

Functional Algebra with the use of the Graphing Calculator

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Algebra is a very important topic in mathematical programs for upper secondary education, but a vast majority of students is weak in understanding and using formal algebraic tools. This paper discusses some ideas about using the graphing calculator to support the learning of algebra in the context of functions and to help students overcome algebra-anxiety. Accepting the graphing calculator as a supportive toolkit in the learning of algebra has far-going consequences for *the way* in which *what* kind of algebra should be learned and taught.

Introduction

In the Netherlands, students can choose for vocational training in upper secondary education (age 16-20), after finishing lower secondary general or vocational education. When choosing for vocational training in the field of engineering, students have to go through a lot of mathematics and science training. Mathematics used to be a subject that kept many students away from being successful, because they were not able to fulfil the demands for algebra in the first year of the course. This was one of the reasons to start a project to redefine the mathematics and science curriculum for engineering courses. The main aim of the project was to define a new mathematics (and science) program that (1) is really supportive for the vocational subjects of study and later practice, (2) integrates the use of IT tools in both the learning process and assessment and (3) takes into full account the poor algebra skills of most students. For mathematics, many interesting findings about algebra can be reported from the project (Van der Kooij, 1999, 2000), but in this paper only the findings about the integration of the graphing calculator in the learning of algebra will be discussed. The main focus is on the functional and language-based aspects of algebra.

Background

Over the last decade, the influences of the graphing calculator on learning algebra of functions are discussed in many papers. Roughly, these articles can be split up in two different groups: the ones that emphasize the danger because a graphing tool should easily cause misconceptions about functions, and the ones that show improvement of students' conceptual knowledge of functions due to the use of graphical tools.

Studies of the first group, discussed in an overview (Dunham, 2000), are mostly focusing on the problems that are caused by the window format and the discrete nature of (the graphs on) the screen. The supposed misconceptions are mainly: mistakes about slopes because of different scaling on the axes; wrong ideas about (dis)continuity and asymptotes because a graph on the screen is drawn with discrete pixels, and the incompleteness of graphs because of the chosen window setting. I added 'supposed' to it, because misconception seems not the right word to use. In my opinion the main problem here is that educators and researchers expect that the use of graphing tools for studying functions should result in the same knowledge and skills as through traditional paper and pencil mathematics education (PPM). But in PPM, textbooks and educators always make a graph have the same scaling along the axes: the misconception about the slope is not due to the use of a graphical tool, but to the rigidity of traditional education in PPM. The same kind of temporary problems are found when graphs from other disciplines or from magazines are used in PPM, because they almost never use same scalings along the axes. About continuity and asymptotes: in PPM, graphs are only the final step in an

extensive process of algebraic work. Considering continuity and discontinuities is part of the work to be done before a reasonable graph can be plotted by hand. The availability of graphing tools can change this order completely: plotting a graph can now be the starting point for the investigations on the behavior of a function. The ZOOM, TRACE and TABLE features of the graphing tools are interesting options that can stimulate further (algebraic) analysis of these phenomena. The definition of a complete graph is also related to PPM: it is a graph that shows all important information that results from the algebraic work, like the position of zeros and extreme values. Seen this way, the left graph in figure 1 is complete and the right one is not complete (the function is $y = x^3 + x^2 - 4$). The left graph has window settings $[-5,5] \times [-5,5]$ and the right one $[-20,20] \times [-5000,5000]$

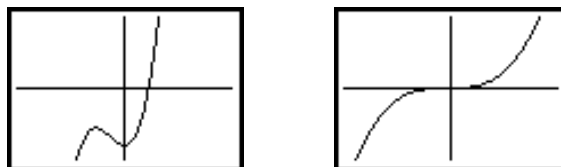


Figure 1. Complete graphs?

The word ‘complete’ is very confusing in this context. On domain $[-\infty, \infty]$ a graph can never be completed and the right one in figure 1 seems to be at least less incomplete than the left one, because it covers a bigger part of the whole domain. For graphing calculator use, the idea of a complete graph should be replaced by the idea that, whatever window setting is chosen, the window only allows you to see a small part of the graph. The ZOOM feature allows to inspect details (zoom in) and global behavior of a function (zoom out).

Mathematical misconceptions related to the use of graphical tools can only be anticipated if the tool is accepted as a supportive tool for the learning of mathematics, bringing its own qualities. Instead of expecting the same learning outcomes that are reached in PPM, the use of graphing tools should lead to reconsideration of curricular and pedagogical issues.

