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Guides for didactical decision making in primary school mathematics education: the focus on the content domain of estimation[1]

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1 Introduction

What mathematics should we teach our students and how should this mathematics be taught? These questions can be considered as the key questions of mathematics education. Although the nineties can be seen as the decade of standards ? in many countries documents are published that prescibe what schools should teach their students[2] ? within the field of research of mathematics education more attention is paid yet to the how of the teaching than to the what. Not having an in-depth discussion about the what of teaching might probably be the main reason for the emergence of the present "Math Wars".[3]

Therefore I think the U.S. National Council of Teachers of Mathematics (NCTM) took a good initiative by devoting one of the issues of their discussion journal Mathematics Education Dialogues to the question "Who should determine what you teach?"[4] In the editorial introduction of this issue it is made clear that there are different ideas about the level on which decisions about the what should be made. Some people believe that teachers, who are the closest to their students and who know their local situation better than outsiders, should made these choices. Others think that these should be made more centrally, at the school level, the school-district level, state or provincial level, national level, or even at the international level.

Interesting was the contribution of William Schmidt to this issue of Dialogues, in which he refered to the lessons we can learn from the Third International Mathematics and Science Study (TIMSS). According to Schmidt (1999) one of the most salient and powerful policy implications from TIMSS is the essential role that curriculum plays in teaching and learning mathematics. Reflecting upon the disappointing results of U.S. students, he thinks these results might be caused by the highly repetitive character of the curriculum over the grade levels, by the fact that the curriculum does not not focus on rigorous content, by the lack of coherence within the curriculum and the splintered vision in the U.S. system. His concern about the latter was also expressed in the title of his contribution: "Toward a national consensus."

It was only recently that I read these conclusions of Schmidt and I was struck by the emphasis he was putting on a focused and coherent curriculum. Actually, in 1997 we had the same guiding rationale for starting the development of learning-teaching trajectories for primary school mathematics, which will be the topic of my lecture. These trajectories are based on a didactical phenomenological approach[5] and give teachers a pointed overview of how children's mathematical understanding can develop from K1 through grade 6 and of how education can contribute to this development. Although these trajectories contain many concrete examples of classroom activities, they cannot be used as a recipe book for everyday teaching. In contrast with other curriculum materials the trajectory descriptions are more theoretical. They give an overview at a more conceptual level. They show how different stages in the longitudinal process are connected to each other and how new stages can evolve from previous ones. As such these learning-teaching trajectories can provide teachers with a "*mental educational map*" for didactical decision making.

In this lecture, my focus will be on the content domain of estimation, but I will start with giving some background information about the main factors that determine the content of the primary-school mathematics curriculum in the Netherlands so far.

2 The determinants of the Dutch primary-school mathematics curriculum

Unlike many other countries, at primary school[6] level the Netherlands does not have centralized decision

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1. Plenary lecture held at the Opening Conference of the Norwegian Center for Mathematics Education, Trondheim, Norway, 18-19 November 2002.
2. Examples of these standards are the NCTM Standards in the United States, and the National Curriculum and the Numeracy Project in the United Kingdom.
3. A good description of the mathematics war developments in the United States can be found in Becker and Jacob (1998).
4. Mathematics Education Dialogues, Volume 2, Issue 2 (April 1999).
5. I will come back to this later.
6. In the Netherlands, primary school is meant for students of ages 4 to 12 and includes eight grade classes: K1-2 and Grade 1-6. In The Netherlands these classes are called "Groep 1-8".

making regarding curriculum syllabi, text books or examinations (see Mullis et al., 1997). None of these need approval by the Dutch government. For instance, the schools can decide which textbook series they use. They can even develop their own curriculum. In general, what is taught in primary schools is, for the greater part, the responsibility of teachers and school teams and the teachers are fairly free in their teaching. To give some more examples, teachers have a key to the school building, they are allowed to make changes in their timetable without asking the school director (who often teaches a class too), and, as a last example, the teacher's advice at the end of primary school, rather than a test, is the most important criterion for allocating a student to a particular level of secondary education.

Despite this freedom in educational decision making—or probably one should say thanks to the absence of centralized educational decision making—the mathematical topics taught in primary schools do not differ much between schools. In general, all schools follow roughly the same curriculum. This leads to the question: what determines this curriculum?

