Design research on how IT may support the development of symbols and meaning in mathematics education

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Introduction

This paper sketches the methodology and some results of three studies that were part of a comprehensive research project called "IT and mathematics education". With the theory of Realistic Mathematics Education as a background, the project investigated the contribution of IT tools for the guided reinvention of mathematical concepts and the influence of IT use on the students' process of symbolising. Each of the studies focussed on a different mathematical topic and used different types of technological tools. The three studies that are discussed in this paper combine the methodology of design research with a prominent role for the Hypothetical Learning Trajectory as a research instrument in all phases of design research (design, teaching experiment, retrospective analysis).
In the first section we state the overarching research question. In section 2 we describe the research methodology. Each of the phases of the research cycles is addressed: the preliminary phase in which a Hypothetical Learning Trajectory and instructional activities are designed, the teaching experiment phase and the phase of the retrospective analysis. In section 3 we address each of the three studies subsequently and present some exemplary results. Section 4 contains a discussion, and section 5 states common conclusions from the three studies.

1. Research Question

One of the salient characteristics of mathematics is the use of symbols. This is not merely an external characteristic, as mathematical symbols are an integral part of mathematics. It is hard to think about algebra without the use of literal expressions, or to understand calculus without pointing to tangents and areas beneath graphs, or to grasp the notion of a normal distribution in statistics without referring to an image such as a bell shape. This intertwinement of meaning and visual representations poses a problem for mathematics education. Experts - like teachers and instructional designers - tend to see these symbols as carriers of meaning. For them, those graphs and literal expressions are transparent; they can "see the mathematics through it", so to speak. The students, however, do not have the necessary mathematical background to interpret those symbolic representations in that manner. As a consequence, teachers will have to explain to the students what there is to see, and how to reason with those symbolic representations. This, Cobb et al. (1992) point out, leads to proceduralising and algorithmetising and the loss of meaning - or to, as van Oers (2000) calls it, "pseudo mathematics".
To find a way out of this dilemma, one may consider the history of mathematics to investigate how meaning and symbols emerged. It turns out that mathematical symbols did not arrive ready-made, with full-fletched meaning. Instead, one can discern a reflexive process in which symbolising and the development of meaning co-evolve (Meira, 1995). Symbolising, here, refers to inventing and using a series of symbols. In relation to this, Latour (1990) and others (e.g. Roth & McGinn, 1998) speak of a "cascade of inscriptions". This notion of a cascade of inscriptions has its counterpart in semiotic concepts as "chain of signification" (Walkerdine, 1988; Whitson, 1997).
The current challenge for mathematics educators is to develop mathematics education that is in line with these dynamic conceptions of symbolising and development of meaning. The task of researchers is to shed light on the key elements of this type of mathematics education.

In 1998 a group of projects was started around the premises that IT could be used to support the aforementioned type of mathematics education. Symbolic representations could be embedded in various IT tools such as computers and hand-held calculators. Furthermore, this type of learning environments might create opportunities for exploration and reflection. This group of projects was funded by the PROO (a branch of the Netherlands Organisation for Scientific Research, NWO) as a comprehensive research project entitled "IT and mathematics education" (grant number 575-36-003). This research encompassed five projects, three of which we will discus in this paper. These three projects focussed on the research question:

How can the use of IT in mathematics education help students reinvent mathematical concepts and representations?

This overarching question is elaborated on for three topics, the notion of distribution in statistics, calculus or, as Kaput calls it, "the mathematics of change and variation", and the concept of parameter in algebra. For each research project, this implies a dual goal:

- On the one hand, answering the question on how to shape the teaching and learning process on this specific topic while integrating IT in it, and
- on the other hand, investigating the role of IT, and, more specifically, the reflexive relations between IT-embedded symbol use and the development of meaning.

Given these goals, we chose what is called design research, or developmental research (Gravemeijer, 1994b, 1998), as our research method. Following Brown (1992), Cobb, Confrey, diSessa, Lehrer and Schauble (2003) refer to this type of research as design experiments, which they elucidate in the following manner:

> Prototypically, design experiments entail both "engineering" particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them. This designed context is subject to test and revision, and successive iterations that result play a role similar to that of systematic variation in experiment.
> (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003, p. 9)

In this description, the two aspects of our research goal come to the fore in (a) the design of means of support for particular forms of learning, and (b) the study of those forms of learning. In each of the research projects under discussion, the backbone of the design is formed by the design, development and revision of a hypothetical learning trajectory (a term that we will elaborate on later). The enactment of the instructional activities that are part of the hypothetical learning trajectory in teaching experiments creates the opportunity to study IT-embedded symbol use and the development of meaning in the context of the actual learning trajectory.

2. Design Research Methodology

The design research approach we follow in the three projects has a cyclic character: each design research study consists of research cycles in which thought experiments and teaching experiments alternate. We distinguish macro cycles that concern the global level of the teaching experiments, and micro cycles that concern the level of subsequent lessons. Gravemeijer argued that the micro cycles lead to a cumulative effect of small steps, in which teaching experiments provide "feed-forward" for the next thought experiments and teaching experiments (Gravemeijer, 1994a). A macro cycle of design research consists of three phases: the preliminary design phase, the teaching experiment phase, and the phase of retrospective analysis.