The second group of papers discusses the improved conceptual knowledge of functions when graphical tools are used in education. These reports discuss well defined experiments in which graphical tools do support the learning of both the operational concept (a function as a process) as well as the structural concept (a function as an object). The type of problems that are offered in these experiments make full use of the qualities that a graphical tool has to offer: visualizations of algebraically defined relationships. Students who learn the algebra of functions in a graphical tool environment (GTE) are more able to approach problems from a multi-representational perspective (Hollar & Norwood, 1999) than traditionally educated students. One reason for this flexibility of students is the fact that a graphical tool like the TI 83 makes it easy to switch between the three representations of a function: the table, the formula and the graph. Another important reason can be found in the kind of activities that students are dealing with in GTE. In PPM the emphasis is on training algorithmic algebraic routines, where in GTE the “why” is at least as important as the “how” (O’Callaghan, 1998). An important claim of O’Callaghan is that GTE improves students’ attitudes and reduces their anxiety toward mathematics.

When graphical tools are used in the learning of the algebra of functions, it is important to realize that investigations on the typical qualities of the tool (*what* can it do and *how* does it work) should be part of the content. It should also be realized that the specific qualities of the tool should define the kind of activities that are presented in classroom. One more aspect that is possible in GTE is hardly discussed until now. The availability of graphical tools makes it possible to learn a mathematical concept in a way that is completely different from the traditional way. The ZOOM option, for example, opens the opportunity to present the slope of a curve through local linearity instead of through the formal algebraic treatment of limits (Tall,

1985). “Students with a strong calculator-based precalculus experience come to calculus with different skills and conceptions of mathematics. It is important to take advantage of these new skills”. This argument was used to define a calculus course based on the local linearity approach (Teague, 1996). The program he suggests is quite different from the traditional ones. Using the specific qualities of graphical tools to define new approaches to traditional topics of algebra is one of the main targets of this paper. The graphing calculator and dynamically defined, interactive Java applets are considered as supportive tools for the learning of algebra. They have the potential of enhancing the flexibility of students to solve problems in which algebra plays a role.

Algebra and the use of the graphing calculator

Given the idea of the graphing calculator (GC) as a supportive toolkit, there are at least three levels on which this toolkit can be used to support the learning of algebra:

1. tools to be used as an alternative for routine work (calculations and plotting) that otherwise will take a lot of time when done by hand or that is too hard to handle for the (majority of) students.
2. tools that support the learning of a mathematical concept in two ways:
 - through the visualization of a concept
 - through an alternative way to introduce the concept.
3. tools that support investigations on problems in a mathematical or real-life context.

These three levels will be illustrated with examples and some discussion.

1. Alternative for routine work.

In the traditional approach of mathematics education, trigonometry is a very difficult subject for most students. Many formulas have to be learned and most students, when they are confronted with a problem like

Given the functions $f(x) = -1+2\sin^2(x)$ and $g(x) = 1+\cos(x)$

a. Solve $f(x) = 0$

b. Solve $f(x) = g(x)$

try all known formulas in a random way, hoping that the right one will pop up sooner or later. Since the availability of new technology, the focus of an educational program can be more on sinusoids with characteristics based on symmetry and periodicity and less on algebraic algorithmic treatment. In the Netherlands, many formulas have disappeared from the curriculum and are replaced by qualitative reasoning, visually supported by graphs. Question b of the problem, for example, can be solved with the help of the GC as follows:

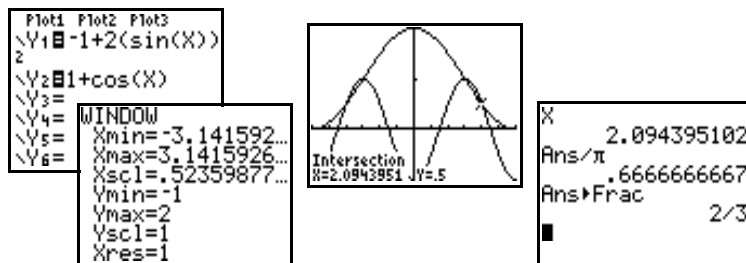


Figure 2. Graphing and reasoning

The other solution ($\pi/2$) can be found in the same way. Now all solutions can be generated, only using reasoning based on symmetry and periodicity. The example is a traditional assessment problem that can also be used with this aim when the GC is at hand. Instead of using algebraic techniques, students should realize that the answering now asks for qualitative reasoning,

supported by sketches of the graphs and some information about how the GC was used in solving the stated problem. An alternative question in this assessment problem could also be:

The function $f(x) = -1 + 2 \sin^2(x)$ can be described as a sinusoid. What formula fits this sinusoid?