Until recently, there were three important determinants for macro-didactic tracking in Dutch mathematics education in primary school:

1. The mathematics textbooks series they use can be seen as the most important tools for guiding the teachers' teaching. If I restrict myself to the most prevailing textbooks, there are six different textbooks between which the schools can choose. All the textbooks are published by commercial publishers, and the schools have a free choice in deciding which textbook they will use. The textbook authors are independent developers of mathematics education, but they can make use of ideas for teaching activities resulting from developmental research at, for instance, the Freudenthal Institute (and its predecessors) and the SLO.
2. Another main determinant of the curriculum content is a series of publications, called the "Proeve."[7] These publications which were published since the late eighties, and of which Treffers is the main author contain descriptions of the various domains of mathematics as a primary school subject. The Proeve books have been very influencial on the development of textbook series, but these in their turn also inspired the domain descriptions.
3. A further influential factor that determines what is taught in primary school is the list of 23 attainment targets for primary school as described by the government. This list of "core goals" was established in 1993 by the Dutch Ministry of Education and was revised in 1998 (see OC and W, 1998).

The list of attainment targets is split into six domains, including general abilities, written algorithms, ratio and percentage, fractions, measurement, and geometry. The goals describe what students have to learn by the end of their primary school career (at age twelve). Table 1 shows the core goals for general abilities and written algorithms. Core goal four explains what the students should have achieved regarding estimation.

Table 1: Part of the core goals for Dutch primary school students in mathematics

By the end of primary school, the students ...		
General abilities	1	Can count forward and backward with changing units
	2	Can do addition tables and multiplication tables up to ten
	3	Can do easy mental-arithmetic problems in a quick way with insight in the operations
	4	Can estimate by determining the answer globally, also with fractions and decimals
	5	Have insight into the structure of whole numbers and the place-value system of decimals
	6	Can use the calculator with insight
	7	Can convert simple problems which are not presented in a mathematical way into a mathematical problem
Written algorithms	8	Can apply the standard algorithms, or variations of these, to the basic operations, of addition, subtraction, multiplication and division in simple context situations

Compared to goal descriptions and programs from other countries this list is a very simple one. It means that there is a lot of freedom in interpreting the goals. At the same time, however, such a list does not give much support for

7 The complete title of this series is "Design of a national program for mathematics education in primary schools" [Proeve van een Nationaal Programma voor het reken- wiskundeonderwijs op de basisschool]. The first part of this series was published in 1989 (see Treffers, De Moor and Feijs, 1989). Note that the title refers to a "national program" although there was no government interference. The authors wanted to label this a national program in order to achieve a communal program. They have clearly succeeded in this aim.

educational decision making. In the years since 1993, there have been discussions about these core goals (see De Wit, 1997). Almost everybody agreed that they can never be sufficient to support improvements in classroom practice or to control the outcome of education.

For several years it was unclear which direction would be chosen for improving the core goals: either providing a more detailed list of goals for each grade, expressed in operationalized terms, or a description which supports teaching rather than pure testing. In 1997, the Dutch Ministry of Education tentatively opted for the latter and asked the Freudenthal Institute to work out the description for mathematics. In September 1997, this decision resulted in the start of the TAL Project[8], which meant that eventually a fourth curriculum determinant would come into being.

4 Learning-teaching trajectories—new guides for didactical decision making

The aim of the TAL project

The aim of the TAL Project is to develop learning-teaching trajectories for all domains of the primary-school mathematics curriculum. In total three learning-teaching trajectories will be developed: a trajectory for whole number calculation, one for measurement and geometry, and one for fractions, decimals and percentages.

The project started with the development of a learning-teaching trajectory for whole-number calculation. This first trajectory description for the lower grades (including K1, K2, and grades 1 and 2) was published in November 1998. The definitive version was released a year later. The following year the whole-number trajectory for the higher grades of primary school (including grades 3 through 6) was published. In 2001, both learning-teaching trajectories were translated in English and published together in one book (Van den Heuvel-Panhuizen (Ed.), 2001).

In 1999, a start was made on the development of a learning-teaching trajectory for measurement and geometry. This will be finished (in Dutch) by the end of 2002.

What is meant by a learning-teaching trajectory?

