

Marja van den Heuvel-Panhuizen

Realistic Mathematics Education as work in progress

Summary

This lecture addresses several “progress” issues related to the Dutch approach to mathematics education, called “Realistic Mathematics Education” (RME). The most important of these issues is the way in which RME facilitates the progress of children’s understanding in mathematics. This topic forms the heart of the lecture. Attention is paid to both the micro-didactic and the macro-didactic perspective of the students’ growth. Progress in achievements, as the result of this learning, is the next progress issue to be dealt with. Finally, the spotlights are turned towards the developments within RME itself. The general focus in the lecture is on primary school mathematics education.

1 Introduction

RME in brief

Realistic Mathematics Education, or RME, is the Dutch answer to the need, felt worldwide, to reform the teaching of mathematics. The roots of the Dutch reform movement go back to the beginning of the seventies, when the first ideas for RME were conceptualized. It was a reaction to both the American “New Math” movement, which was likely to flood our country in those days, and the then prevailing Dutch approach to mathematics education, which often is labeled as “mechanistic mathematics education.”

Since the early days of RME much design work connected to developmental research (or design research) has been carried out. If anything is to be learned from the Dutch history of the reform of mathematics education, it is that such a reform takes time. It looks a superfluous statement, but it is not. Again and again, too optimistic thoughts are heard about educational innovations. Our experience is that reforms in education take time. The development of RME is thirty years old now, and we still consider it as “work under construction.”

*F. L. Lin (Ed.) Common Sense in Mathematics Education, 1-43.
Proceedings of 2001 The Netherlands and Taiwan Conference on Mathematics Education, Taipei, Taiwan, 19 – 23 November 2001.*

That we see it in this way, however, has not only to do with the fact that until now the struggle against the mechanistic approach to mathematics education has not been conquered completely — especially in classroom practice much work still has to be done in this respect. More determining for the continuing development of RME is its own character. Inherent to RME, with its founding idea of mathematics as a human activity, is that it can never be considered a fixed and finished theory of mathematics education.

“Progress” issues to be dealt with

This self-renewing feature of RME was an important reason for choosing “work in progress” as the title for this lecture. But there were more reasons for this choice. The title also refers to another significant characteristic of RME, namely its focusing on the growth of the children’s knowledge and understanding of mathematics.

The way in which RME continually works on the progress of children is the first progress issue to be dealt with in this lecture. This progress work is distinguished in two levels of working on the mathematical development. Attention is paid to both the micro-didactic perspective and the macro-didactic perspective of the students’ growth. The micro-didactic perspective clarifies how within the context of one or two lessons shifts in comprehension and abilities can happen. In this process, models which originate from context situations and which function as bridges to higher levels of understanding have a key role. The macro-didactic perspective deals with the progress in understanding over a longer period of time. The focus here is on learning-teaching trajectories – including the attainment targets to be reached at the end of primary school and the landmarks along the route – that serve as a longitudinal framework for teaching mathematics. The coherence between the various levels of mathematical understanding that is made apparent in this trajectory description plays a key role in stimulating students’ growth.

A following progress issue has to do with the students’ achievements in mathematics. The question is whether RME brought Dutch primary students to the top level of mathematics achievements. Although the TIMSS results and results from other comparative studies are suggesting this, there are also arguments against.

Finally, the lecture deals with the progress in the RME approach to mathematics education. Although this approach is already some thirty years old it is still “under construction.”

2 More about RME

History and founding principles

As I said already, almost thirty years ago the development of what is now known as RME started. Freudenthal and his colleagues laid the foundations for it at the former IOWO, which is the earliest predecessor of the Freudenthal Institute. The actual impulse for the reform movement was the inception, in 1968, of the

Wiskobas project, initiated by Wijdeveld and Goffree. The present form of RME has been mostly determined by Freudenthal's (1977) view about mathematics. According to him, mathematics must be connected to reality, stay close to children and be relevant to society, in order to be of human value. Instead of seeing mathematics as subject matter that has to be transmitted, Freudenthal stressed the idea of mathematics as a human activity. Education should give students the "guided" opportunity to "re-invent" mathematics by doing it. This means that in mathematics education, the focal point should not be on mathematics as a closed system but on the activity, on the process of mathematization (Freudenthal, 1968). Later on, Treffers (1978, 1987) formulated the idea of two types of mathematization explicitly in an educational context and distinguished "horizontal" and "vertical" mathematization. In broad terms, these two types can be understood as follows. In horizontal mathematization, the students come up with mathematical tools, which can help to organize and solve a problem located in a real-life situation. Vertical mathematization is the process of reorganization within the mathematical system itself, like, for instance, finding shortcuts and discovering connections between concepts and strategies and then applying these discoveries. In short, one could say — and here I am quoting Freudenthal (1991) — "horizontal mathematization involves going from the world of life into the world of symbols, while vertical mathematization means moving within the world of symbols." Although this distinction seems to be free from ambiguity, it does not mean, as Freudenthal (ibid.) said, that the difference between these two worlds is clear-cut. Freudenthal (ibid.) also stressed that these two forms of mathematization are of equal value. Furthermore one must keep in mind that mathematization can occur on different levels of understanding.

Misunderstanding of "realistic"

Despite of this overt statement about horizontal and vertical mathematization, RME became known as "real-world mathematics education." This was especially the case outside the Netherlands, but the same interpretation can also be found in our own country. It must be admitted, the name "Realistic Mathematics Education" is somewhat confusing in this respect. The reason, however, why the Dutch reform of mathematics education was called "realistic" is not just the connection with the real world, but is related to the emphasis that RME puts on offering the students problem situations which they can imagine. The Dutch translation of the verb "to imagine" is "zich REALISERen." It is this emphasis on making something real in your mind that gave RME its name. For the problems to be presented to the students this means that the context can be a real-world context but this is not always necessary. The fantasy world of fairy tales and even the formal world of mathematics can be very suitable contexts for a problem, as long as they are real in the student's mind.

The realistic approach versus the mechanistic approach

In any way, the use of context problems is very significant in RME. This is in contrast with the traditional, mechanistic approach to mathematics education, where programs mostly only contain problems with bare numbers. If in the

mechanistic approach context problems are used, they are mostly used to conclude the learning process. The context problems function only as a field of application. By solving context problems the students can apply what was learned earlier in the bare format. In RME this is different. Here, context problems function also as a source for the learning process. In other words, in RME, context problems and real-life situations are used both to constitute and to apply mathematical concepts. While working on context problems the students can develop mathematical tools and understanding. First, they develop strategies closely connected to the context. Later on, certain aspects of the context situation can become more general which means that the context can get more or less the character of a model, and as such give support for solving other but related problems. Eventually, the models give the students access to more formal mathematical knowledge. In order to fulfill the bridging function between the informal and the formal level, models have to shift from a “model of” to a “model for.” Talking about this shift is not possible without thinking about our colleague Leen Streefland, who died in 1998. It was he who in 1985 detected this crucial mechanism in the growth of understanding.¹ His death means a great loss for the world of mathematics education.

Another notable difference between RME and the traditional approach to mathematics education is the rejection of the mechanistic, procedure-focused way of teaching in which the learning content is split up in meaningless small parts and where the students are offered fixed solving procedures to be trained by exercises, often to be done individually. RME, on the contrary, has a more complex and meaningful conceptualization of learning. The students, instead of being the receivers of ready-made mathematics, are considered active participants in the teaching-learning process, in which they develop mathematical tools and insights. In this respect RME has a lot in common with socio-constructivist based mathematics education. Another similarity between the two approaches to mathematics education is that crucial for the RME teaching methods is that students are also offered opportunities to share their experiences with others.

This concludes a brief overview of the characteristics of RME. Now, I will continue with the issue of progress in understanding and will start with the micro-didactic perspective.

3 Progress in understanding — the micro-didactic perspective

In this section several aspects of RME are discussed that all give support to elicit shifts to a higher level of comprehension.

1 Streefland (1985). Later on, this idea of a shift in models became a significant element in RME thinking about progress in students' understanding of mathematics (see Streefland, 1991; Treffers, 1991; Gravemeijer, 1994; Van den Heuvel-Panhuizen, 1995).

3.1 Progressive schematization

For the first example we have go to the domain of algorithms and to return to the early years of RME. In the beginning of the eighties, in Dutch fourth-grade classrooms it was customary that, for instance, long division was taught by starting to explain the procedure with small numbers and gradually increasing the degree of difficulty. The students were taught long division according to the teaching method of “progressive complexity” (see Figure 1).

$$\begin{array}{r}
 2 \overline{) 6} \backslash 3 \\
 \underline{6} \\
 0
 \end{array}
 \quad
 \begin{array}{r}
 3 \overline{) 72} \backslash 24 \\
 \underline{6} \\
 12 \\
 \underline{12} \\
 0
 \end{array}
 \quad
 \begin{array}{r}
 5 \overline{) 342} \backslash 68 \\
 \underline{30} \\
 42 \\
 \underline{40} \\
 2
 \end{array}$$

Figure 1 Teaching long division according to the method of progressive complexity

Contrary to this approach, RME pleaded for “progressive schematization” as the leading precept for organizing a teaching unit on algorithms. In this approach the student have to cope with large numbers immediately, as is the case in the *Stickers* problem² (see Figure 2).

2 This problem, or variations of it, can be found in Dekker, Ter Heege, and Treffers, 1982; Treffers and Goffree (1985); Treffers (1987). At the same time, a different problem was also used for learning long division, namely the *Supporters* problem (see Gravemeijer, 1982). In this problem the students have to find out how many buses are needed to transport a large number of supporters to a football match. Compared to the *Stickers* problem the *Supporters* problem contained more context information to take into account in answering the question. Another, major difference is that the *Stickers* problem implies division by distribution (quotitive division) while the *Supporters* problem involves division by partitioning (partitive division). Later, the latter was considered as being more suitable in the beginning of the process of long division.

> *342 match stickers are fairly distributed among five children. How many does each of them get?*

Figure 2 *Stickers problem*

This method of “progressive schematization” implies that $342 \div 5$ is not situated at the final stage of the process of learning long division but at the very beginning of it. That the students could solve these difficult problems from the introduction on has all to do with the context in which the problems are presented. It is the context of “fairly sharing” that makes that they have access to problems like these. As a start the students can apply a natural strategy of sharing out. Mathematically this means the repeated carrying out of a subtraction (see Figure 3).

The image shows a hand-drawn long division for $342 \div 5$. The dividend is 342 and the divisor is 5. The student has written the quotient 68. The work is organized into four stages of subtraction, each with a visual aid:

- Stage 1:** $342 - 300 = 42$. The 300 is represented by five hands (50 each).
- Stage 2:** $42 - 40 = 2$. The 40 is represented by five circles (8 each).
- Stage 3:** $292 - 250 = 42$. The 250 is represented by five hands (50 each).
- Stage 4:** $42 - 40 = 2$. The 40 is represented by five circles (8 each).

The final result is 68, with a remainder of 2.