A second characteristic of design research is the importance of the development of a learning trajectory that is made tangible in instructional activities (Gravemeijer, 1994a). The design of instructional activities is more than a necessity for carrying out teaching experiments. The design process forces the researcher to make explicit choices, hypotheses and expectations that otherwise might have remained implicit. The development of the design also indicates how the emphasis within the theoretical development may shift and how the researcher's insights and hypotheses develop. As Edelson argues, design is a meaningful part of the research methodology:

> (...) design research explicitly exploits the design process as an opportunity to advance the researchers understanding of teaching, learning, and educational systems. Design research may still incorporate the same types of outcome-based evaluation that characterise traditional theory testing, however, it recognizes design as an important approach to research in its own right. (Edelson, 2002, p.107)

This is particularly the case when the theoretical framework involved is under construction.
One approach to design research is to immediately adapt the instructional activities after every lesson according to the experiences. This leads to many short micro cycles and can be an effective method, as shown by the following quotation that describes a situation where adaptations "on the fly" led to a micro cycle method:

> The time pressure was considerable, but it had one advantage: experiences in the classroom could be incorporated immediately. Consequently, practical experience assumed great significance as a 'feed forward'; the components of the curriculum, which were still to be developed, could be adjusted directly on the basis of classroom experiences. (Gravemeijer, 1994a, p. 36)

In these studies, however, we started the teaching experiments with a complete set of instructional activities. Due to practical circumstances in two of the projects, adaptations were limited to decisions on skipping tasks or stressing other assignments. Therefore, the cyclic character of the research primarily concerns the macro cycles.

2.1. Hypothetical learning trajectory

Within each macro level research cycle, we distinguish three phases: the preliminary design phase, the teaching experiment phase, and the phase of retrospective analysis. The first phase of preliminary design includes two related parts, the development of a Hypothetical Learning Trajectory (HLT) and the design of instructional activities. In the next four sections we elaborate on each of these (partial) phases. Of course, each phase was slightly different in the three projects; here we describe common aspects of each phase within the different projects.
The first phase of each research cycle includes the development of a "Hypothetical learning trajectory" – a term that is taken from Simon (1995). Originally, Simon used the HLT for designing and planning short teaching cycles of one or two lessons. In our study, however, we developed HLTs for teaching experiments that lasted for sequences varying from 8 to 20 lessons. As a consequence, the HLT comes close to the concept of a local instruction theory (Gravemeijer, 1994a). Also, this interpretation of the HLT is close to the scenario concept as it is used by Klaassen (1995). A second difference with Simon's approach is that Simon took a teacher's perspective, whereas we take a researcher's perspective.
The development of an HLT involves the choice or design of instructional activities in relation to the assessment of the starting level of understanding, the formulation of the end goal and the conjectured mental activities of the students. Essential in Simon's notion of a HLT is that it is hypothetical; when the instructional activities are acted out, the teacher – or researcher in our case – will be looking for evidence of whether these conjectures can be verified, or should be rejected.
For the design of the student activities, their motivation and the estimation of their mental effects, the designer makes full use of his domain specific knowledge, his repertoire of activities and representations, his teaching experience, and his view on the teaching and learning of the topic. After

a field test by means of a teaching experiment, the HLT will usually be adapted and changed. These changes, based on the experiences in the classroom, start a new round through the mathematical teaching cycle, and, in terms of the design research method, the next research cycle.
The concept of the HLT may seem to suggest that all students follow the same learning trajectory at the same speed. This is not how the HLT should be understood. Rather than a rigid structure, the HLT represents a learning route that is broader than one single track and has a particular bandwidth.
Because of its stress on the mental activities of the students and on the motivation of the expected results by the designer, the HLT concept is an adequate research instrument for monitoring the development of the designed instructional activities and the accompanying hypotheses within design research. It provides a means of capturing the researcher's thinking and its development throughout the research and helps in getting from problem analysis to design solution.

2.2. Design of instructional activities

The preliminary design phase of the design research cycles includes the development of the HLT and the instructional activities. Of course, the development of an HLT and the design of instructional activities are closely related: the HLT guides the design of the instructional activities, but choices made in the design process may lead to reconsideration of the HLT. The expectations of the students' mental activities established in the HLT are elaborated into specific key activities in the instructional materials. In this section, we describe our design method and principles.
The design of instructional activities in these studies included the development of student text booklets, teacher guides, solutions to the assignments, tests and software. While designing these materials, choices and intentions were captured and motivated, to inform the teacher and to keep track of the development of the designer's insights. When the materials were about to be finalised, these aims and expectations were described at the task level. Key items, that embodied the main phases in the HLT, were identified. These items reflected the relevant aspects of the intended learning process and were based on the conceptual analysis of the topic. The identification of key items guided observations and prepared for the retrospective data analysis. Finally, teacher guides as well as observation instructions were written, to make intentions and expectations clear to teachers and observers. During the design phase, products were presented to colleagues, teachers and observers. This led to feedback that forced the researcher to be explicit about goals and aims, and that provided opportunities for improving all the materials.
While designing instructional activities, the key question is what meaningful problems may foster students' cognitive development according to the goals of the HLT. Three design principles guided the design process: guided reinvention, didactical phenomenology and mediating models.
The design principle of *guided reinvention* involves reconstructing the natural way of developing a mathematical concept from a given problem situation. A method for this can be to try to think how you would approach a problem situation if it were new to you, or, as Gravemeijer phrases it, "think how you might have figured it out yourself" (Gravemeijer 1994, p. 179). In practice, this is not always easy to do, because as a domain expert it is hard to think as if you were a freshman. The history of the domain can be informative on specific difficulties concerning concept development (e.g. Bakker, 2003; Gravemeijer & Doorman, 1999).
The second design principle, *didactical phenomenology*, was developed by Freudenthal (Freudenthal, 1983; Gravemeijer, 1994a; Gravemeijer & Cobb, 2001). Gravemeijer explains it as follows:

> Didactical phenomenology points to applications as a possible source. Following on the idea that mathematics developed as increasing mathematisation of what where originally solutions to practical problems, it may be concluded that the starting points for the re-invention process can be found in current applications. (Gravemeijer 1994, p. 179)

Didactical phenomenology aims at confronting the students with phenomena that “beg to be organised” by means of mathematical structures. In that way, students are invited to build up mathematical concepts. Meaningful contexts, from real life or “experientially real” in another way, are sources for generating such phenomena (de Lange, Burrill, Romberg, & van Reeuwijk, 1993; Treffers, 1987). The question, therefore, is to find meaningful problem contexts that may foster the development of the targeted mathematical objects. The context should be perceived as natural and meaningful, and offer an orientation basis for mathematisation.
The last remark leads to the third design principle, the use of *mediating models* (Gravemeijer, 1994a). In the design phase we try to find problem situations that lead to models that initially represent the concrete problem situation, but in the meantime have the potential to develop into general models for an abstract world of mathematical objects and relations.

2.3. Teaching experiments

The second phase of the design research cycle is the phase of the teaching experiment, in which the prior expectations embedded in the HLT and the instructional activities are confronted with classroom reality. The term “teaching experiment” is borrowed from Steffe (Steffe & Thompson, 2000). The word “experiment” is not referring to an experimental group - control group design. In this section we explain how the teaching experiments were carried out; in particular, we pay attention to the data sampling techniques used during the teaching experiments.
The research questions share a process character: they concern the development of understanding of mathematical concepts. Therefore, we focussed on data that reflected the learning process and provided insight into the thinking of the students. The main sources of data, therefore, were observations of student behaviour and interviews with students. The observations took place on three levels: classroom level, group level and individual level. Observations at classroom level concerned classroom discussions, explanations and demonstrations that were audio and video taped. These plenary observations were completed by written data from students, such as handed in tasks and notebooks.
Observations at group level took place while the students were working on the instructional activities in pairs or small groups. Short interviews were held with pairs of students. In addition to this, the observers made field notes.
The lessons were evaluated with the teachers. These discussions led to consequences for the following lessons, as is characteristic with design research micro cycles. In particular, the organisation of the next lesson and the content of the plenary parts were discussed. Also, decisions were taken about skipping (parts of) tasks because of time pressure. Such decisions were written down in the teaching experiment logbook.

2.4. Retrospective analysis

The third phase of a design research cycle is the phase of retrospective analysis. It includes data analysis, reflection on the findings and the formulation of the feed-forward for the next research cycle. In this section we focus on the data analysis method.
The first step of the retrospective analysis concerned *elaborating on the data*. A selection from video and audiotapes was made by event sampling. Criteria for the selection were the relevance of the fragment for the research questions and for the HLT of the teaching experiment in particular. Data concerning key items was always selected and these selections were transcribed verbatim. From the videotaped work of pairs of students, only a limited part of the registered time was transcribed verbatim, as students often proceeded slowly while working on their own, and did not always explain their method. In such cases, the tapes were summarised into descriptions. Long plenary explanations and classroom discussions were summarised as well, unless they contained interesting student contributions. Registrations where students were not working at all, or where organisational matters were discussed, were not elaborated on. The written work from the students was surveyed and

analysed, especially the work on key items, tests and hand-in tasks. Results were summarised in partial analyses.
The first phase of the analysis consisted of *working through the protocols* with an open approach that was inspired by the constant comparative method (Glaser & Strauss, 1967; Strauss & Corbin, 1988). Remarkable events or trends were noted as conjectures and were confronted with the expectations based on the HLT and the instructional activities. The second phase of analysis concerned *looking for trends* by means of sorting events and analysing patterns. The findings were summarised illustrated by prototypical observations. These conjectures were tested by surveying the data to find counterexamples or other interpretations, and by data triangulation: we analysed the other data sources, and in particular the written student material, to find instances that confirmed or rejected the conjectures. Analysis of the written materials often evoked a reconsideration of the protocols. Analysis was continued in this way until saturation, which meant that no new elements were added to the analysis and no conclusions were subject to change.
The third phase in analysing the data was the *interpretation of the findings* and the comparison with the preliminary expectations of the HLT. Also, explanations for the differences between expectations and findings were developed. These conclusions and interpretations functioned as feed-forward for the formulation of new hypotheses for the next cycle in the research.