Giving the teachers a pointed overview of how children's mathematical understanding can develop from K1 through grade 6 and of how education can contribute to this development, is the main purpose of this alternative to the traditional focus on strictly operationalized goals as the most powerful engine for enhancing classroom practice. In no way, however, is the trajectory meant as a practical recipe book. It is, rather, intended to provide teachers with a *mental educational map* which can help them, if necessary, to make adjustments to the textbook. The learning-teaching trajectory serves as *a guide at a conceptual level*. Having an overview of the process the students go through is very important for working on progress in students' understanding; see for instance Freudenthal's (1981) plea for providing teachers with experience in the long-term learning processes. To make adequate decisions about help and hints, a teacher must have a good idea of the goals, the route that can lead to these goals and the landmarks the students will pass one way or another along this route, when selecting new problems. Without this outline in mind it is difficult for the teacher to value the strategies of the students and to foresee where and when one can anticipate the students' understandings and skills that are just coming into view in the distance (see also Streefland, 1985). Without this longitudinal perspective, it is not possible to guide the students' learning.

Another remark has to be made about the name "trajectory". Although a learning-teaching trajectory puts the learning process in line, it should not be seen as a linear and singular step-by-step regime in which each step is necessarily and inexorably followed by the next one. A learning-teaching trajectory should be seen as being broader than a single track and should have a particular bandwidth. It should do justice to differences in learning processes between individual students and to the different levels at which children master particular skills and concepts.

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8 The TAL Project is carried out by the Freudenthal Institute and the SLO (the Dutch Institute for Curriculum Development), in collaboration with CED (school advisory center for the city of Rotterdam). TAL is a Dutch abbreviation and stands for Intermediate Goals Annex Learning-Teaching Trajectories. Since the beginning of the project the following people have contributed to the development of the learning-teaching trajectory: Joop Bokhove (FI), Jan van den Brink (FI), Arlette Buter (FI/CED), Kees Buys (SLO), Nico Eigenhuis (CED), Erica de Goeij (FI), Marja van den Heuvel-Panhuizen (project leader) (FI), Jan Hochstenbach (FI), Christien Janssen (FI), Julie Menne (FI), Ed de Moor (FI), Jo Nelissen (FI), Anneke Noteboom (FI), Markus Nijmeijer (FI), Adri Treffers (FI), Ans Veltman (FI), Jantina Verwaal (FI). The total size of the TAL Team has been the equivalent of three fully employed persons.

A new educational phenomenon

Compared to the goal descriptions that were traditionally supposed to guide education and support educational decision making, the learning-teaching trajectory as it is worked out in the TAL Project has some new elements that makes it a new educational phenomenon.

First of all, the trajectory is more than an assembled collection of the attainment targets of all the different grades. Instead of a checklist of isolated abilities, the trajectory clarifies how abilities are built up in connection with each other. It shows what is coming earlier and what is coming later. In other words, the most important characteristic of the learning-teaching trajectory is its *longitudinal perspective* which has a long history in the Dutch didactical (subject-matter connected) approach to mathematics education.[9]

A second characteristic is its *double perspective of attainment targets and teaching methods*. The learning-teaching trajectory does not only describe the landmarks in student learning that can be recognized en route, but it also portrays the key activities in teaching that lead to these landmarks.

The third feature is its *inherent coherence, based on the distinction of levels*. The description makes it clear that what is learned at one stage, is understood and performed on a higher level in a following stage. A recurring pattern of interlocking transitions to a higher level forms the connecting element in the trajectory. It is this level characteristic of learning processes, which is also a constitutive element of the Dutch approach to mathematics education, that brings longitudinal coherence into the learning-teaching trajectory. Another crucial implication of this level characteristic is that students can understand something on different levels. In other words, they can work on the same problems without being on the same level of understanding. The distinction of levels in understanding, which can have different appearances for different sub-domains within the whole number strand, is very fruitful for working on the progress of children's understanding. It offers footholds for stimulating this progress.

The fourth attribute of the TAL learning-teaching trajectory is the new description format that has been chosen for it. The description is not a simple list of skills and insights to be achieved, nor a strict formulation of behavioral parameters that can be tested directly. Instead, *a sketchy and narrative description, completed with many examples*, of the continued development that takes place in the teaching-learning process is given.

The main purpose of a learning-teaching trajectory is to give the teachers a pointed overview of how children's mathematical understanding can develop from K1 and 2 through grade 6 and of how education can contribute to this development. It is intended to provide teachers with a "mental educational map" which can help them to make didactical decisions, for instance making adjustments to the textbook that they use as a daily guide.