Figure 3 Doing long division by repeated subtraction³

Later on, this repeated subtraction strategy in which a small number of stickers is shared out each time, can be curtailed by doing the sharing more efficiently in less steps by taking larger numbers. In this way the students gradually arrive at the standard procedure of long division (see Figure 4).

³ The picture of the student work is taken from Rengerink (1983).

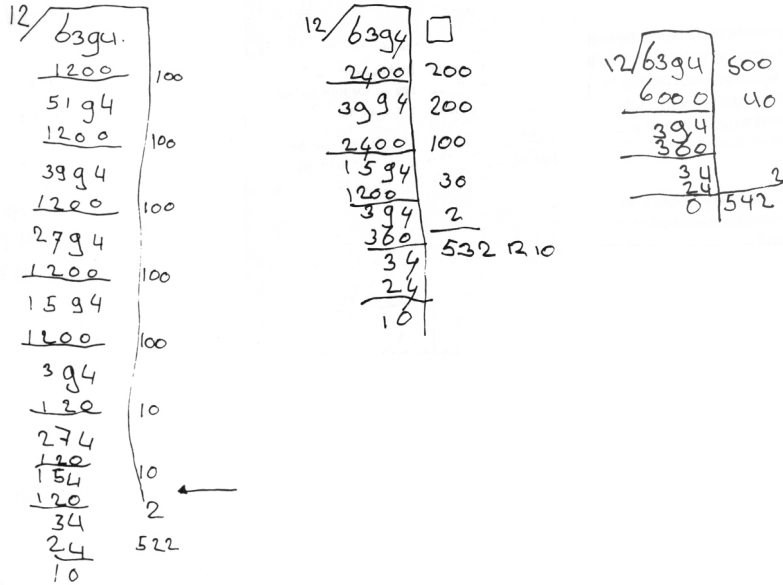


Figure 4 Progressive schematization in long division⁴

This approach of progressive schematization, here illustrated by an example of long division, formed, in the days when it was formulated, a real break with the past. Instead of step by step increasing the complicatedness of the problems, the problems remain the same, but the strategies become more and more advanced. In other words, the students' growth is basically on the part of the applied strategies and is in essence not necessarily prompted by offering problems of rising intricacy. There are several advantages to this approach: the students can solve the problems on their own level, they can start at a context-connected informal level, and all different levels are "within reach" in the classroom.

3.2 Contexts as vehicles for growth

While the *Stickers* problem made clear how a context can endow the students with cues for developing strategies – handing over an informal repeated subtraction strategy as a pre-stage of the standard long division strategy – the following example demonstrates even more strongly the role of the context in building up mathematical knowledge. This example involves the context of a city bus.⁵ The *Bus* problem that is based on this context turned out to be a very powerful learning environment for first graders. First of all, this problem offers students opportunities to develop a formal mathematical language. The teaching starts with a

⁴ The picture of the student work is taken from Dekker, Ter Heege, and Treffers (1982).

⁵ The use of this context was an idea of Van den Brink.

“real life” situation in which the students have to act as the driver of a city bus. The passengers are getting on and off the bus, and at each stop the students have to determine the number of passengers in the bus. Later the same is done on paper (see the worksheet in Figure 5).

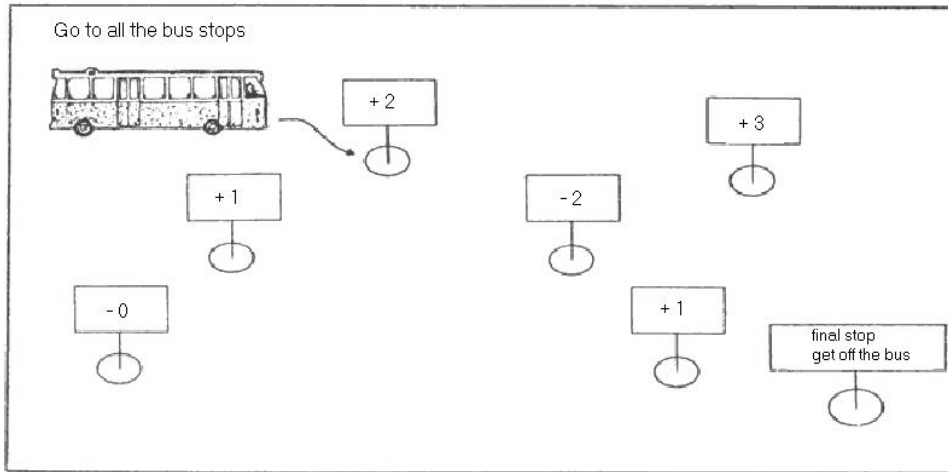
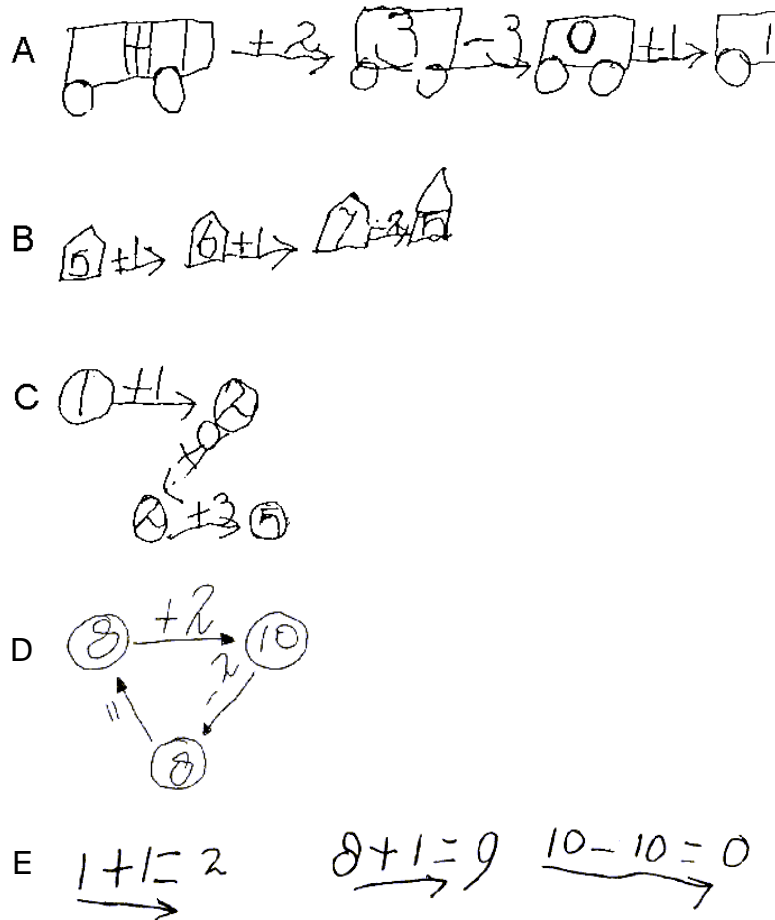


Figure 5 Bus problem⁶

The development of mathematical language is elicited by the need to keep track of what happened during the ride of the bus. Initially the language is closely connected to the context, but later on it also used for describing other situations. Gradually, the bus context loses its narrative feature and takes on more of a model character. The following student work (see Figure 6) reflects how the context-connected mathematical language can evolve progressively to a more general formal mathematical language.

⁶ Picture is an adapted version from Van den Brink (1989).

Figure 7 Student work bus problem⁷

What started as a context-connected report of the story of the bus (A), is later used for numerical operations in other contexts, e.g. keeping track of the number of customers that are in a shop (B), and for expressing operations with pure numbers (C and D).⁸ In (E) the transition to the standard way of notating number sentences is visible.

In addition to offering the students a learning environment for developing a formal mathematical language that makes sense to them, the bus context – and

⁷ Picture is taken from Van den Brink (1989).

⁸ Although in (C) pure numbers are used, the shape of the notation betrays that it still represents an operation with context numbers.

particular the context of the bus stop – is also very suitable to elicit mathematical reasoning. Evidence for this suitability was illuminated by developmental research on negative numbers carried out by Streefland (1996). In a Dutch grade five-six classroom he presented the students – who never had dealt with negative number before – the problem that is pictured in Figure 8 and asked them first how many passengers were in the bus after the bus left the bus stop. Then he asked them the challenging question of what else could have happened at the bus stop with the same result in terms of passengers in the bus after the bus stop. Figure 9 shows how the students worked out this question.

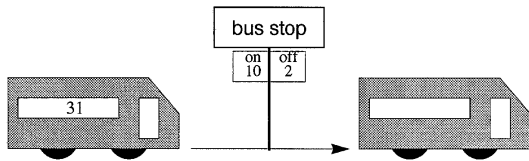


Figure 8 Getting on and off the bus

in	out
15	7
9	1
8	0
12	5
11	3
20	12
17	9
18	10

Figure 9 Bus stop stories

Later on in the lesson, the students could even use the bus stop model to generate problems with a fixed starting number and a fixed result (see Figure 10).

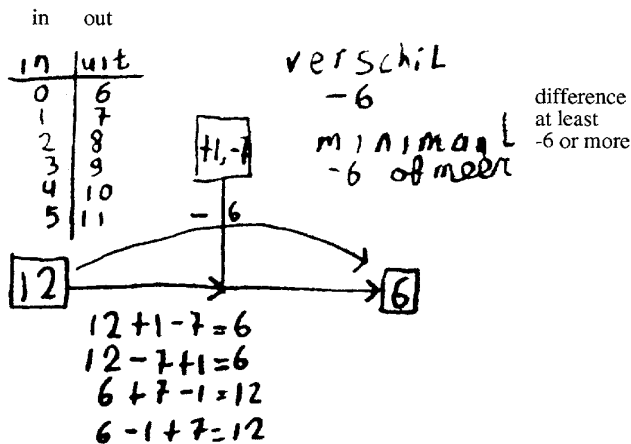


Figure 10 Student-generated problems based on experiences with the bus stop context

The bus and the bus stop context are an example of how experiences from a “daily life” situation can be the impetus for growth in mathematical understanding.

Compared to the context of the stickers this bus context evolves to a model to support mathematical thinking. An important requirement for models functioning in this way is that they are rooted in concrete situations and that they are also flexible enough to be useful in higher levels of mathematical activities. This means that the models will provide the students with a foothold during the process of vertical mathematization, without obstructing the path back to the source.

3.3 Connected models as the backbone of progress

As became already visible in the previous examples, progress implies that students arrive at more general solutions from context-related solutions. Contexts that have model potential serve as an important device for bridging this gap between informal and more formal mathematics. First, the students develop strategies closely connected to the context. Later on, certain aspects of the context situation can become more general, which means that the context more or less acquires the character of a model and as such can give support for solving other, but related, problems. Eventually, the models give the students access to more formal mathematical knowledge.