3. Exemplary cases from the three projects

3.1. A case from the statistics project

Background
The aim of this project, entitled “symbolising in statistics education”, was to develop an instruction theory for early statistics education, with a focus on the notion of distribution of univariate data in relation to sampling. Students tend to perceive a data set as a collection of *individual* data points (Hancock, Kaput, & Goldsmith, 1992), whereas statistical data analysis is mainly about describing and predicting *aggregate* features of data set. The end goal of the instructional unit was that students should come to view a data set as a whole and develop a notion of distribution as an object that can have different features. The notion of distribution is a key concept of statistics and it is intimately tied to the idea of *shape*. A common representation of the normal distribution, for instance, is the bell shape. Such shapes are, of course, not just visual images, but signify how the data are distributed in terms of frequency or density. This means that students need to develop the meaning of such distributions before they can interpret conventional shapes such as the bell shape as symbols with meaning.
The teaching experiment that we report on here was in an eighth-grade class (second class havo-vwo) with 30 students, and lasted for ten lessons of 50 minutes. The students had learned about means and bar graphs, but no other statistical concepts or graphs, before the start of the teaching experiment.

Design of the HLT
The representational backbone of the Hypothetical Learning Trajectory (HLT) was the series of graphs in Figure 1. Students started with value-bar graphs of minitool 1; they could hide the bars, so that only the endpoints remained. If these endpoints were dropped (mentally), they got the dot plot of minitool 2. In minitool 2, students can organise data sets into equal intervals (precursor to histogram), two equal groups (median), four equal groups (box plots), and so on. While solving statistical problems with these minitools, students reasoned with different aspects of distribution, such as mean, spread, majority and outliers. We then wanted students to reason with continuous shapes, as opposed to discrete dot plots. Our conjecture was that this transition from a discrete plurality of data values to a continuous entity of a distribution is important for fostering a notion of distribution as an object. Once shape would be an object that students could reason about, the next step was to distinguish different shapes that are precursors to different distributions (uniform, symmetrical, skewed).

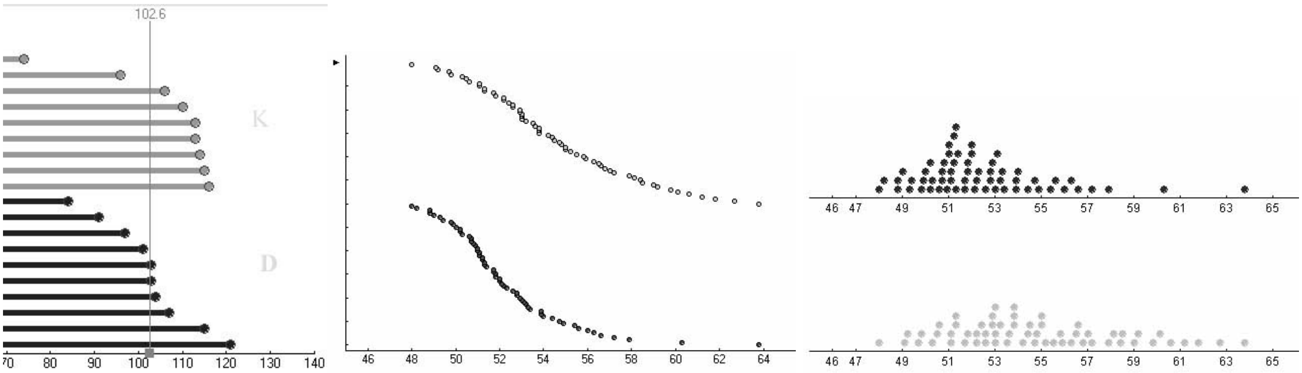

Figure 1. Minitool 1, with and without bars, and minitool 2 (the last two are the same data set).

Teaching experiment phase

One of the conjectures that emerged during an earlier cycle of design research was that experience with a so-called *growing sample* would help the students to reason about shapes of distributions, which is seen as a pre-stage to working with distribution as an object. The task of the students was to predict graphs of data of hypothetical situations with a particular sample size and compare their predictions with actual data. Then a larger sample had to be predicted and compared with actual data. In this way the shape of a distribution in relation to particular distribution aspects could become a topic of discussion.

In the first cycle of growing a sample, students had to predict weight data of ten eighth graders. We give examples of the work of three students that form a representative group of the class. Figure # shows the different graphs that were made: value-bar graphs, only the endpoints of the bars, and dot plots. These graphs correspond to the minitool graphs students had worked with in previous lessons (Figure 1). When comparing their own graphs with real data graphs, students often used informal attributes such as “together” and “spread out” to indicate range and spread.