Development of the TAL learning-teaching trajectories

In the development of the TAL learning-teaching trajectories "didactical phenomenological analyses"? as Freudenthal (1983) called them? play a crucial role. These analyses reveal what kind of mathematics is worthwhile to learn and which actual phenomena can offer possibilities to develop the intended mathematical knowledge and understanding. Important is that one tries to discover how students can come into contact with these phenomena, and how they appear to the students. This means that problems and problem situations that give students opportunities to develop insight in mathematical concepts and strategies must be identified. Therefore a team, containing all kinds of specialisms in primary school mathematics, has been formed. The group contains experience in research and development of mathematics education, assessment, teacher educating, teacher advice, and teaching mathematics in primary school. The core of the work is formed by the (almost) weekly discussions in the project team, for which input comes from a variety of sources: analyses of textbook series, analyses of research literature, investigations in classrooms, and extensive consultations of experts in mathematics education. An earlier example of such an approach, but aimed at finding the long term learning process for the domain of ratio, can be found in Streefland (1984/1985).

The TAL trajectory for whole number calculation

In the TAL trajectory for calculation with whole numbers (see Figure 1), calculation is interpreted in a broad sense, including number knowledge, number sense, mental arithmetic, estimation and algorithms. The trajectory description gives an overview of how all these number elements are related to each other.

[9] This longitudinal characteristic makes that the TAL learning-teaching trajectories differ significantly from what Simon (1995) called a "hypothetical learning trajectory" which covers only a couple of lessons and which moreover refers to a teacher's plan within the context of his or her own classroom.

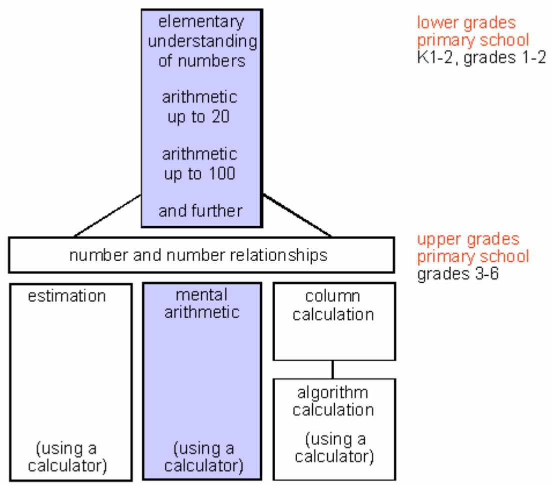


Figure 1: The TAL learning-teaching trajectory for whole number calculation in primary school.

The scheme reflects that the students gradually come from a non-differentiated way of counting-and-calculating to calculations in more specialized formats that fit particular kinds of problems in a particular number domain. Mental arithmetic is considered to play a central role in whole number calculation. It is seen as an elaboration of the arithmetic work that is rooted in the lower grades and forms the backbone for the upper grades.

5 Estimation as an example[10]

New in this trajectory is also the proposed didactics for estimation. Although estimation is now widely acknowledged as an important goal of mathematics education, in most textbooks a framework for how to learn to estimate is lacking. The textbooks at most only contain several problems on estimation, but doing a few estimation problems from time to time is not enough to develop real understanding of how an estimation works, and it is not sufficient to develop comprehension of what is ‘allowed’ and what is not when estimating.

The learning-teaching trajectory offers a first proposal for a phased structure that the students can go through for developing estimation skills. Therefore a subdivision is made into the *four sub-domains*:

? rounding off numbers
? estimations in addition and subtraction
? estimations in multiplication and division
? estimations in case of incomplete data.

The above sequence reflects, in a general sense, the trajectory followed by the children. The basis of estimation is formed by rounding off numbers, which is followed by estimating in calculation problems.

Of the four basis operations, addition and subtraction are offered first. Among other reasons, this is because the consequences of rounding off for division and multiplication are often more difficult to perceive. As long as the students are being asked to make only a very global estimate, estimation in multiplication and division is indeed comparable with estimation in addition and subtraction. This changes when a more refined estimation is involved and the students are also required to indicate the magnitude of the deviation. This difference has mainly to do with the fact that the effects of rounding off are not as clear cut in multiplication and division, because deviations and imprecision become magnified. One effect of this magnification is that it becomes more difficult to understand which type of rounding off results in the best estimates. This is especially true when multiple numbers must be rounded off in a single problem. At this point, the students have crossed over into the terrain of the more skillful practitioner of arithmetic. Learning to estimate in multiplication and division is a process that goes beyond primary school. The foregoing makes that, generally speaking, addition and multiplication take a more important place in the primary school curriculum for estimation than subtraction and division do.