In order to make the shift from the informal to the formal level possible, the models have to modify from a “model of” a particular situation (e.g. a scheme that represents the situation of passengers getting on and off the bus at a bus stop) to a “model for” all kinds of other, but equivalent, situations (e.g. a scheme that can be used for expressing shop attendance, but that also can be used to find number pairs that give the same increase or decrease as a result).⁹

This important role of models has all to do with the level principle of RME. This principle implies that learning mathematics is considered as passing through various levels of understanding: from the ability to invent informal context-related solutions, to the creation of various levels of shortcuts and schematizations, to the acquisition of insight into the underlying principles and the discernment of even broader relationships. Crucial for arriving at the next level is the ability to reflect on the activities conducted before.¹⁰ This reflection can be elicited by interaction. I will come back to this later.

The strength the level principle is that it guides growth in mathematical understanding by giving the curriculum a longitudinal coherency. This long-term perspective is characteristic of RME. There is a strong focus on the relation between what has been learned earlier and what will be learned later. This means also that the use of models should not be considered as isolated activities. On the contrary, the progressive power of the models is based on the connections between them. The longer a model – or adaptations of it – can “keep pace” with the development in mathematical understanding, the more it can prompt and elicit the student’s progress.

9 For more about the idea of “model of” and “model for” see Section 2 of this paper and Note 1.

10 This idea of accumulated reflection was based on the work of the Van Hiele (see Freudenthal, 1991).

A powerful example of such a “longitudinal” model is the number line. Figure 11 shows the various ways in which the number line can appear in the different stages of the learning process.

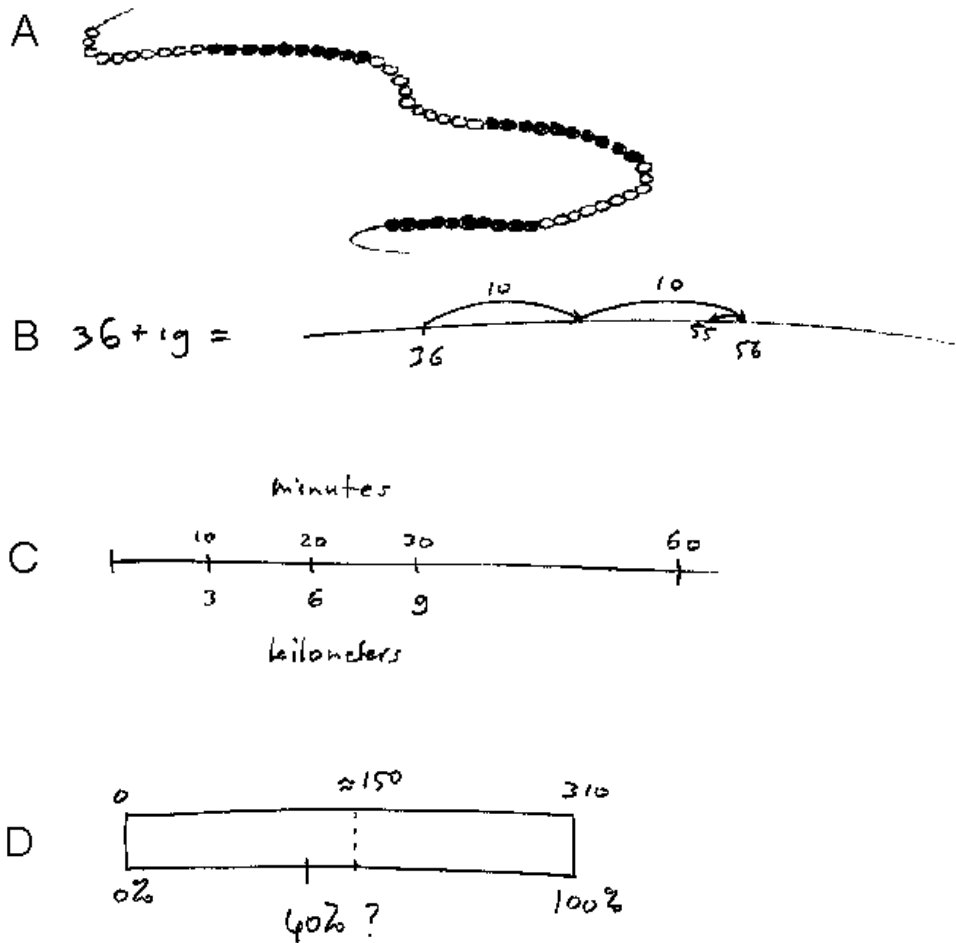


Figure 11 Various ways in which the number line can appear

The number line begins in first grade as (A) a beaded necklace on which the students can practice all kind of counting activities. In higher grades, this chain of beads successively becomes (B) an empty number line for supporting additions and subtractions, (C) a double number line for supporting problems on ratios, and finally (D) a fraction/percentage bar for supporting working with fractions and percentages.

3.4 Interactive whole-class teaching as a lever

Apart from the models that give a hold for progress, the teaching methods are also an important lever for students to make steps forward in mathematical understanding.

Within RME, the learning of mathematics is considered as a social activity. Based on this idea the *interaction principle* is one of the major characteristics of RME. Education should offer students opportunities to share their strategies and inventions with each other. By listening to what others find out and discussing these findings, the students can get ideas for improving their strategies. Moreover, the interaction can evoke reflection, which is necessary to reach a higher level of understanding.

The significance of the interaction principle implies that whole-class teaching plays an important role in the RME approach to mathematics education. However, this does not mean that the whole class is proceeding collectively and that every student is following the same track and is reaching the same level of development at the same moment. On the contrary, within RME, children are considered as individuals, each following an individual learning path. This view on learning often results in pleas for splitting up classes into small groups of students each following their own learning trajectories. In RME, however, there is a strong preference for keeping the class together as a unit of organization¹¹ and for adapting the education to the different ability levels of the students instead. This can be done by means of providing the students with problems that can be solved on different levels of understanding.

The next example involves a lesson¹² in which the interaction principle of RME is made concrete. It gives an impression of how a whole-class setting can contribute to students' progress in the applied mathematical strategies. Starting point is the exploration of a context problem which — and this is very essential — can be solved on several levels of understanding. By discussing and sharing solution strategies in class the students who first solved the problem by means of a longwinded strategy can come to a higher level of understanding and new mathematical concepts can be constituted.


The scene of the action is a third-grade classroom. The students are eight through nine years old. The teacher starts with the presentation of a problem, called the *Parents evening* problem (see Figure 12).

11 Within the structure of keeping the group together, a variety of teaching methods can be applied: ranging from whole-class teaching, to group work, and to individual work.

12 This classroom activity originates from Van Galen and Feijs (1991); the vignette was also used by De Lange in his plenary lecture at ICME 1996 in Seville, Spain.

"...Tonight, a parents' evening will be held...
 ...The slips I received from you told me that 81
 persons will attend...
 ...The meeting will take place in the large
 conference hall...
 ...The parents will be seated at large tables...
 ...At each table six persons can be seated..."

Then the teacher makes a drawing of such a table on the blackboard:



After doing this, the teacher concludes with the following question: "How many tables do we need for 81 persons?"

Figure 12 *Parents evening* problem

The students begin to work and the teacher is walking around the classroom. Whenever necessary she gives some help. Some ten minutes later she asks the students to show their work and explain their solutions.

Badr drew as many tables as he needed to have all the parents seated (see Figure 13).

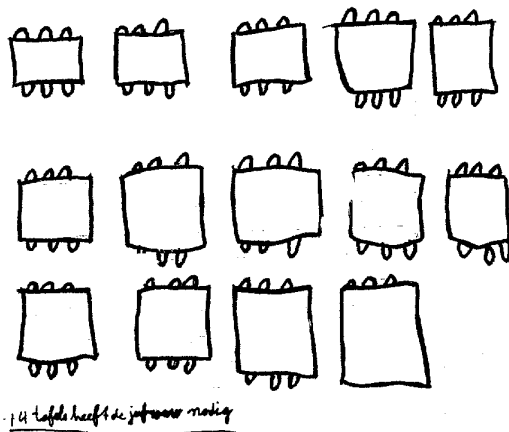


Figure 13 Badr's work

Roy started in the same way, but after he drew two complete tables he drew two rectangles and put the number six in them. While he was drawing more of these rectangles he suddenly realized that if you have five tables, 30 parents can be seated. He continued drawing rectangles and after another five he wrote down 60. Then he drew another two, wrote down 72, and another one, and wrote down 78. He finished with a rectangle with the number 3 in it (see Figure 14).

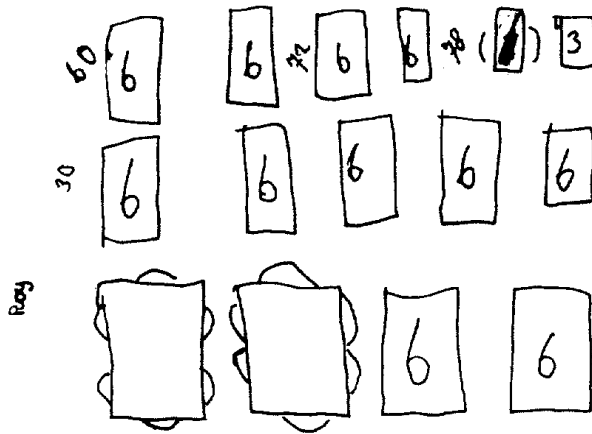


Figure 14 Roy's work

A third student, Abdelaziz, was even a little bit more advanced in mathematizing the problem situation. Although he also started with drawing a copy of the table that was on the blackboard, he immediately moved up to a more formal solution by using his knowledge about multiples of six. He wrote down: $6 \times 6 = 36$, doubled this number and came to 72, and then he added two more tables to 72 and came to the answer of 84 (see Figure 15).

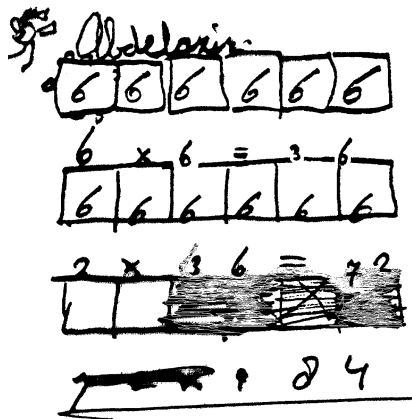


Figure 15 Abdelaziz' work

When we look at these three different solutions, we can see that at each level some mathematization took place. Even in the work of Badr, since visualization and schematization are also powerful tools for mathematizing. In the other two examples of student work the mathematics is more visible, but not yet at the level that is aimed at. This problem was namely meant as a start for learning long division. In order to achieve this goal, the problem should be followed by other problems.

Therefore, after the class discussion about the different strategies had been finished, the teacher presented a new problem: the *Coffee pots* problem (see Figure 16).

"...The 81 parents will be offered a cup of coffee...
 ...With one coffeepot you can fill 7 cups...
 ...How many coffee pots will be needed?"

