more than two kinds of items. The notebook is similar to a notebook used by a waiter in a restaurant. The columns represent the items that can be ordered (taco, salad, drink), and the price of an order is recorded in the last column. Each line of the notebook represents an order; a combination of items with the corresponding price. Lines in the notebook (the orders) can be combined (added, subtracted, multiplied or divided by a factor) to create new combinations (new orders).

ORDER	TACO	SALAD	DRINK	TOTAL
1	2	4	--	$10
2	1	2	3	$8
3	3	--	3	$9
4	1	2	--	
5	1	--	1	
6	2	2	1	
7	4	2	3	
8				
9				
10				

Mario runs a Mexican restaurant and he is very busy. He goes from one table to another, writing down all the orders. On the left you see how he writes the orders on his order pad.

1 Some of the orders do not have total prices.
What are the prices of the new orders?

2 Make up two new orders.
What will be the prices of your orders?

3 What are the prices of the individual items?

Figure 4, Notebook to write down combinations.

In the last section, the *formal equation* notation is used. Students translate information form a problem situation into an equation and make connections between the different strategies they used in earlier sections.

Some prices in Mario's restaurant have changed. One taco, two salads and a drink now cost $6.50. For one taco, four salads, and three drinks you have to pay $11.50. For $4.50 you can get one taco and two drinks

1. You can make an equation that corresponds to each of he orders. Write down the three equations.
2. By combining the orders, you can make new equations. What equation do you get when you add the last two orders?
3. Make up two other equations by combining orders.
4. Show how you can combine equations to get the equation 1S + 1D = 2.50. Explain your answer.
5. Find the new price for each of the three items.

Comparing Quantities in the Classroom

The unit was developed and tried out in several cycles (pilot, field test, commercial version), and in each cycle data was collected to capture the teaching learning process: Student work (notebooks of individual students, results of group work, work of individual students on the assessment tasks), field notes and observations reports, and interviews with teachers.

For the purpose of this paper we restricted ourselves to data that illustrates students' progress over the course of the unit. From four classes in the field test with very complete data we distinguished six different groups of students. Each group demonstrates another way of learning. In the following we only briefly summarize the learning of the students, using exemplary students from each group. Extensive reports on student learning will be published in an upcoming full report on this study.

Andy and Michael

Andy and Michael were bright students who thought everything was easy for them, sometimes even too easy. Especially Michael got bored. His work showed that he had a somewhat arrogant or superior attitude towards the use of contexts and pre-formal strategies.

Andy and Michael seemed to know the formal mathematics of equations from the very beginning of the unit. They were not interested in the pre-formal strategies that are used in the unit. Michael used letters to stand for words already in the first section. He was not interested in combination charts and notebooks. When equations were finally introduced in section E, the unit did not seem to be boring any more. From then on, he wrote everything down very carefully, and he even explained what he was doing.

Andy's work showed a similar progress through the unit. He did not do much, but he sort of woke up in the last section. He suddenly started to write equations and he also made the connection with the combination chart and the notebook.

Amanda and Mary

Amanda and Mary had problems with organizing their work as they had so many things on their mind. They needed structure.. Amanda and Mary worked very concentrated on the combination charts and the notebooks However, they did not seem not to understand the meaning of these strategies and why they were used. In the last section though, Amanda used equations and explained how to combine two equations, but it is not clear if she really understood what the equations stood for.

Mary followed a similar path through the unit: nicely filling out combination charts and notebooks, but it is not clear if she understood the strategies. In section E, Mary did not get to the level of formalization; she did not how to write equations using the formal mathematical notation. Amanda and Mary liked the pre-formal strategies, and preferred these strategies to solve problems above the formal equations.

Louise and Beth

Louise and Beth did everything as they were told to do, by the teacher or by the unit. They were average students who did even the simplest calculations on paper in their notebooks.

Their notebooks provide examples of careful reasoning and they gave clear explanations for their conclusions. The process Louise and Beth went through – from the level of verbal reasoning to level of using formal mathematical notation and reasoning – can be easily followed. They used the combination chart and the notebook for solving problems in sections B, C, and D, and they understood these strategies. In section E, both students made the connection between the pre-formal strategies and the formal equations.

Erik and Jeff

Erik and Jeff had difficulty concentrating on the problems and finishing their work. They were somewhat sloppy, and did not explain much of their way of thinking.

Erik preferred pictures and tried to draw equations with pictures from section B on. For Jeff and Erik, combination charts and notebooks seemed to be just a nice way of writing down the solutions to the problems, that they got through reasoning with (picture) equations. They did not seem to use these strategies (combination chart, notebook) to solve the problems. Erik and Jeff wrote and used formal equations in the last section, but did not make the connection with the chart and notebook explicit.

Harry

It seemed that Harry took his own path in many ways; he had some original ways of writing information down. He sometimes created something which had nothing to do with the problems. These additions were correct but not related to the question. Harry made progress during this unit but he did not seem to achieve the formal level that most students got to. He did not understand the structure of systems of equations.