A question like this is closer to the goals of the GC-supported way of learning mathematics. It is good to realize that the attainment targets of mathematics education should prescribe what is to be assessed and in which way. If mastering of all kinds of trigonometric formulas and their algebraic manipulation are still among the goals, students should be told that they have to solve the problem without the use of a GC (other than checking the answers they found). The availability of technology can cause a shift from algebraic, merely algorithmic activities towards a more conceptual approach. It takes time for students to arrive at the level of understanding concepts, but this needed time can be found by accepting that extensive, very specific algebraic training is no longer needed with the GC or any other technological tool at hand.

2. Alternative ways to investigate a mathematical concept.

In most countries the study of quadratic functions is an important topic in traditional programs. In the new Dutch programs parabolas are just one of the types of functions that are studied. A nice way of introducing parabolas was suggested some years ago by Judah Schwartz:

'Multiplication of two straight lines results in a parabola'

This geometry-oriented idea opens many opportunities to investigate families of functions in a way that is not even worth trying with only pencil and paper.

Some examples of curves that appear after combining two straight lines:

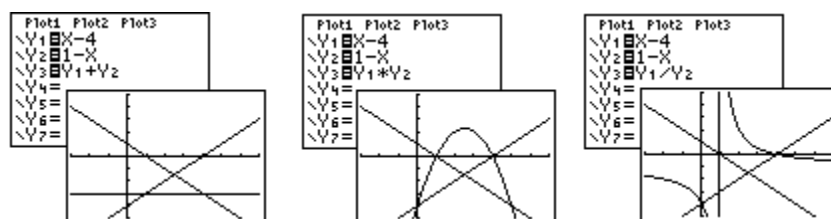


Figure 3. Combining straight lines into curves

Many interesting problems can be offered to students, helping them to get insight in functions in a way that is completely different from the algebraic way it was taught in the traditional programs. When this kind of functions are investigated in an alternative way like this, a different concept of polynomials is developed. A zero of a function now means that the polynomial can be split in two parts, of which one is linear. And that is a different wording of the Factor Theorem: "The binomial $x - c$ is a factor of $P(x)$ if and only if $P(c) = 0$ " For most students the graph-supported definition makes more sense, because it comes with a visualization of the rule.

A nice type of problem that uses this geometrical version of the Factor theorem:

The third degree polynomial described in Y1 can be seen as the multiplication of a straight line and a parabola. Find the formula for both the straight line and the parabola.

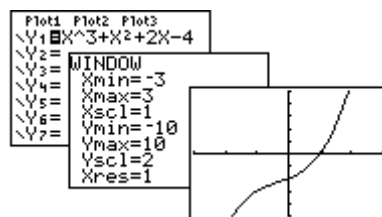


Figure 4. Factorizing a cubic function

A question like this can be answered in different ways. Students who are good in algebra may use the GC only to see that $y = x - 1$ has to be the straight line. The formula for the parabola can then be found by a long division or from the multiplication $x^3 + x^2 + 2x - 4 = (x - 1)(x^2 + ax + 4)$. For students who are not that good in algebraic manipulation, the GC can be used to find the graph of the parabola and from there it is possible to find a formula for the parabola:

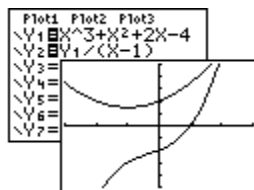


Figure 5. Finding the parabola graphically

The parabola has vertex $(-1, 2)$ (this can be checked with the CALC-key) and from here the investigations on parabolas as the product of two straight lines can be used to arrive at the formula $y = (x + 1)^2 + 2$.

Instead of using the vertex of the parabola, the intersection with the y -axis can be used: after shifting the parabola down for 4 units, the new one intersects the x -axis in $(-2, 0)$ and $(0, 0)$. Therefore the formula of the original parabola is $y = x \cdot (x + 2) + 4$.

The last option is to find the parabola through quadratic regression. This results in the formula $y = x^2 + 2x + 4$.

Of course, the different formulas that are found are a good reason to discuss algebraic manipulations to show that all formulas are (in an algebraic way) the same.

This way of introducing functions, fully supported by graphical tools, opens possibilities for own initiatives of students. This is only possible when they are given opportunities to investigate problems in non-algorithmic ways. The availability of machines like the GC can surely help to create such opportunities in classrooms. A nice side-effect is that in this way the factorizing of polynomials can be studied in a natural way. In traditional courses, factorizing is never connected directly to the graphing of functions.