Figure 16 *Coffee pots* problem

From a mathematical point of view this problem is the same as the previous one. Instead of dividing by six, now, the students have to divide by seven. For the students, however, this is (still) a completely different problem. Moreover, it is more difficult to make a visual presentation of this new problem. Tables are easier to draw than coffeepots. Yet, Badr tried to draw them (see Figure 17).

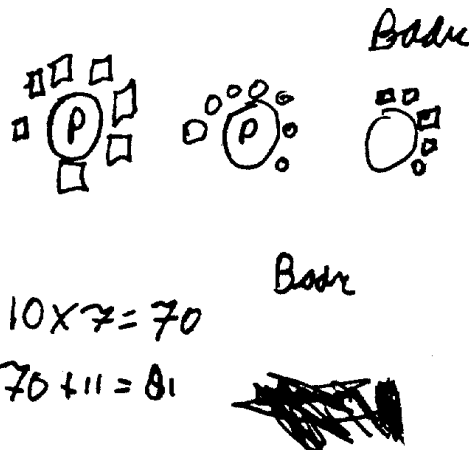


Figure 17 Badr's work of the *Coffee pots* problem

After he drew two pots he remembered the discussion about how the answer can be found more quickly by multiplying. He continued with $10 \times 7 = 70$ followed by $70 + 11 = 81$, and decided that 12 pots are needed.

Both the constitution of mathematical tools (the representation of the problem situation, the schematization, the repeated addition, the application of number fact knowledge, the way of keeping track of the results, and the communicating about the strategies) and Badr's level shift were evoked by the problems given to students, and more precisely, by the "cluster of problems." In a way the contexts in this cluster of problems – that can be considered as a micro learning teaching trajectory – prompted him to this growth.

4 Progress in understanding — the macro-didactic perspective

The previous vignette showed an example of progress in understanding in one lesson. In the following section I will move to progress in understanding over a longer period of time. RME has a high concern with the longitudinal perspective of the learning-teaching process. In our view a theory of mathematics education cannot be complete if it is restricted to the micro perspective of an instructional environment. It should also cover the longitudinal macro perspective. This perspective goes together with a strong focus on the subject matter content and the goals to be achieved. For RME the *what* of the teaching is at least as important as the *how* of it. Because of its strong connection to mathematics and the central place that the content of the curriculum has in its thinking about mathematics education, RME can be considered as a didactical-oriented approach to mathematics education.¹³

For answering these macro-didactical what-questions "didactical phenomenological analyses" – as Freudenthal (1983) called them – have a crucial value. These analyses reveal what kind of mathematics is worthwhile to learn and which actual phenomena can offer possibilities to develop the intended mathematical knowledge and understanding. Important is that one tries to discover how students can come into contact with these phenomena, and how they appear to the students. This means that problems and problems situations must be identified that give students opportunities to develop insight in mathematical concepts and strategies. To have the power of stimulating and guiding the students' learning the problems should be embedded in a long-term learning-teaching trajectory. Therefore the longitudinal perspective of problems should never be neglected.

13 In this respect, RME differs from approaches to mathematics education that are more psychological-oriented, like the constructivist approaches to mathematics education. Whereas on the micro-didactic level RME has a lot in common with the constructivistic way of thinking, on the macro-didactic level of the curriculum some major differences between the two become apparent. As a matter of fact, constructivistic approaches do not have a macro-didactic level in which decisions are made about the goals for education and the learning-teaching trajectories that need to be covered in order to reach these goals. In contrast with RME, the constructivistic approaches are more a learning theory than a theory of education.

4.1 Learning-teaching trajectories as a framework for didactical decisions

In line with this emphasis on the longitudinal perspective, the TAL Project has worked on the development of learning-teaching trajectories for primary school mathematics since 1997. In 2001 an English version was published of the first product of this project, a learning-teaching trajectory for calculation with whole numbers (Van den Heuvel-Panhuizen (ed.), 2001).

What is meant by the TAL learning-teaching trajectories? To put it briefly, a learning-teaching trajectory describes the learning process the students follow. It should not be concluded from this, however, that it *only* contains the learning perspective. In our view, the term learning-teaching trajectory¹⁴ has three interwoven meanings:

- a *learning* trajectory that gives a general overview of the learning process of the students
- a *teaching* trajectory, consisting of didactical indications that describe how the teaching can most effectively link up with and stimulate the learning process
- a *subject matter* outline, indicating which of the core elements of the mathematics curriculum should be taught.

A learning-teaching trajectory puts the learning process in line, but at the same time it should not be seen as a strictly linear, singular step-by-step regime in which each step is necessarily and inexorably followed by the next. A learning-teaching trajectory should be seen as having a certain bandwidth, instead of being a single track. It is very important that such a trajectory description is doing justice to:

- the learning processes of individual students
- discontinuities in the learning processes; students sometimes progress by leaps and bounds and at other times can appear to relapse
- the fact that multiple skills can be learned simultaneously and that different concepts can be in development at the same time, both within and outside the subject
- differences that can appear in the learning process at school, as a result of differences in learning situations outside school
- the different levels at which children master certain skills.

The main purpose of a learning-teaching trajectory is to give the teachers a pointed overview of how children's mathematical understanding can develop from K1 and 2 through grade 6 and of how education can contribute to this development. It is intended to provide teachers with a "mental educational map" which can help them to make didactical decisions, for instance making adjustments to the textbook that they use as a daily guide. The learning-teaching trajectory serves as a guide at a meta level. Having an overview of the process the students go through is very important for working on progress in students' understanding. To make adequate decisions about help and hints, a teacher must have a good idea of the goals, the

14 In Dutch, a learning-teaching trajectory is called "leerlijn". The Dutch verb "leren" has a double meaning: It stands for "to learn" and for "to teach".

way that can lead to these goals and the landmarks the students will pass in one way or another along the route, when selecting new problems. Without this outline in mind it is difficult for the teacher to value the strategies of the students and to foresee where and when one can anticipate the students' understandings and skills that are just coming into view in the distance (see also Streefland, 1985). Without this longitudinal perspective, it is not possible to guide the students' learning.

The TAL learning-teaching trajectories are meant to give this longitudinal perspective. Compared to the goal descriptions that were traditionally supposed to guide education and support educational decision-making, the TAL learning-teaching trajectories are a new educational phenomenon.

First of all, the trajectory is more than an assembled collection of the attainment targets of all the different grades. Instead of a checklist of isolated abilities, the trajectory makes it clear how the abilities are built up in connection with each other. It shows what comes earlier and what comes later. In other words, the most important characteristic of the learning-teaching trajectory is its *longitudinal perspective*.

A second characteristic is its *double perspective of attainment targets and teaching framework*. The learning-teaching trajectory does not only describe the landmarks in student learning that can be recognized en route, but it also portrays the key activities in teaching that lead to these landmarks.

The third feature is its *inherent coherence, based on the distinction of levels*. The description makes clear that what is learned in one stage, is understood and performed on a higher level in a following stage. A recurring pattern of interlocking transitions to a higher level forms the connecting element in the trajectory. It is this level characteristic of learning processes, which is also a constitutive element of the Dutch approach to mathematics education, that brings longitudinal coherence into the learning-teaching trajectory. Another crucial implication of this level characteristic is that students can understand something on different levels. In other words, they can work on the same problems without being on the same level of understanding. The distinction of levels in understanding, which can have different appearances for different sub-domains within the whole number strand, is very fruitful for working on the progress of children's understanding. It offers footholds for stimulating this progress.

The fourth attribute of the TAL learning-teaching trajectory is the new description format that is chosen for it. The description is not a simple list of skills and insights to be achieved, nor a strict formulation of behavioral parameters that can be tested directly. Instead, *a sketchy and narrative description, completed with many examples*, is given of the continued development that takes place in the teaching-learning process.

4.2 The TAL trajectory for whole number calculation

In the TAL trajectory for calculation with whole numbers, calculation is interpreted in a broad sense, including number knowledge, number sense, mental arithmetic, estimation and algorithms. In fact the trajectory description is meant to

give an overview of how all these number elements are related to each other. The following I will zoom in on this trajectory of whole number calculation. To begin with, some general characteristics of the trajectory are discussed. Consequently, some snapshots will be presented from the trajectory for the lower grades with the spotlight on the *coherence between the different levels of counting and calculating*.

Some general characteristics of the trajectory for whole number calculation

As can be seen in Figure 18, the TAL trajectory for whole number calculation contains two parts: one for the lower grades and one for the upper grades. Although the learning-teaching process in both parts forms a continuous process, it cannot be neglected that each has its own characteristics. The students gradually come from a non-differentiated way of counting-and-calculating to calculations in more specialized formats that are dedicated to particular kinds of problems in a particular number domain. In other words, in the lower grades, all activities with numbers can be generally labeled by “arithmetic”, whereas in the upper grades different forms of calculations can be distinguished, like mental arithmetic, estimation, column arithmetic, algorithms, and calculation by using a calculator.

Another characteristic of the trajectory is the central role of mental arithmetic. It is seen as an elaboration of the arithmetic work that is rooted in the lower grades and forms the backbone in the upper grades.

Another feature of this trajectory is the explicit attention that is paid to numbers and number relations. The idea is that if the students are familiar with the context of numbers, their position in terms of magnitude and their internal structure, an important foundation is laid for the development of their calculation abilities. The more the students know about numbers, the easier the problems become for them. Or put it differently, if one invests in the numbers one gets the operations, so to say, for free.

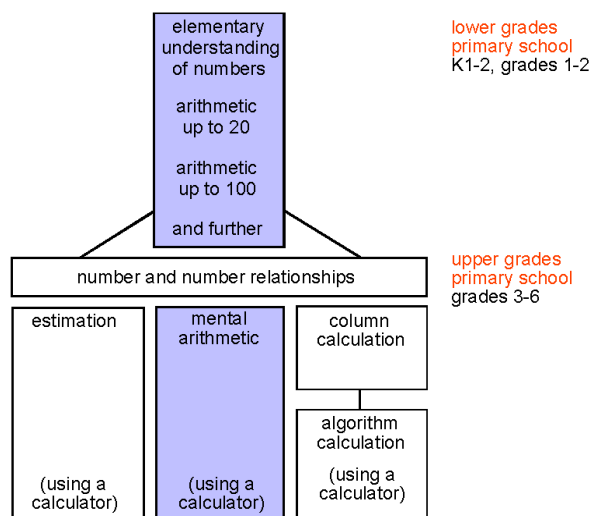


Figure 18 The TAL learning-teaching trajectory for whole number calculation in primary school

New in this trajectory is also that it contains a didactic for learning estimation. Although estimation is considered an important goal of mathematics education, in most of the textbooks a structure for how to learn to estimate is lacking. The textbooks at most only contain some problems on estimation, but doing some estimation problems from time to time is not enough to develop real understanding in how an estimation works and it is not sufficient to comprehend what is possible and what not when estimating.

Another novelty is the distinction that is made between algorithms and a less curtailed way of calculating in which whole numbers are processed instead of digits, which is called “column calculation”.