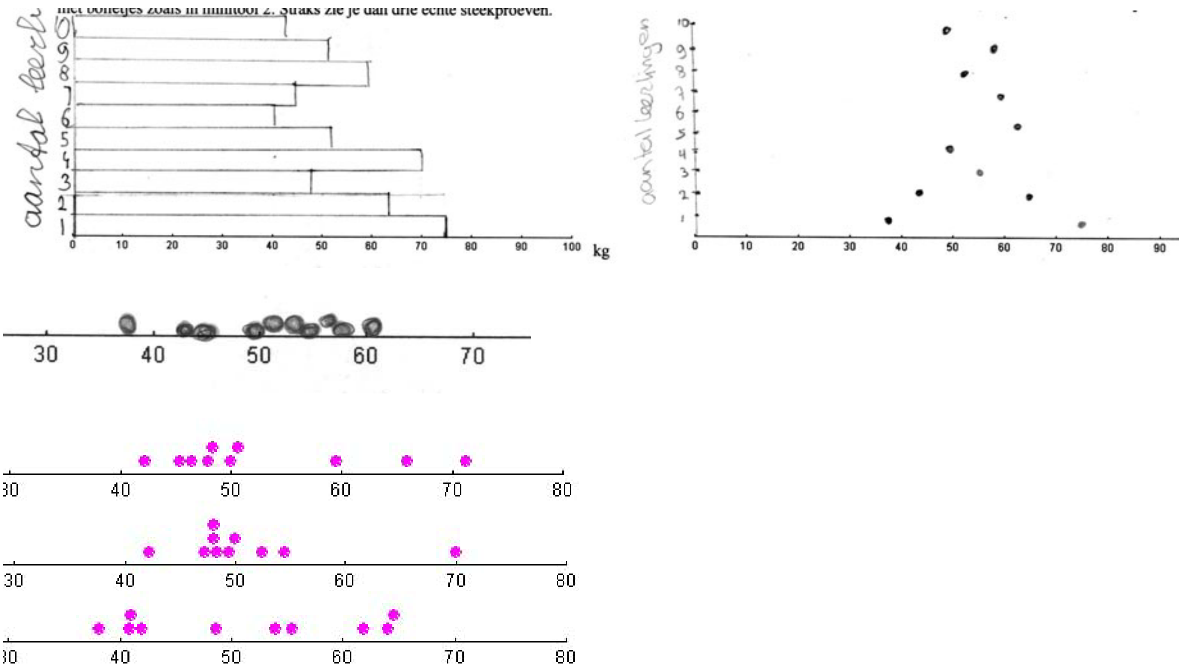

Figure 2. Students’ predictions of sample size 10, and three actual samples of 10.

Christel: The middle-most [graph with actual data] best resembles mine because the weights are closely together and that is also the case in my graph. It lies between 35 and 75 [kg].

Susan: The other [real data] are more weights together and mine are further apart.

In the second cycle of growing a sample, students had to predict data for a whole class and for three classes after the feedback of the samples of ten data points in dot plots.

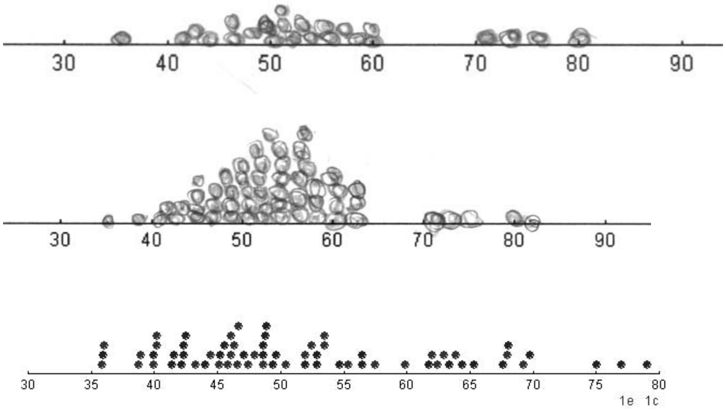

Figure 3. Susan’s predictions for one and three classes and an actual sample of three classes.

Rikkert: My spread is different.
Christel: Mine resembles the sample, but I have more people around a certain weight and I do not really have outliers because I have 10 about the 70 and 80 and the real sample has only 6 around the 70 and 80.
Susan: With the 27 there are outliers and there is spread; with the 67 there are more together and more around the average.

In contrast to the first cycle, students used nouns instead of just attributes for comparing the graphs. This might seem trivial, but statistically it makes a difference whether we say, “the dots are spread out” or “the spread is large”. Once spread is an object that can be talked about, it can be measured, for instance with range or interquartile range.

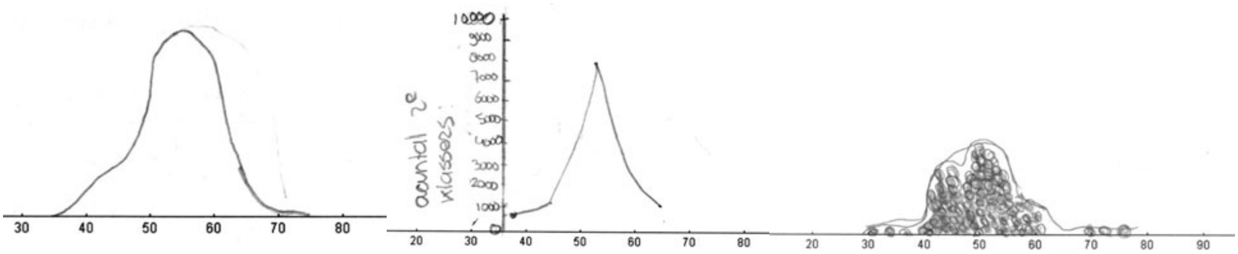

Figure 4. Students’ predictions of shapes in the third cycle of growing a sample.