[10] This part of the paper gives a summary of my chapter on Estimation in the TAL book (Van den Heuvel-Panhuizen (Ed.), 2001).

Within the area of estimation *two different types of estimation problems* can be distinguished:

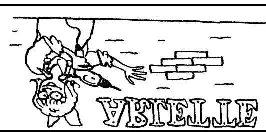
? calculations with rounded off numbers

? calculations with estimated values.

An example of the first type is the *Bread* problem (Figure 2). In this type of problems, in which only a global calculation is needed, the precisely given numbers can be rounded off followed by an exact calculation with these round numbers. The *Arlette* problem (Figure 3) is an example of the second type of estimation problem. In this type of problem the necessary data is incomplete or unavailable. To solve calculations with estimated values the students should have good insight into the number system, and they should be familiar with measures and sizes from daily life. Moreover this type of problems often requires making specific assumptions as a starting point.

Loaves of raisin bread cost 1.98 euro each. Will 10 euro be enough to buy four?

Figure 2: *Bread* problem



One ornamental letter costs 3 euro and so many cents each. The exact price is not readable. How much do you think it would cost to buy all the letters to write the name ARLETTE?

Figure 3: *Arlette* problem

The central theme of the learning-teaching trajectory in estimation is characterized by three types of *key questions* that can be asked in order to elicit estimation and make this process of estimation sensible. In fact, the following types of questions are suitable for this:

? Are there enough?

? Could this be correct?

? Approximately how much is it?

It is these questions—which in themselves can take on all kinds of different forms—that are the driving force behind learning to estimate and which, moreover, are anchored in estimation as it occurs in everyday life. Although these three questions can be asked in every grade, the first two types belong more to the initial phases of the learning process, while questions of the third type should be offered later on. As a matter of fact, the latter type of question is a direct estimation question where the children themselves must arrive at an estimation. The others are more indirect estimation questions.

The most basic structure in the learning-teaching trajectory that guides the learning process is the distinction in different phases in learning estimation:

? In the informal phase the students can globally determine answers without using the standard rounding off rule.

? In the rule-directed phase the students arrive at the standard rounding off rule for operating with numbers and learn to apply this rule.

? In the flexible and critical phase the students are capable of applying more balanced estimation methods when operating with numbers and in which they can deal in a critical way with rounded off and exact numbers.

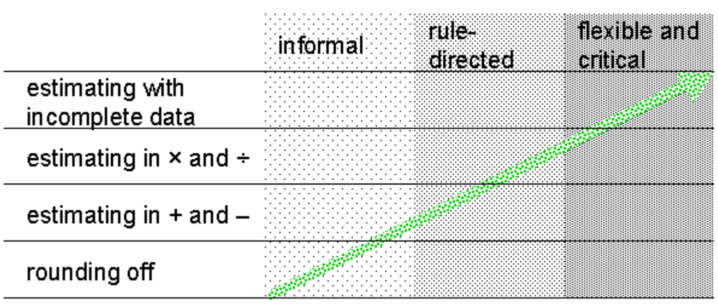


Figure 5: The conceptual didactical structure of the trajectory for estimation (grade 3-6)

These different learning phases can be found in each of the earlier mentioned sub-domains (Figure 5). The arrow indicates the global direction of the conceptual didactical structure of the trajectory for estimation from grade 3 through 6.

In the following sections I will clarify this structure a little bit further by focusing on the trajectory that has been developed for learning to estimate in addition and subtraction problems.

Estimating in addition and subtraction problems

After an initial base is created regarding rounding off numbers and understanding rounded off numbers the students start applying estimation in addition and subtraction problems. In the informal phase this means that the numbers are selected in such a way that a very global approach is sufficient. It is sufficient if the children know that the total of 2,113 and 3,389 is more than 5,000. This becomes clear even when the students only look at the value of the largest position. Instead of rounding off to the closest round number, the solution is then found by literally—or mentally—covering the other position values with one's thumb (Figure 6).