Finally, learning to calculate with whole numbers, of course, should include being able to use a calculator. The trajectory describes how this ability can be built, but at the same time it reflects to be very cautious. The main goal is that the students can in the end make sensible decisions about whether to use the calculator or not. Therefore, in the scheme, calculator use is placed between parentheses.

4.3 Coherence between the different levels of counting and calculating

The coherence between the various levels of mathematical understanding that is made apparent in the trajectory description plays a key role in stimulating students’ growth. It clarifies how learning in one stage is related to learning in other stages, and how the understanding gradually can evolve.

To give an idea of this coherence, I first zoom in on a teaching activity in grade 1. In this grade the *Restaurant lesson* is set up to offer the students a learning environment in which they can develop strategies for solving addition problems up to twenty in which they have to bridge the ten. The lesson that is described in the TAL book actually was given in a mixed class containing K2 and grade 1 children, aged five and six years. The teacher, Ans Veltman, is one of the staff members of the TAL team. She also designed the lesson, although she would disagree with this — Ans feels that her student Maureen was the developer of this lesson. Maureen opened a restaurant in a corner of the classroom and everybody was invited to have a meal. The menu card shows the children what they can order and what it costs. The prices are in whole guilders (see Figure 19).

The teacher’s purpose with this lesson is, as said before, to work on a difficult addition problem bridging the number ten. The way she does this, however, reflects a world of freedom for the students. The teacher announced that two items could be chosen from the menu and asked the children what items they would choose and how much this would cost. In other words, it appeared as if there was no guidance from the teacher, but the contrary was true. By choosing a pancake and an ice cream, costing 7 guilders and 6 guilders, respectively, she knew in advance what problem the class would be working on; namely the problem of adding above ten, which is what she wanted them to work on.



Figure 19: The menu card for Maureen's restaurant

And there is more, there is a purse with some money to pay for what is ordered. The teacher had arranged that the purse should contain five-guilder coins and one-guilder coins. This again shows subtle guidance from the teacher. Then the students start ordering. Niels chooses a pancake and an ice cream. Jules writes it down on a small blackboard. The other children shout: "Yeah ... me too." They agree with Niels' choice. Then the teacher asks what this choice would cost in total. Figure 20 shows a summary of what the children did.

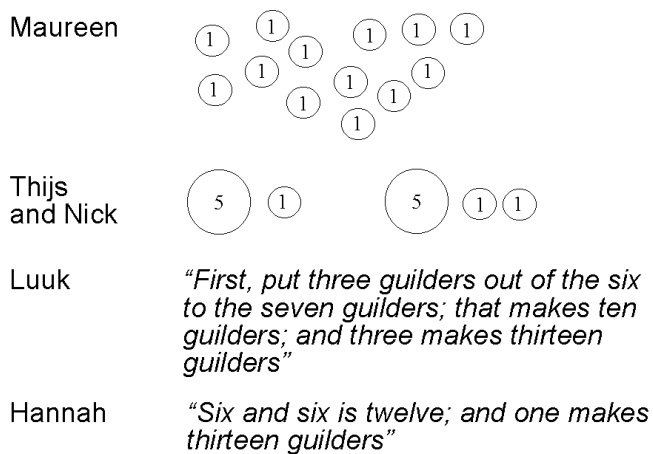


Figure 20: The students' strategies to solve $7+6$

Maureen counted 13 one-guilder coins. Six coins for the ice cream and seven coins for the pancake. Thijs and Nick changed five one-guilder coins for one five-guilder coin and pay the ice cream with "5" and "1" and the pancake with "5"

and “1” and “1”. Then they saw that the two fives make ten and the three ones make 13 in total. Later Nick placed the coins in a row: “5”, “5”, “1”, “1”, “1”. Luuk came up with the following strategy: “First put three guilders out of the six to the seven guilders, that makes ten guilders, and three makes thirteen.” Hannah did not make use of the coins. She calculated: “6 and 6 makes 12, and 1 makes 13 guilders.” Another student came with: “7 and 7 makes 14, minus 1 makes 13.”

This Restaurant lesson makes it clear that children who differ in skill and level of understanding can work in class on the same problem. To do this, it is necessary that problems that can be solved on different levels be presented to the children. The advantage for the students is that sharing and discussing their strategies with each other can function as a lever to raise their understanding. The advantage for teachers is that such problems can provide them with a cross-section of their class’s understanding at any particular moment. Such a cross-section includes the different levels on which the students can solve the problem:

- calculating by counting (calculating $7+6$ by laying down seven one-guilder coins and six one-guilder coins and counting the total one by one)
- calculating by structuring (calculating $7+6$ by laying down two five-guilder coins and three one-guilder coins)
- formal (and flexible) calculating (calculating $7+6$ without using coins and by making use of one’s knowledge about $6+6$).

The power of this cross-section is that it also offers the teachers a longitudinal overview of the trajectory the students need to go along (see Figure 21). The cross-section of strategies at any moment indicates what is coming within reach in the immediate future. As such, this cross-section of strategies contains handles for the teacher for further instruction.

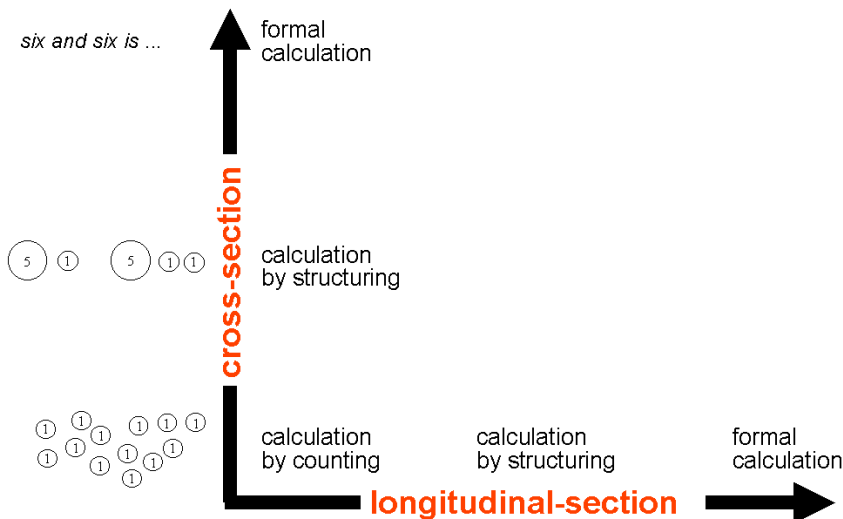


Figure 21 The cross-section shows “the future”

The next question – and this is really what the learning-teaching trajectory is about, namely showing the coherence of the whole process – is what this calculating in grade 1 has to do with resultative counting in Kindergarten and calculation up to one hundred in grade 2?

Regarding the ability of resultative counting¹⁵ that the children attain in Kindergarten, again different levels of working can be differentiated (see Figure 21). In the very beginning of the children’s learning process – when the concept of number is not very thoroughly established – they can meet difficulties in answering direct “how many” questions. To overcome this problem, a context-related question can be asked instead, like:

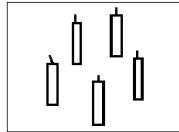
- how old is she (while referring to the candles on a birthday cake)?
- how far may you jump (while referring to the dots on a dice)?
- how high is the tower (while referring to the blocks of which the tower is built)?

• *context-bound*
counting
(calculating)



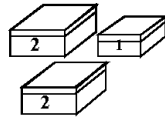
How old?

• *object-bound*
counting
(calculating)



How many
candles?

• (towards) *formal*
counting
(calculating)



How many?

Figure 21 Different levels of counting (and calculating) in Kindergarten

In the context-related questions, the context gives meaning to the concept of number. This context-related counting precedes the level of object-related counting in which children can handle the direct ‘how many’ questions in relation to a collection of concrete objects without any reference to a meaningful context. Later on, the presence of the concrete objects is also no longer needed to answer ‘how many’ questions. Via symbolizing, the children have reached a level of understanding in which they are capable of what might be called formal counting, which means that they can reflect upon number relations and that they can make use of this knowledge.

¹⁵ By resultative counting the ability to determine the number of objects in a group is meant. One could also say that resultative counting signifies the finding of the cardinal number. To indicate that resultative counting at Kindergarten level is already a preliminary stage of calculation, in the learning-teaching trajectory, the skill of counting is called “counting-and-calculating.”

Realistic Mathematics Education as work in progress

Regarding calculation up to one hundred in grade 2 the following main strategies have been distinguished in the learning-teaching trajectory:

- the stringing strategy in which the first number is kept as a whole and the final answer is reached by making successive jumps
- the splitting strategy in which use is made of the decimal structure and the numbers are split in tens and ones and processed separately
- the varying strategy in which use is made of knowledge of number relations and properties of operations.

Examples of these different strategies for solving 48+29 are shown in Figure 22.

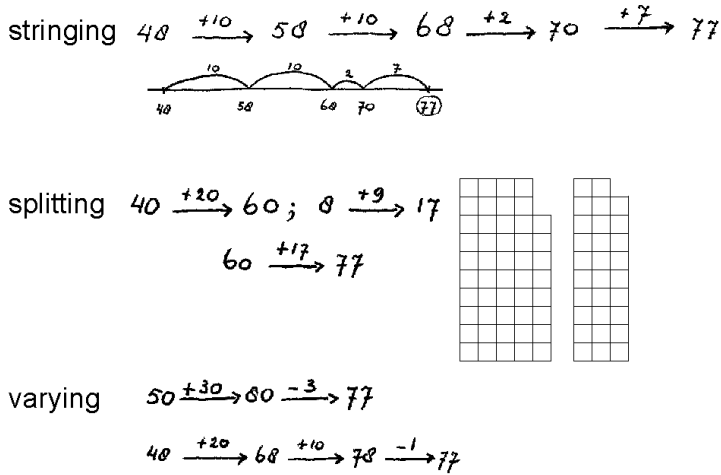


Figure 22 Different strategies for solving 48+29

The scheme in Figure 23 shows how the different calculation strategies are connected to each other, and that they all are based upon counting.

K1-2	Grade 1 (and 2)	Grade 2 (and 3)
	+ AND - UP TO 20	+ AND - UP TO 100
	- formal	- varying
	- structuring	- splitting
COUNTING	- counting	- stringing
- formal		
- object-bound		
- context-bound		

Figure 23 Coherence of the different levels of counting and calculating

And, of course, the feasibility of these strategies does not stop at the end of the lower grades. The strategies keep their importance when the students, for instance, have to solve problems with larger numbers and operations other than addition and subtraction. In Figure 24 the same basic strategies can be recognized in the student work of grade 4 students when they had to explain how they solved the problem about the coffee boxes.