In the third cycle of growing samples students made their first remarks about shape. The continuous shapes that they proposed were pyramid, semicircle and bell shape. Using these shapes in the sixth lesson, students came to reason about different aspects of distributions, including mean, outliers and skewness, in relation to sampling.

Retrospective analysis

From the analysis we concluded that students’ experience with the minitools supported their reasoning about growing samples and that this activity of growing samples in turn supported the development of a notion of distribution in relation to sampling. In the lessons after this fourth lesson, students could work with growing samples in the second minitool. We claim that the activity that students worked on in the fourth lesson fitted well in the HLT. One indication for this is that a considerable part of the HLT was retrieved in one lesson. In fact, Treffers and colleagues use this as a criterion for the quality of an activity in a learning trajectory that aims at guided reinvention.

> What becomes visible in the cross section of solutions of the complete group at a certain moment is the longitudinal section of the learning trajectory that has to be followed by each student. (Treffers, Streefland, & de Moor, 1994, p.147; translation from Dutch)

And indeed we encountered the whole representational backbone of the HLT in the fourth lesson:
1. Value-bar graph of minitool 1;
2. Only the endpoints of the bars;
3. Dot plot of minitool 2;
4. Dot plot with continuous shape on top of it;
5. Continuous shape.

From the analyses of students' reasoning with these graphs and shapes (symbols), we can infer that students did indeed start to develop meanings as intended. From a systematic analysis that was inspired by the constant comparative method of Glaser and Strauss (1967), we concluded that students in all seventh and eighth-grade classrooms tended to think of distributions as consisting of low, average, and high values. While working with diagrams both on paper and the computer screen, they started to develop a more subtle view on distribution.

The semiotic framework of diagrammatic reasoning (Peirce, 1976) helped in describing the process of meaning development in relation to statistical diagrams, in particular in the formation of mathematical objects. Not all of the students developed a notion of distribution as an object, but they did learn to reason about shapes in relation to frequency and density. Diagrammatic reasoning consists of constructing diagrams, experimenting with them, and reflecting on the results. It proved useful to let students invent their own diagrams and let them compare these with other students' diagrams. The minitools were used optimally during the experimentation phase, which formed the basis for reflection during class discussions.

The insights that have been developed in this project are of course domain-specific and situated in the Dutch context. In that sense, the theoretical contribution of such design research is "humble", as Cobb et al. phrase it (2003, p. 9). Yet, the fact that the HLT is theoretically and empirically grounded and is tailored to a specific domain makes it directly applicable to educational practice. Because the HLT has proved viable in different classrooms in different variants (also in the USA) we assume it can form the basis for a more general instruction theory on early statistics education.

3.2. A case from the calculus study

Background and design of HLT

The aim of this project, entitled "design research for the teaching and learning of calculus and kinematics with ICT", is to find out how students can learn the basic principles of calculus and kinematics by modelling motion. Many alternatives to the traditional transfer method stress the importance of modelling activities. In these alternatives two approaches are recognised. One uses simulations for exploratory modelling and the other uses (computer) tools for expressive modelling. Exploratory approaches stress the discovery of concepts while students work with formal representations in computer simulations. In the expressive approach, students' expressions during their modelling activities are the starting points for evolving concepts. This approach has many similarities with the importance of mediating models in realistic mathematics education. The expressive approach also fits the discussion about a dialectic relation between symbolising and development of understanding. With symbolising we refer to theories on how we create, use and adapt symbols and how this relates to concept development as presented in the previous statistics study. In this project we focus on the role of graphical representations in the process of learning the principles of calculus and kinematics.

Nowadays, graphs are used in calculus and kinematics education as representations for describing change of velocity or distance travelled during a time interval. Students are expected to give meaning to the relation between distance travelled and velocity through characteristics of these graphs such as area and slope. The use of such instructional materials is based on a representational view (Cobb et al., 1992), which assumes that instructional materials can represent scientific knowledge, and that scientific concepts can be made accessible without fully taking into account the limitations of the knowledgebase of the students into which they have to be integrated. Elements of this view can still be found in exploratory approaches.

Cobb et al. oppose this view. In line with their reasoning, we claim that symbolisations and knowledge of motion can co-evolve in a learning process. Theories on symbolising give rise to heuristics for designing a learning route within which the mathematical and scientific knowledge emerges from the activity of the students (Gravemeijer et al. 2000). In this route, the creation, use and adaptation of various graphical representations are interwoven with learners' activities in a series of science-practices, from modelling discrete measurements to reasoning with continuous models of motion. Mathematical and physical aspects are integrated in these activities. Our focus is on students' contributions during these practices. Consequently, for understanding their reasoning we use the design research approach of planning and testing the envisioned trajectory in classroom situations, a research approach that aims at creating an educational setting for investigating *how* a trajectory works, instead of trying to decide *whether* it works.

The learning route – inspired by the domain history - is tried out and revised during teaching experiments in three tenth-grade classes. We collected data by video and audio taping whole class discussions and group work. The videotapes were used to analyse students' discourses and students' written materials with respect to the conjectured teaching and learning process.