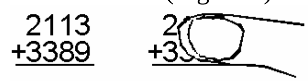


Figure 6: Rounding off by only looking at the value of the largest position

By assigning problems where the very global method can lead to incorrect conclusions, the students can arrive at the rule-directed phase. In such problems more precise rules about rounding off are needed in order to come to a sensible answer, as, for instance, is the case in the *Televisions* problem (Figure 7).

Number of televisions sold in two years: 4,896 and 5,987. Is the total more than 10,000?

Figure 7: *Televisions* problem

By "cutting off" these numbers and only looking at the thousands, one could conclude that the total is less than 10,000. However, if one slides one's thumb to the right, then it becomes immediately obvious that this estimation is too low; 800 plus 900 is more than 1000, which would make the total more than 10,000.

The number of positions the students have to slide their thumbs to the right depends on the specific problem. In this case, the positional values of the tens and ones do not matter. After all, as one reaches the hundreds it becomes immediately clear that the total must be above 10,000.

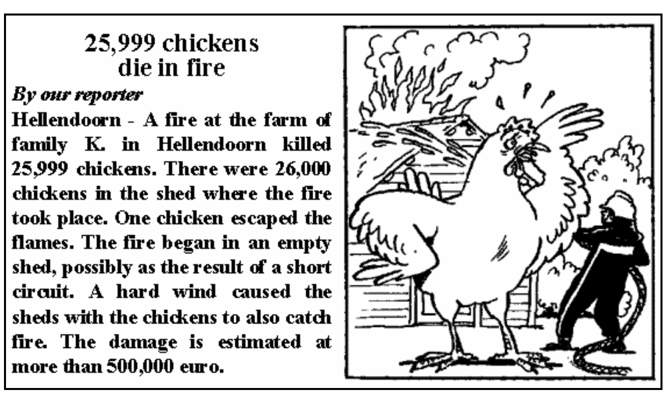
During the phase of rule-directed rounding off, the children are expected to be able to round off numbers when estimating in addition and subtraction problems according to the standard rounding off rule.

The next step is the discovery that this rounding off rule does not always have to be applied strictly when estimating in addition and subtraction problems. Then, the students arrive in the phase of flexible and critical estimation. They realize that the standard procedure of rounding off must be modified, especially if the numbers in an addition or subtraction problem are close to the turning points of fifty, five-hundred, and so on. The *Tickets* problem (Figure 8) makes this obvious.

The following number of tickets for the championship game were sold at three different outlets in the city: 3587, 2574 and 3928. Approximately how many tickets were sold in total?

Figure 8: *Tickets* problem

If these numbers are rounded off according to the standard rule 3587 becomes 4 thousand, 2574 becomes 3 thousand, and 3928 becomes 4 thousand. The total is then 11 thousand, even though 10 thousand is a better estimation. By comparing the children's estimates and checking them against the exact answer, refinements in the rounding off method can be brought up for discussion.



25,999 chickens die in fire

By our reporter

Hellendoorn - A fire at the farm of family K. in Hellendoorn killed 25,999 chickens. There were 26,000 chickens in the shed where the fire took place. One chicken escaped the flames. The fire began in an empty shed, possibly as the result of a short circuit. A hard wind caused the sheds with the chickens to also catch fire. The damage is estimated at more than 500,000 euro.

Figure 9: Chicken problem

Another example of flexible and critical estimation is the *Chicken* problem (Figure 9). It is an striking example of what can go wrong when exact calculation is used with rounded off numbers. At first glance it may seem strange that the reporter who wrote the newspaper clipping would know exactly how many chickens had died, but it soon becomes clear how this total was arrived at. Obviously, if only one chicken escaped and there were 26,000 chickens in the shed, this means that 25,999 were killed. Nonetheless, a serious mistake has been made in this calculation. Although this is not explicitly stated in the newspaper article, the total of 26,000, of course, stands for "approximately 26,000 chickens." The number of chickens was probably rounded off to the nearest thousand. This is why it is incorrect to subtract the single escaped chicken from this total number. Children who understand how silly this calculation is will probably not have any difficulty solving the *Billion-million* problem (see Figure 10).

Explain why the following answers are correct	approximately 1 billion + 1 million = approximately 1 billion approximately 1 billion ? 1 million = approximately 1 billion

Figure 10: *Billion-million* problem

In order to understand that these answers are indeed correct, a certain amount of thinking is necessary. The description "approximately 1 billion" indicates that this round number is in fact an estimate. This is still reasonably simple to understand. It is more difficult to understand that this is an estimation with a rounding off margin of one hundred million and that—for the total—it does not matter whether one million is added or subtracted from the estimate of one thousand million. A number line can help the children understand this.