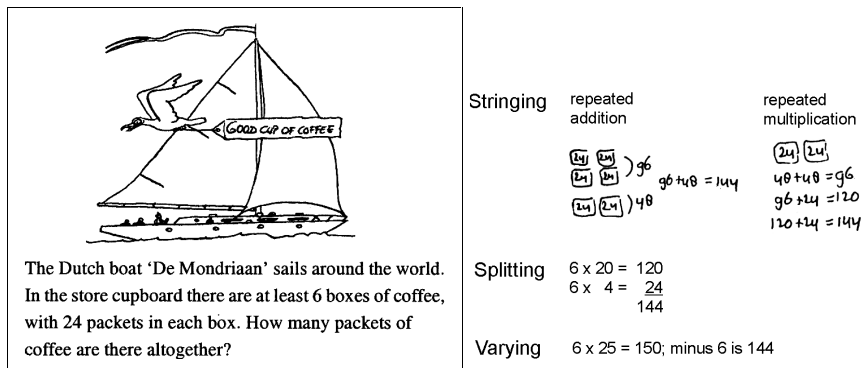


Figure 24 Strategies that grade 4 students applied for finding the number of coffee packets

The above examples of strategies show a clear longitudinal coherence of counting and calculating. It explains how learning to calculate in one grade is connected to the learning process in another grade. Insight into these levels in strategies provides teachers with a powerful foundation for gaining access to children's understanding and for working on shifts in their understanding.

Whether RME can really offer students a fertile soil for progress in mathematics achievements will be dealt with in the next section.

5 RME as an elicitor of progress in mathematics achievements?

According to the TIMSS results, Dutch students achieve very well in mathematics. Restricting us to the primary school study, the Dutch fourth grade students can be seen to lead the field of Western countries, including East-European countries (see Figure 25). Regarding the third-grade results, the Czech students took this place, leaving the Dutch students just behind them.

It should be acknowledged that in the ranking order as shown in Figure 25, it is not taken into account that The Netherlands did not satisfy the guidelines for sample participation rates. But suppose for a longer or a shorter moment that this shortage did not affect the reliability of the findings. Could these results then be interpreted as the yield of 30 years of RME?

Realistic Mathematics Education as work in progress

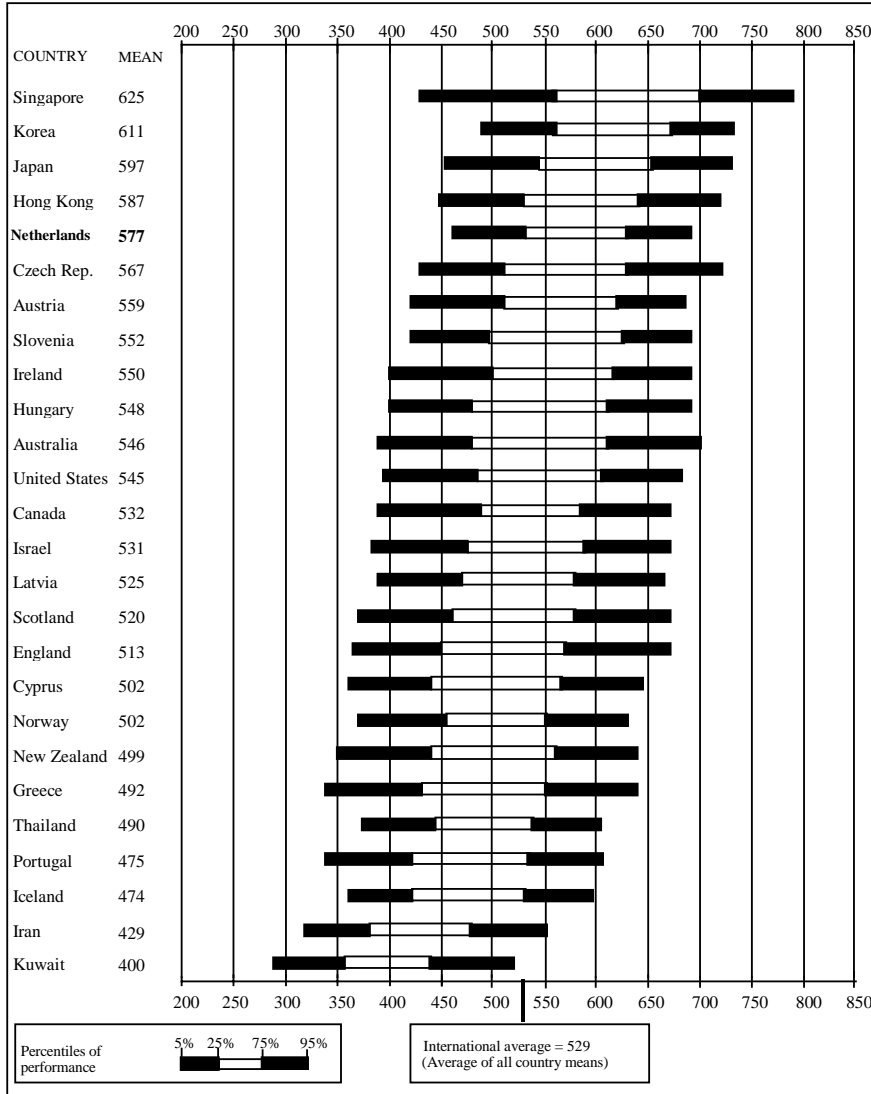


Figure 25 TIMSS results mathematics achievements grade 4 (adapted from Mullis et al., 1997, p.24)


I am afraid they can not. RME is namely striving for a different output. The problem with the TIMSS results is that the test does not fit our approach to mathematics education nor the content of our curriculum. Some topics in the test, like decimals, are not addressed until grade 5 in our curriculum. Other topics, like formal geometry, probability and making true statements and sentences are simply missing in our primary school curriculum. If anything from the results could be interpreted as an outcome of RME then it would be the ability of Dutch students to

use their common sense to solve problems. Often, high scores were found on topics which are not taught in school. Take, for instance, the following probability item about the chance of picking a red marble out of a bag (see Figure 26).


Example 18

Chance of picking red marble.


There is only one red marble in each of these bags.



10 Marbles



100 Marbles



1000 Marbles

Without looking in the bags, you are to pick a marble out of one of the bags. Which bag would give you the greatest chance of picking the red marble?

A. The bag with 10 marbles
 B. The bag with 100 marbles
 C. The bag with 1000 marbles
 D. All bags would give the same chance.

Figure 26 TIMSS item about probability (from Mullis et al., 1997, p. 94)

On this item the Dutch third graders had the highest percentage (56) correct scores after Japan (64) and the Dutch fourth graders had the highest percentage correct of all countries (74; the scores for fourth-grade students in Japan was 70, in Hong Kong 69, in Singapore 61, and in Korea 39). Moreover, the Dutch fourth-grade score even equaled the seven graders' international average score. Nevertheless, this result cannot be seen as the direct result of our curriculum. If there is any influence it might be the emphasis in the Dutch curriculum on common sense thinking and strategies to solve problems.

Another reason that the TIMSS results cannot be regarded as an indication of our supposed progress in mathematics achievements is that in the TIMSS test several topics are missing which have a prominent role in our curriculum — or the topics are not in the test in the way they should be. This is especially true of the topics mental arithmetic, estimation, and number sense.

Problems like, for instance, the ones which are shown in the Figures 27 and 28 are not a part of the TIMSS test.

**4000 students
in 48 classrooms**

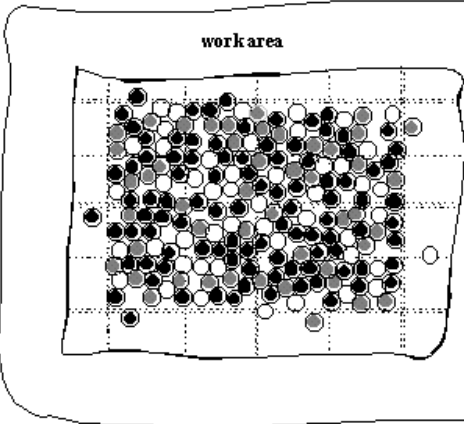
Woodrock has nine schools in total. Two of them have a dislocated annex somewhere else in a nearby quarter. Four of the schools are middle schools. The others are elementary schools including a kindergarten department. In total, almost four thousand children go to the nine schools. They are distributed over

What do you think of the headline in the newspaper clipping?
Can that be true, 4000 students in 48 classrooms?
Explain why or why not?

Figure 27 4000 students problem (adapted from Buys, 1998)

Marbles

work area



question-and-answer area

About how many marbles are on the table cloth?


 Explain how you came to your answer.

Figure 28 Marbles problem¹⁶

16 This item I designed for the AMI Pilot test. AMI is a comparative study on mathematics achievements and stands for “Applying Mathematics International”. In this study “applying mathematics” is taken as a criterion for

More in general, what is especially missing in the TIMSS test is “applying mathematics” in the RME sense of “mathematization”. This ability, however, can only be assessed if the students are provided with problems that are suitable for mathematization. In other words, they should not be presented problems in which everything is already prepared and the only thing a student has to do is finding — or rather fishing for— the one and only correct solution. Instead, the problems handed out to the students should offer them a rich context that can be organized, analyzed, and elaborated by means of mathematical tools. These tools can be standard ones or informal ones. As said earlier, mathematization can occur on different levels. These levels are connected to the various levels of understanding through which students can pass: from the ability to invent informal context-related solutions, to the creation of various levels of shortcuts and schematization, and finally, to the acquisition of insight into the underlying principles and the discernment of even broader relationships. So, important for the choice of problems is that they can be solved on different levels.

The problems in Figures 27 and 28 make clear that assessment tasks like these that can make the ability of mathematization assessable do not quite meet the requirements for standardized testing. Not only can these problems be solved by different solution strategies, even worse is that the answers can differ. In other words, RME has consequences for assessment (De Lange, 1987; Van den Heuvel-Panhuizen, 1996) and asks for rethinking the psychometric model of assessment.¹⁷

If we restrict ourselves to Dutch studies about the effect of mathematics education in primary school then the picture is also not very clear. Comparisons of mathematics scores on the PPON¹⁸ tests of grade 6 students in 1987, 1992 and 1997 do not show a very convincing progress (Janssen et al., 1999). On the contrary, it looks as if the output of Dutch mathematics education is deteriorating (see Figure 29).¹⁹

comparing mathematics achievements of fourth graders in different countries.

The participating countries all agreed that applying mathematics is an important goal of mathematics education. The research report is not yet published.

17 More about this topic in my lecture about assessment.

18 PPON stands for Periodical Assessment of Educational Achievements. It is a NAEP-like study carried out by the CITO, the National Institute for Educational Assessment.

19 Qualifications of the effect sizes: -0.8 (large negative effect); -0.5 (moderate negative effect); -0.2 (small negative effect); 0.0 (no effect); 0.2 (small positive effect); 0.5 (moderate positive effect); 0.8 (large positive effect).