Teaching experiment phase

We illustrate the change in how students think and talk about a model with the following episode. The trajectory starts with questions about a weather forecast. The teacher discusses the change of position of a hurricane with students: when will it reach land? This problem is posed as a leading question throughout the unit as a context for the need of grasping change. After the emergence of time series as useful tools for describing change of position, students work with situations that are described by stroboscopic photographs. The idea is that students come up with measurements of displacements, and that it makes sense to display them graphically for finding and extrapolating patterns. Two types of discrete graphs are discussed: graphs of displacements (distances between successive positions) and graphs of the total distance travelled. Note that discrete graphs are not introduced as an arbitrary symbol system, but emerge as models of discrete approximations of a motion, that link up with prior activities and students' experiences. By using the computer program Flash students are able to investigate many situations. During these activities their attention shifts from describing specific situations to properties of these discrete graphs and the relation with kinematical concepts.

Our findings confirm such a change in reasoning. In the beginning students refer to distances between successive positions. After a while they reason using the global shapes and properties of graphs and motion. An example of such reasoning concerns an exercise about a zebra that is running at constant speed and a cheetah that starts hunting the zebra. The question is whether the cheetah will catch up with the zebra. In the graphs the successive measurements of the zebra and the cheetah are displayed. The following discussion takes place between an observer and two students (Rob and Anna).

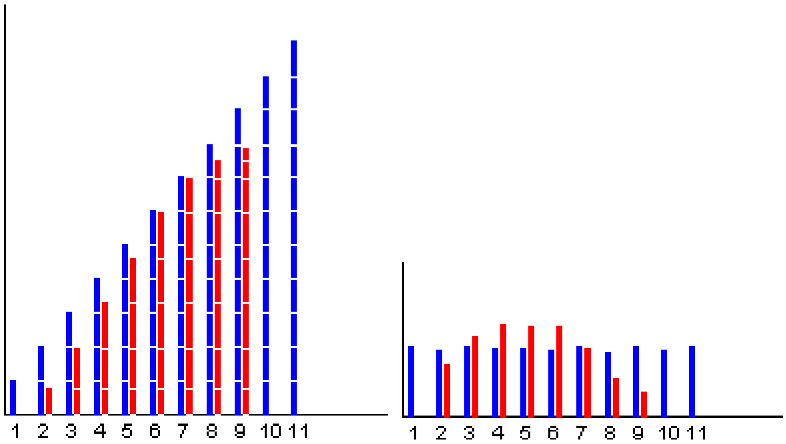

Figure 5. Distance-travelled and displacement graphs in Flash.

Observer:	Oh yes. So why did you choose the one for the total distance [left graph in Fig. 5]?
Rob:	Because it's the total distance that they cover and then you can-
Anna:	Then you can see if they catch up with each other.
Observer:	And can't you see that in the other [right graph]? There you can also see that the red [grey] catches up with blue [dark grey]?
Rob:	Yes, but -
Anna:	Yes, but that's at one moment. That only means that it's going faster at that moment but not that it'll catch up with the zebra.

Retrospective analysis

A difference between the displacements graph and the distance-travelled graph is the difference between the interpretations of the horizontal (time) axis. A value in the distance-travelled graph represents a distance from the start until the corresponding time, while a value in the displacements graph represents a distance in the corresponding time interval. Anna's last observation is an important step in the process of building the model of a velocity-time graph (and everything that comes with it).
The qualitative analyses show that during the practices students re-invent and develop graphical symbolisations, as well as the language and the scientific concepts that come with them. However, these inventions only became explicit after interventions by an observer or by the teacher. Additionally, we found that the teacher had a crucial role during the classroom discussions. It was not always easy to organise the discussions in line with the intended process. Sometimes the teacher reacted to students' contributions in terms of the inscriptions or concepts aimed at. In those cases students awaited further explanation. The discussions appeared to be especially productive when the teacher organised classroom discussions about students' contributions in such a way that the students themselves posed the problems that had to be solved, and reflected on their answers. In a second teaching experiment we arranged a setting where the teacher had more information about the possible contributions of the students and the way in which they could be organised. Additionally, we designed activities for classroom discussions. In this experiment we found that this resulted in discussions where students anticipated on the intended concepts and reflected on their contributions.

In this approach the construction and interpretation of graphs and the scientific concepts are rooted in the activities of the students through emerging models. This ensures that the mathematical and physical concepts aimed at are firmly rooted in students' understanding of everyday phenomena. On the basis of our findings we conclude that classroom discussions where students discuss their solutions and pose new problems to be solved, are essential for a learning process during which symbolisations and knowledge of motion co-evolve.

3.3. A case from the algebra study

Background

The aim of the third project, entitled 'Learning algebra in a computer algebra environment', was to investigate the contribution of computer algebra use to the development of algebraic concepts and techniques. A computer algebra system (CAS) is a software package, available on both PCs and hand-held devices, which is able to carry out algebraic procedures such as solving equations, substituting expressions and simplifying formulas. The availability of such powerful tools for algebra raises the question of how this kind of technology can be integrated into algebra education and how this integration affects the role of by-hand algebra skills. What is the relation between the work in the computer algebra environment, the paper-and-pencil work, and the students' mental understanding?