Restrictions in estimating in multiplication and division problems

Once the students have acquired a lot of experience with rounding off numbers and are comfortable with estimation in addition and subtraction, it might appear that learning to estimate in multiplication and division is only a small step. However, this is only partly true. As long as the students are being asked to make only a very global estimate, estimation in multiplication and division is indeed comparable with estimation in addition and subtraction. This changes when a more refined estimation is involved and the students are also required to indicate the magnitude of the deviation. Take, for instance, the following *Seats* problem (Figure 11). The A-part of the problem is very easy to answer: 20 ? 30 would give a close estimate. So at first glance this appears to be a simple estimation problem. The B-part, however, makes it clear that this problem actually is quite difficult. Butterworth (1999) found that this kind of problems which he called "false-compensation" problems even are difficult for university students.

A.	In the hall, there are 18 rows of 32 seats each. Approximately how many seats together?
B.	Is your estimate lower or higher than the exact calculation of 18 ? 32?

Figure 11: *Seats* problem

The difficulty of estimating in multiplication and division problems has to do with the fact that the effects of rounding off are not as clear cut as in addition and subtraction problems. In the multiplications and divisions the deviation and imprecision become magnified which make that it becomes more difficult to understand which type of rounding off results in the best estimates. This is especially true when multiple numbers must be rounded off in a problem. At this point, the students have crossed into the terrain of the more skillful practitioner of arithmetic. Learning to estimate in multiplication and division is a process that goes beyond primary school. Therefore in the TAL learning-teaching trajectory we advise to restrict the estimating in multiplication and division problems to rule-directed rouding off and to problems in which only one number has to be rounded off. Teachers should be aware that presenting the students more complex estimation problems with numbers that "automatically" give a good estimate will in fact lead to a type of mock knowledge in the domain of estimaton.

6 TAL trajectory for estimation compared to the PPON results

To make the picture more complete I will conclude with comparing the TAL trajectory that has been developed for the domain of estimation? and that was based on a didactical phenomenological approach? with the empirically established sequence of estimation problems that resulted from the Dutch PPON study. This study is a large-scale assessment of students' performance in school subjects that is carried out by CITO (the Dutch National Institute for Educational Measurement). The latest assessment of achievements in mathematics was conducted in 1997.