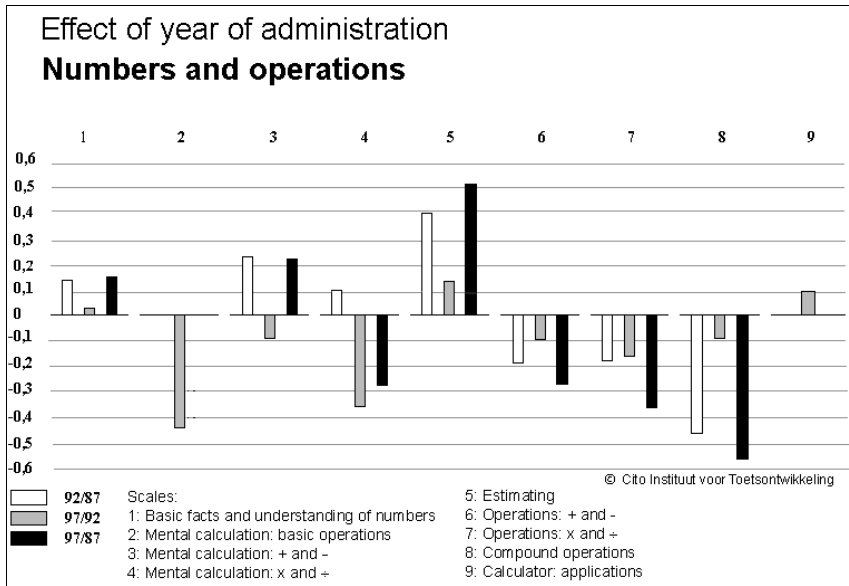


Figure 29 Comparison of PPON mathematics scores on number scales²⁰

In the Netherlands, there was a lot of discussion about these results. People wondered whether the analyses were done in the right way²¹ and whether the problems are suitable for revealing the students' mathematical knowledge and understanding.

A more positive result came recently from an international comparison between England and the Netherlands, carried out by Anghileri and Beishuizen (in press). In their research, in which English and Dutch fourth-grade students were tested in January and June, they revealed a remarkable progress of the Dutch students in the way they carry out division problems.²² In January the Dutch students were successful with 47% of the problems and the English students with 38%. In June the results were 68% and 44% respectively. When individual students were compared for the two test results the changes in scores of the Dutch students ($n = 259$) were much better than those of the English students ($n = 276$) (see Figure 30).

²⁰ Picture from Janssen et al. (1999).

²¹ Very questionable is, for instance, that in the analyses the text books used in the school were considered as a control variable, while the change and improvement of text books is an important result of the implementation on RME.

²² The test contained ten division problems (five context problems and five bare number problems). The problems that were "bare" in January were presented in context in June, and vice versa. The numbers and the contexts remained the same.

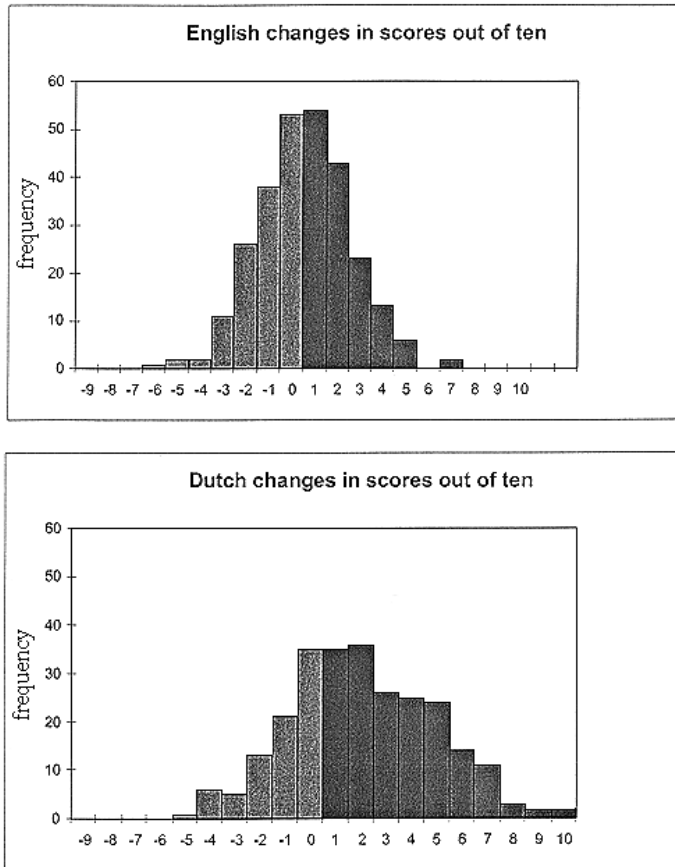


Figure 30 Changes on test scores of English and Dutch fourth-grade students

In January both groups applied informal strategies but in the English sample no clear progression was evident from these informal strategies to the standardized procedure. Although the English students showed sound informal approaches they were often disorganized in their recording. The Dutch students, however, showed a clearer organization in their recording methods that could be associated with the taught procedure based on repeated subtraction. The study revealed that “the Dutch approach [...] leads to a chunking procedure that the pupils are confident to use and that they used effectively. Because this procedure can be used at different levels of efficiently [(see Figure 31)] an element of choice is retained so the pupils continue to have some ownership of thinking within the structured approach. This appears to achieve a smooth transition from strategy to procedure which avoids the mechanical application of taught rules.” (Anghileri and Beishuizen, in press). Again, these results make it apparent that making progress detectable makes demands on assessment.

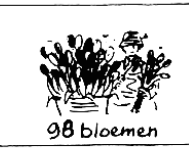
<p>Division test in Dutch Year 5</p>	 <p>98 bloemen</p>	<p>98 flowers are bundled in bunches of 7. How many bunches can you make ?</p>										
<p>repeated subtraction:</p> $\begin{array}{r} 98 \\ \underline{7} \\ 89 \\ \underline{7} \\ 784 \\ \underline{7} \\ 77 \\ \textcircled{14} \end{array}$	<p>using doubling:</p> $\begin{array}{r} 98 - 14 = 2 \\ 14 \\ \underline{84} - 2 \\ 14 \\ 670 - 4 \\ 28 - 4 \\ 342 - 4 \\ 28 - 4 \\ 14 - 2 \\ 14 \\ \underline{0} \end{array}$ <p>$6 + 8 = \textcircled{14}$</p>	<p>high level chunking:</p> $\begin{array}{r} 98 \\ \underline{70} - 10x \\ 28 \\ \underline{28} - 4x \\ 0 \\ \textcircled{14} \end{array}$ <table style="display: inline-table; vertical-align: middle;"> <tr><td>7</td><td>1x</td></tr> <tr><td>14</td><td>2x</td></tr> <tr><td>28</td><td>4x</td></tr> <tr><td>56</td><td>8x</td></tr> <tr><td>70</td><td>10x</td></tr> </table> <p>(using list of multiples)</p>	7	1x	14	2x	28	4x	56	8x	70	10x
7	1x											
14	2x											
28	4x											
56	8x											
70	10x											

Figure 31 Similar procedures used at different levels of efficiency by individual students

To conclude this lecture I will say some words about the developments within RME as a theory of mathematics education by discussing some trends in RME.

6 Progress in the RME theory of mathematics education

In the previous sections I gave an impression of what is meant by RME. Of course it is not doable to give a complete picture of RME in one lecture. RME is too complex for that. Moreover — and that might be a surprise —, another “difficulty” is that RME is not a unified approach to mathematics education. That means that the various lectures about RME given at this conference will contain different accentuations and interpretations of RME. At first glance this might be confusing and not workable. After rethinking this phenomenon it becomes obvious how important these differences were — and still are — for the development of RME. As a matter of fact, the different accentuations are the impetus for the continuing progress of the RME theory.

Rather than being a clear-cut theory of mathematics education, RME consists of some shared basic ideas about the what-and-how of teaching mathematics. These ideas have been developed over the past thirty years and the accumulation and repeated revision of these ideas has resulted in what is now called “Realistic Mathematics Education”. During the development of RME, emphasis has been placed on differing aspects of the theoretical framework that is connected with RME, and this has guided the research and developmental work in the field of mathematics education. Along with this diversity, the theoretical framework itself was also subject to a constant process of renewal. Inherent to

RME, with its basic idea of mathematics as a human activity, is that RME can never be considered as a fixed or finished theory of mathematics education. In other words, not only the work with the students is seen as “work in progress” but also the RME theory itself (see also Van den Heuvel-Panhuizen, 1998).

In the beginning stage of RME, the main framework by means of which the RME approach was explained consisted of the five characteristics of RME curricula: (1) the dominating place of context problems; (2) the broad attention paid to the development of models; (3) the contributions of students by means of own productions and constructions; (4) the interactive character of the learning process; and (5) the intertwining of learning strands (see Treffers and Goffree, 1985). Later on, these characteristics of RME curricula and teaching units evolved into a set of principles that formed a framework for an instruction theory. The characteristics were interpreted as the five instruction principles that guided the process of “progressive mathematization” (Treffers, 1987). A further development was the distinction between the learning aspect and the instruction aspect of the principles: (L-1) the concept of learning as construction and (I-1) starting with a concrete orientation basis; (L-2) level character of learning and (I-1) the provision of models; (L-3) the reflective aspect of learning and (I-3) the assignment of special tasks in particular own production tasks; (L-4) learning as a social activity and (I-4) interactive instruction; (L-5) the structural or schematic character of learning and (I-5) the intertwining of learning strands (see Treffers, 1991).

As is indicated by Treffers (1991), the relationships between the learning aspect and the instruction aspect of the RME principles must not be seen as one-to-one connections. Actually, each learning aspect can be connected with each instruction aspect; resulting in a very complex pattern of learning-instruction principles. Although this complex model will be more in tune with the complex process of learning and instruction in reality, I prefer to stick to the five principles that were originally formulated; of which some are more connected to teaching and some to learning (see figure 32). Moreover, I like to add the guidance principle.²³ A more detailed description of this list of six founding principles of RME can be found in Van den Heuvel-Panhuizen (2001a and 2001b).

The scheme in Figure 32 also reflects that presently there is more and more awareness of a distinction between a global theory and a local theory for different content domains.

23 In this respect differs the RME approach from the constructivist approach. See Note 13.

GENERAL RME THEORY		
WHAT	HOW	
Meaningful human activity Horizontal and vertical mathematization Low-level skills and high-level skills	Teaching	Context or reality principle Intertwinement principle Guidance principle
	Learning	Activity principle Level principle Interaction principle

LOCAL RME THEORIES	
WHAT	HOW
Whole number calculation	Understanding of numbers as a basis Progressive schematization Connected strategies Productive practicing
ETC	ETC

Figure 32 Theoretical framework of the RME approach

Compared to the RME approach in the seventies and the eighties, today’s RME approach to mathematics education is more differentiated and more in balance.