3. Investigative work on mathematical problems

A problem like the one that is described below (a function with a parameter) can be investigated with the use of the GC, because the machine can easily plot a lot of different graphs that support or contradict students' conjectures.

For different values of the parameter a , graphs are drawn of the function $f(x) = \frac{x^2 - x}{ax + 1}$

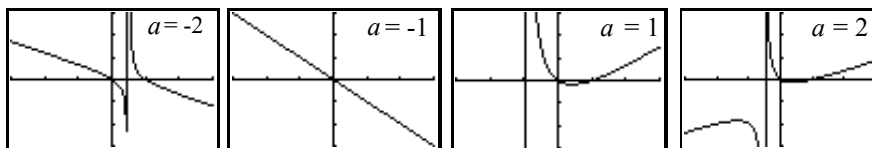


Figure 6. A family of graphs

All graphs are plotted in a window with setting $[-3, 3] \times [-3, 3]$

a. Explain the shape of the graph for $a = -1$

The graph changes for every new value of a . The vertical asymptote moves. But there is also another asymptote for certain values of a . Furthermore the graph has extreme values or it has not.

b. Investigate the shape of the graphs for different values of a and discuss your findings.

These investigations are even more challenging when a Java applet is used in which the shape of the curves change continuously when changing the value of a . In fact, looking at the changes in the graph evokes questions that never come up when this kind of functions are analyzed only from an algebraic perspective(see <http://www.fi.uu.nl/applets/transformations/welcome.html>).

Discussion

In the design of a new mathematics program for engineering courses in secondary vocational education, the use of the graphing calculator was fully integrated in the learning process. Because the students are quite poor in using formal algebra, the GC was firstly used to replace difficult algebraic manipulations. Like is found in other experiments, the availability of the GC with its different ways to represent a function stimulates students' flexibility in solving problems. They become familiar with the operational concept of a function in a dynamical way. The possibility to plot data stored in lists, and the search for a formula describing the relationship that fits the data, helps to understand the modeling aspects of functions. The discrete nature of the screen with all consequences for the graphs, was used as a challenging problem posing feature of the GC: depending on chosen window settings, graphs of one function can look different. In fact, the discrete nature of graph plots corresponds very nicely with the activity of sampling. This is an often used technique in electronics to find the pattern of a wave. Because of its' discrete nature, the same problems can arise in sampling as with the plotting of a graph on a GC. Instead of focusing on the 'poor' behavior in a formal mathematical way, this supposed disadvantage of the tool can turn out to be a good source for nice investigations and to discuss the differences between discrete and continuous behavior.

The use of the graphing calculator does not only replace algebra. A closer look shows that there is a lot of algebra included, but in a non traditional way. In traditional programs there was hardly any attention paid to the structure of formulas. The structure of the trigonometric formula in figure 1, for example should be clear to somebody who wants to enter it in the GC. In fact, the language aspect of algebra is becoming more important if the use of graphing tools in mathematics education is allowed. For that reason, much attention is paid in the new program on how a formula is built, including notations and rules for algebraic priorities. The training of basic techniques has been de-emphasized, but geometrical aspects of algebra have greater emphasis now: patterns in graphs, like periodicity; symmetry and related to that even and odd functions. The features of the TI 83, shown in figure 7 are asking for substitution of expressions in formulas, to be able to understand the connection between the (geometry-based) transformations and the influence on the (algebraic) formula and vice versa.

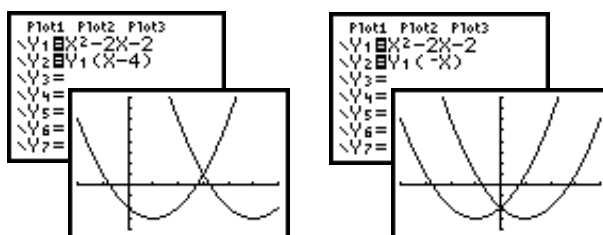


Figure 7. Geometry-based algebra

Based on the experiences in the project, I see a future for algebra of functions that makes sense to most students. Such a program should

- de-emphasize training on basic algorithmic skills,
- pay more attention to the language aspect of algebra,
- use the nice possibilities of graphical tools to visualize algebraic concepts (the geometry of algebra),
- stimulate own investigations by students on algebraic subjects, supported by graphical tools,

- use non-traditional approaches to mathematical concepts and methods that graphical tools allow
- introduce new topics because of the specific qualities of the tool, like continuous vs discrete and curve fitting (regression).

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