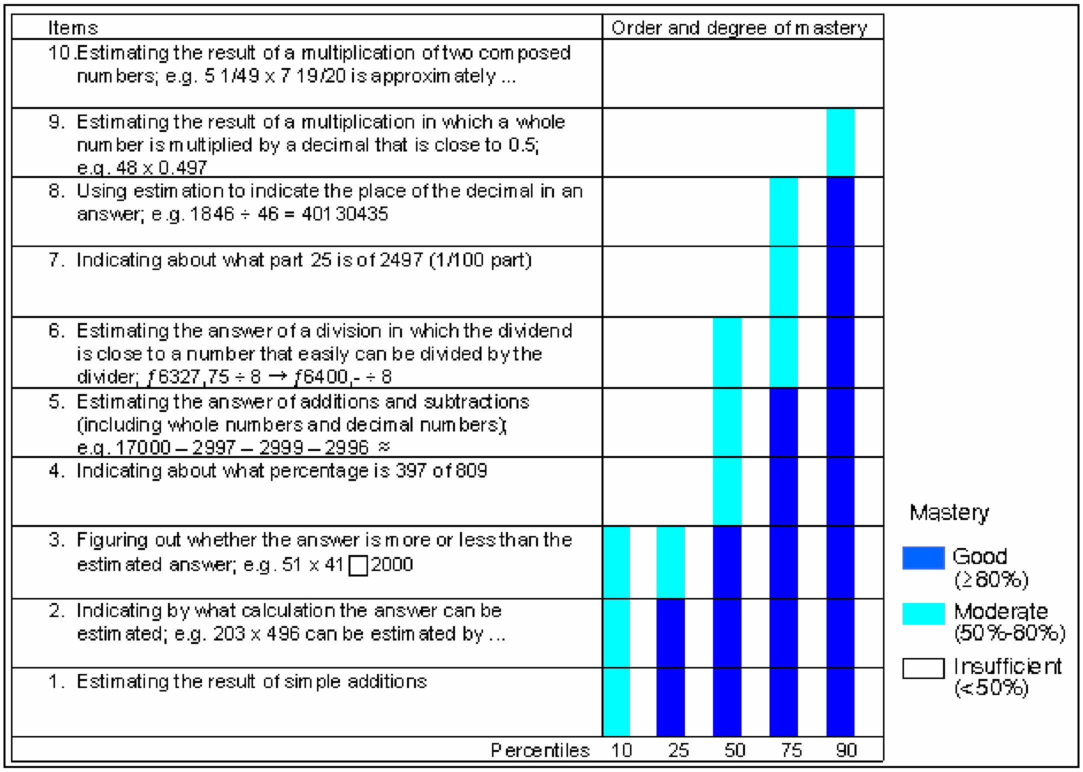


Figure 12: PPON 1997 results on the subdomain estimation

Figure 12 shows the results (taken from Janssen et al. (1999)) on estimation problems that were found at the time for sixth-grade students (twelve year old students) at different ability levels. Problems like problem 1 turned out to be most easiest ones, while problems like problem 10 were the most difficult. Even the 10% best students had an insufficient mastery of this problem, which means a chance of less than 50% to solve it correctly.

A closer look at these results reveals that some problems are identified here as easy, but actually refer to a category of problems that requires a high level understanding of estimation. This is especially true for the problems 2 and 3. Similar to the *Seats* problem, discussed earlier, it is not easy to understand whether 200 ? 500 gives a larger or a smaller result than the precise answer of 203 ? 496. In the same way problem 3 belongs to a more difficult category of problems than is indicated here. Changing the numbers a little bit? for instance, changing 51 ? 41 into 52 ? 38? would alter this problem into a tough problem.

At the other hand, problems like problem 9 might be less difficult if presented in context. Look, for instance, at the two problems in Figure 13.

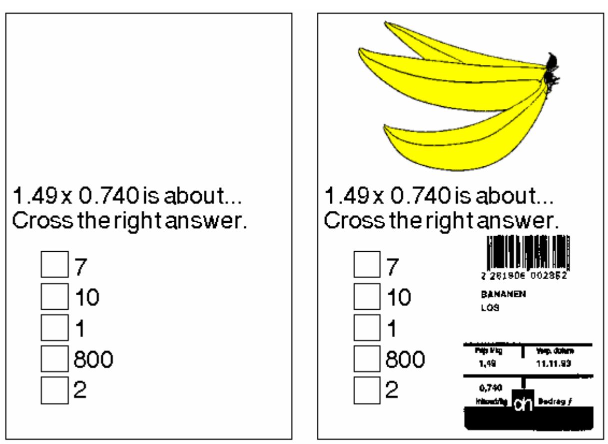


Figure 13: Two versions of the *1.49 ? 0.740* problem

In both versions the students have to give an estimate of 1.49 ? 0.740 which is even a more difficult problem than problem 9 that in the PPON study was administered to sixth-grade students. When I gave the two versions of the *1.49 ? 0.740* problem to a class of fifth-grade students the version with only numbers was answered correctly by only 4% out of the 29 students, wheras 46% of the students came with the correct answer to the context version of the problem (Van den Heuvel-Panhuizen, 1997).

7 Concluding remark

Answering the question about what mathematics should be taught to students in primary school is a crucial one, but not an easy one to answer. A critical reflection on the results from the PPON study made clear that it is not without risks to take empirically-based results from large-scale achievement studies as a criterion to determine what students can or should learn and in what order. The empirical evidence might give teachers a flawed picture of what is attainable and what is worthwhile to achieve (the hidden message that comes from what is assessed). Scores as such cannot be translated directly into a difficulty level for the problems, and scores on their own cannot serve as a guide for making decisions about the *what* of mathematics education. Didactical-phenomenological analyses as Freudenthal proposed may not be lacking here and can open our eyes to blind spots regarding the mathematics behind problems and the *what* of our curriculum. On the other hand, learning-teaching trajectories based on a didactical phenomenological approach, as they have been developed within the TAL project, do certainly not give the final answer. For the content domain of estimation we see this as a first long-term description that might guide teachers (and textbook authors and assessment developers) when they plan a short-term lesson sequence on this topic, but this trajectory description certainly needs continuous revision based on further didactical phenomenological analyses fed by empirical evidence.

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