Joni

Joni went through the unit following the intended learning trajectory as intended and expected. She was also able to explicate her thinking on paper in her notebook. She used figures and pictures in very efficient and informative way. Joni wrote her answers carefully down and explained them also. Her notebook is a very nice example of the learning process she went through.

Joni started to solve problems with the help of pictures and during the unit she developed ways to put her statements in more mathematical form. The following example of Joni's work comes from section D where she created her own kinds of equations, before they were formally introduced in the unit. The problem is about three chickens (Small, Medium, Large) and their weights. Given is: small and large together weigh 8.5 kg; Medium and Large weigh 10.6 kg; and Medium and Small weigh 6.1 kg.

Pg 23 - chickens

1.) s + m + L = ?
2s + 2m + 2L = 25.2
s + m + L = 12.6
m + L =
s = 12.6 − 10.6 = 2.0
m = 6.1 − 20 = 4.1
L = 8.5 − 20 = 6.5

Figure 11. Example of Joni's reasoning for the chicken problem

At the end of the unit, Joni understood the combination chart and notebook, as well as the formal equations. Joni showed that she understood the connection between the different strategies and produced her own examples of these kinds of problems.

Results and conclusions

The teachers were pleased with students' mathematical development during the three weeks they worked on the unit. The interviews with the teachers and the observations confirm that most students had really learned mathematics, that they had progressed to at least pre-formal levels, and that they had a good time with the mathematics itself as well.

The unit builds on existing intuitive and informative knowledge students bring in. The realistic problem situations in the unit brings the mathematics to students' own world, and over the course of the unit, their world is extended with, combination chart, notebook, and equations. The teachers orchestrated this learning process. With the instructional materials they guided the students in the process of (re-)inventing the algebra of equations. Teachers a difficult time to accept that students had to do the work. After a couple weeks, however, they recognized the positive effects on student learning, that students took ownership.

Student Learning

The student work gives a comprehensive picture of students' learning.. Different kinds of students (weak, average, good performers) were analyzed so we had information about the average students' progress as well as the progress of other students. We have learned how students developed the mathematical concepts related to the algebra of equations, and they developed a conceptual understanding insightfully.

The group of *average* students (like Amanda, Mary, Louise, Beth, Jeff, Erik) seemed to benefit the most of the realistic approach. To the majority of the students the mathematics made sense, and they learned how to solve (systems) of equations. Students like Louise, who needed a lot of structure and wrote much in their notebooks, were able to understand the different strategies and eventually invented their own ones.

Bright students (like Michael, Andy, and Joni) also developed new mathematical thinking and reasoning skills. Joni went through the unit very consciously. She did all the problems and got the opportunity to develop and use more abstract and formal strategies before they were introduced in the unit. Andy and Michael were already thinking at a formal level at the beginning of the unit. Michael got bored. However, students like Michael also got something valuable from this unit: the realistic contexts helped to get a more thorough understanding of the formal equations, and he learned to reflect on the meaning of the formal mathematics.

Weaker students like Harry followed their own ways, and did not develop a good understanding of combination chart, notebook, or equations, but these students made progress: The very informal and intuitive ways they used in the beginning of the unit evolved to more systematic notations.

From Informal to Formal

Students used the strategies that are introduced in *Comparing Quantities* in various ways. Some students used the exchange strategy, combination chart, or notebook as "puzzles"; as tools on themselves, not connected to the problem situation that they were solving. Other students used the strategies only as schemes to record their solutions in. Solutions that they got by using guess & check and reasoning strategies, or by using equations. However, the majority of the students used the strategies to develop a conceptual understanding of the mathematics of solving (systems of) equations in the context of shopping problems.

At the end of the unit, when the formal equation is introduced, almost all students understood this formal notation and what the equations mean. Most of the students made an explicit connection of the equations with the combination chart and notebook notation, although it was not always clear if and how they used these strategies in earlier sections.

The first four sections of *Comparing Quantities* focus on the informal and pre-formal strategies, and lay the foundation for the formal mathematics of equations introduced in the last section. At the end of the unit – when the students had gone through the process of step by step formalizing the informal strategies – the step to the formal equation was not big. Students were ready to complete the progressive formalization process of solving (systems of) equations.

Concluding

It is not completely clear how students used the various strategies in the learning of the mathematics of equations, but it is clear that a slow process of starting with informal solving methods through pre-formal strategies towards the formal equations helps students to develop a conceptual understanding of equations. Progressive formalization in a learning environment in which students reinvent the mathematics guided by the teacher and the instructional material contributes to learning for understanding.

From the study with *Comparing Quantities*, from curriculum development in The Netherlands over the past decades, and from the MiC project we can conclude that starting at an early age with algebra activities – intuitive, concrete, and informal – helps to lay a thorough foundation for understanding more abstract algebra in later years. Thus, we encourage to start early with sense making algebra activities.

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