Let me start with the growth in balance. One crucial point of evolution has to do with the interpretation of the concept of mathematization. As I said in the beginning, Freudenthal already laid the foundation for this vital focal point of RME in the sixties. But it was Treffers’ distinction into “horizontal” and “vertical” mathematization that gave this concept the key role it now has in RME. Although these two ways of mathematization were formulated halfway through the seventies, it took much longer to get the balance between the two. Reflecting about the history of RME, Treffers (1992) called the period from 1972 through 1982 the “horizontal period”. In that time there was no sharp view on the function of context and model situations for the vertical aspect. There was no balanced interplay between the two ways of mathematization. Nevertheless, at that time, there was already some awareness (see Treffers 1978; De Lange, 1979, 1987) that in this interplay the heart of RME can be found. According to Treffers (1992) this balanced view came into being during the eighties. Therefore he called the years from 1982 through 1992 the “vertical period”. As the most paradigmatic example of this he mentioned long division. Although “progressive schematization” is a clear example of this vertical aspect of mathematization (the process of reorganization within the mathematical system itself, by, for instance, finding shortcuts), in the beginning of the nineties there was not really a full breakthrough of vertical aspect. Moving within the world of symbols by doing problems with bare numbers and doing investigations into

properties of numbers were not really accepted as belonging to RME, but were in that time still associated with the traditional “mechanistic” approach to mathematics education. However, these ideas are changing now (see Van den Heuvel-Panhuizen (ed.), 2001). Actually, now – at least at theory level – the intended balance is reached.²⁴

Another balance that is reached is the one between insight and skills. This also includes a revised opinion about exercising. Instead of simply rejecting the traditional drill-and-practice methods, new ideas have been developed about basic abilities and how students can build up these abilities (Van den Heuvel-Panhuizen and Treffers, 1998; Menne, 2001).

A further point of growth is that there is more awareness of the different requirements that particular groups of students have, like immigrants’ children (Van den Boer, 1995;), children with learning difficulties (e.g. Boswinkel and Moerlands, 2001), gifted children (e.g. Goffree, 2000a; 2000b), girls and boys (e. g. Van den Heuvel-Panhuizen, 1998; Van den Heuvel-Panhuizen and Vermeer, 1999) and students in vocational education (e. g. Van der Kooij, 2001).

RME is undeniable a child of its time and can also not be isolated from the present worldwide concern about the improvement of mathematics education. This implies that RME has a lot in common with other reform movements in mathematics. In other words, you will certainly recognize similarities with your own ideas on teaching and learning mathematics and your own agenda for further development. On the other hand, however, there might also be some dissimilarities. Again, it is worthwhile to reflect on these differences. It can provide us with clues for further improvement of the what-and-how of mathematics education.

References

- Anghileri, J and M. Beishuizen (in press). *From informal strategies to structured procedures: Mind the gap!*
- Boswinkel, N. and F. Moerlands (2001). Speciaal Rekenen – een onderzoek- en ontwikkelingsproject rond de invoering van realistisch reken-wiskundeonderwijs in het sbo. *Tijdschrift voor Nascholing en Onderzoek van het Reken-wiskundeonderwijs*, 19, 3, 3-13.
- Buys, K. (1998). Schattend rekenen moet je leren [Estimation you have to learn]. *Willem Bartjens*, 17, 5, 16-22.
- De Lange, J. (1987). *Mathematics, Insight and Meaning*. Utrecht: OW & OC, Utrecht University.

24 It is interesting that already at the end of the eighties Freudenthal concluded that in the early days of RME (from 1971 through 1981) too much energy was spent on the design of themes (by means of which domains of reality could be disclosed to the learner by horizontal mathematization), and too little on model contexts (that can offer possibilities for vertical mathematization) (see Treffers, 1993).

- De Lange, J. (1979). Contextuele problemen [Contextual problems]. *Euclides*, 55, 50-60.
- Dekker, A., H. ter Heege, and A. Treffers (1982). *Cijferend vermenigvuldigen en delen volgens Wiskobas* [Algorithmic multiplication and division according to Wiskobas]. Utrecht: OW&OC, Utrecht University.
- Freudenthal, H. (1968). Why to Teach Mathematics so as to Be Useful. *Educational Studies in Mathematics*, 1, 3-8.
- Freudenthal, H. (1977). Antwoord door Prof. Dr. H. Freudenthal na het verlenen van het eredoctoraat [Answer by Prof. Dr. H. Freudenthal upon being granted an honorary doctorate]. *Euclides*, 52, 336-338.
- Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: Reidel Publishing Company.
- Freudenthal, H. (1991). *Revisiting Mathematics Education. China Lectures*. Dordrecht: Kluwer Academic Publishers.
- Gravemeijer, K. (1982). Het gebruik van contexten [The use of contexts]. *Willem Bartjens*, 1 (4), 192-197.
- Gravemeijer, K.P.E. (1994). *Developing Realistic Mathematics Education*. Utrecht: CD-β Press / Freudenthal Institute.
- Goffree, F. (2000a). Bollebozen vroeger en nu. Heeft de wiskunde bollebozen iets te bieden? *Willem Bartjens*, 20, 1, 40-42.
- Goffree, F. (2000b). De andere wereld van (hoog)begaafde rekenaars. *Nieuwe Wiskrant. Tijdschrift voor Nederlands Wiskundeonderwijs*, 19, 3, 26-29
- Janssen, J., F. van der Schoot, B. Hemker, and N. Verhelst (1999). *Balans van het reken-wiskundeonderwijs aan het einde van de basisschool 3* [An account of mathematics education in the final grade of primary school, part 3]. Arnhem: Cito.
- Mullis, I.V.S., Martin, M.O., Beaton, A.E., Gonzalez, E.J., Kelly, D.L., and Smith, T.A. (1997). *Mathematics Achievement in the Primary School Years: IEA's Third International Mathematics and Science Study (TIMSS)*. Chestnut Hill, MA: Boston College.
- Rengerink, J. (1983). *De staartdeling, een geïntegreerde aanpak volgens het principe van progressieve schematisering* [The long division, an integrated approach according to the principle of progressive schematization]. Utrecht: Vakgroep Onderwijskunde/IPAW, Rijksuniversiteit Utrecht.
- Streefland, L. (1985). Wiskunde als activiteit en de realiteit als bron [Mathematics as an activity and reality as source], *Nieuwe Wiskrant*, 5, 1, 60-67.
- Streefland, L. (1991). *Fractions in Realistic Mathematics Education. A Paradigm of Developmental Research*. Dordrecht: Kluwer Academic Publishers.
- Streefland, L. (1996). Negative numbers – Reflections of a learning researcher. *Journal of Mathematical Behavior*, 15, 1, 57-79.
- Treffers, A. (1978). *Wiskobas doelgericht* [Wiskobas goal-directed]. Utrecht: IOWO.
- Treffers, A., and Goffree, F. (1985). Rational analysis of realistic mathematics education – the Wiskobas program. In L. Streefland (ed.), *Proceedings of the*

- Ninth International Conference for the Psychology of Mathematics Education* (Vol. II, pp. 97-121). Utrecht: OW&OC, Utrecht University.
- Treffers, A. (1987). *Three Dimensions. A Model of Goal and Theory Description in Mathematics Instruction – the Wiskobas Project*. Dordrecht: Reidel Publishing Company.
- Treffers, A. (1991). Realistic mathematics education in the Netherlands 1980-1990. In L. Streefland (ed.), *Realistic Mathematics Education in Primary School*. Utrecht: CD-β Press / Freudenthal Institute, Utrecht University.
- Treffers, A. (1992). *Terug naar de toekomst. Reken-wiskundeonderwijs voor de basisschool 1972 – 2002* [Back to the future. Mathematics education in primary school 1972 – 2002]. In F. Goffree, A. Treffers, J. de Lange, Rekenen anno 2002 (pp. 11-34). NVORWO.
- Treffers, A. (1993). Wiskobas and Freudenthal: Realistic Mathematics Education. In L. Streefland (ed.), *The legacy of Hans Freudenthal*. Educational Studies in Mathematics, 25, 1-2. Dordrecht: Kluwer Academic Publishers.
- Van den Boer, C. (1997). Allochtone leerlingen in het wiskundeonderwijs [Immigrant children in mathematics education]. *Tijdschrift voor Nascholing en Onderzoek van het Reken- wiskundeonderwijs*, 15, 4, 27-34.
- Van den Brink, J.F. (1989). *Realistisch rekenonderwijs aan jonge kinderen* [Realistic arithmetic education to young children]. Utrecht: OW & OC, Utrecht University.
- Van den Heuvel-Panhuizen, M. (1995). *A representational model in a long-term learning process — the didactical use of models in Realistic Mathematics Education*. Paper presented at the 1995 AERA in San Francisco.
- Van den Heuvel-Panhuizen, M. (1996). *Assessment and realistic mathematics education*. Utrecht: CD-β Press / Freudenthal Institute, Utrecht University.
- Van den Heuvel-Panhuizen, M. (1998). Gender differences in mathematics achievements in Dutch primary schools — On the search for features of mathematics education that are important for girls. In Chr. Keitel (ed.), *Social Justice and Mathematics Education* (p.135 -149). Berlin: Freie Universität Berlin.
- Van den Heuvel-Panhuizen, M. (1998). Realistic Mathematics Education: Work in progress. In T. Breiteig and G. Brekke (Eds.), *Theory into practice in Mathematics Education*. Kristiansand, Norway: Faculty of Mathematics and Sciences.
- Van den Heuvel-Panhuizen, M. and A. Treffers (1998). *Oefenen. Module behorende bij de Nationale Cursus Rekencoördinator* [Exercising. In-service training module for mathematics coordinators]. Utrecht: Freudenthal Institute, Utrecht University.
- Van den Heuvel-Panhuizen, M. and H. Vermeer (1999). *Verschillen tussen meisjes en jongens bij het vak rekenen-wiskunde op de basisschool* [Differences between girls and boys in primary school mathematics education]. Utrecht: CD-β Press / Freudenthal Institute, Utrecht University.
- Van den Heuvel-Panhuizen (ed.), M. (2001). *Children learn mathematics*. Utrecht/Enschede: Freudenthal Institute, Utrecht University / SLO.

- Van den Heuvel-Panhuizen, M. (2001a). *Mathematics education in the Netherlands: A guided tour*. Utrecht: Freudenthal Institute (paper published on FI cd-rom).
- Van den Heuvel-Panhuizen, M. (2001b). Realistic mathematics education in the Netherlands. In J. Anghileri (ed.), *Principles and Practices in Arithmetic Teaching*. Buckingham: Open University Press.
- Van der Kooij, H. (2001). Mathematics and key skills for the workplace. In FitzSimons, O'Donoghue, Coben (eds.), *Adult and Lifelong Education in Mathematics* (pp. 231-242). Language Australia.
- Van Galen, F. and E. Feijs (1991). A mathematics lesson on videodisc. In L. Streefland (ed.), *Realistic Mathematics Education in Primary School*. Utrecht: CD-β Press / Freudenthal Institute, Utrecht University.
- Van Galen, F., M. Dolk, E. Feijs, V. Jonker, N. Ruesink, and W. Uittenbogaard (1991). *Interactieve video in de nascholing reken-wiskunde* [Interactive video in the in-service training of teachers]. Utrecht: CD-β Press / OC&OW, Utrecht University.