One perspective for investigating this question is the instrumental approach to using technology. By instrumentation we mean the process of developing techniques in the technological environment (Artigue, 2002). Such a technique is built up by the students for solving a specific type of task, and is condensed into so-called instrumentation schemes. An instrumentation scheme includes technical knowledge and skills for manipulating a technological tool, in this case the computer algebra environment, as well as conceptual knowledge that gives meaning to these technical manipulations. This intertwinement of technical aspects related to the tool, and mental aspects related to the students' conceptual understanding makes the instrumentation perspective relevant for our purpose.

Our claim here is that this instrumental approach allows for understanding the interaction between student and technological tool, and in particular the difficulties that are encountered (Drijvers, 2000). Furthermore, we argue that a carefully orchestrated instrumentation of the technological tool, both by the instructional activities and the teacher, may improve conceptual understanding, which also affects paper-and-pencil techniques.

The teaching experiments in which we found evidence for these claims took place in ninth- and tenth-grade classes of high achieving students (in the Dutch school system: vwo3 and vwo4) and lasted 15 – 20 lessons each. Before the experiments, the students had some experience with algebra, but little experience in using technology for learning mathematics. During the teaching experiments, they used the TI-89 symbolic calculator as computer algebra environment. We illustrate our claims with an example on substituting expressions.

Design of HLT

The pre-test of one of the teaching experiments revealed that hardly any of the students were able replace y in the equation $x^2 + y^2 = 10$ by $a - x$, in other words to substitute $y = a - x$ into the equation $x^2 + y^2 = 10$.

During the teaching experiment the students carried out such substitutions using the computer algebra device. Fig. # shows the syntax of the substitution using the vertical substitution bar. Entering *expression1* | *variable* = *expression2*
results in replacing all instances of *variable* in *expression1* by *expression2*.

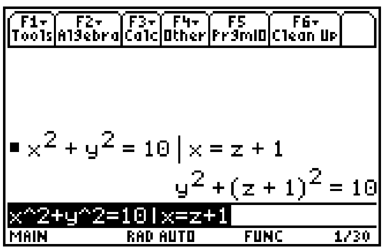

Figure 6. Substituting an expression in the CAS

While carrying out such substitutions, the students had to build up the substitute instrumentation scheme. This scheme involved:

1. Noticing that a substitution is required to obtain a result;
2. Remembering the way substitution is carried out in the computer algebra environment: the meaning of the vertical bar symbol and the syntax of the command;
3. Realising which expressions play the roles of *expression1* and *expression2*, and in particular considering expression2 as an object instead of a process;
4. Being able to type in the command correctly;
5. Accepting the resulting expression as a satisfying answer in spite of its "lack-of-closure" (see below).

Steps 2 and 4 of this scheme have a primarily technical character, whereas the other steps are mainly conceptual. One of the first examples of substitution concerned substituting the expression for the area of the bottom of a cylinder into the volume function: $V = A*H \mid A = \pi*R^2$. The fact that the problem situation was meaningful to the students helped them to understand the substitution. Meanwhile, the fact that the result, $V = \pi*H*R^2$, still contained multiplication symbols was not satisfying to some of the students, as they wanted to be able to carry out the multiplications to get a numerical answer. This is the so-called lack-of-closure obstacle that was referred to in step 5 of the scheme (Tall & Thomas, 1991).

Teaching experiment phase

By means of practising substitution in the CAS, the students got used to expressions as answers and seemed to consider expressions as sensible results. However, their notion of substitution often was somewhat vague, as the following observation indicates:

Observer: And what happens now for example in $V = A*H$ wherein $A = \pi*R^2$?
Maria: Well that it just makes one formula out of it.
Observer: Yes, and how does it do that then?
Maria: By means of joining them or something?

Later in the teaching experiment, students tried to substitute non-isolated forms, such as for example $x^2 + y^2 = 10 \mid x + y = a$. This error was not only a syntactical matter, but also revealed an imprecise concept of substitution in step 3 of the scheme. In the teaching, a proper view on substitution was supported by means of visualisation:

$$x^2 + y^2 = 10 \mid x = z + 1$$

The instrumentation of substitution within the computer algebra environment led to an improved conceptual insight on the issues of steps 3 and 5 of the scheme. This was confirmed by data from the post-experiment interviews, during which the computer algebra device was no longer available to the students. Most students showed to be able to substitute $y = a - x$ into the equation $x^2 + y^2 = 10$. As fig. # shows, one of them used the vertical bar to indicate the substitution, which clearly was inspired the notation in the computer algebra environment. He wrote down $x^2 + (a - x)^2 = 10$ and explained his result by saying: "Here is already a y and here it says what y is, so you just have to replace this with that, I was just thinking too difficultly".

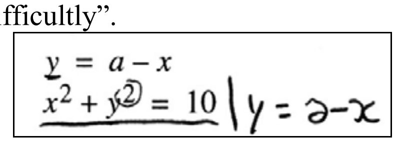

Figure 7. Transfer of notation

Retrospective analysis
Looking back at the results of this study, we conclude that the perspective of instrumentation was helpful for understanding the interaction between student and technological tool and for studying the effect of computer algebra use on conceptual development. The question is how these findings link up with theories on symbolising, which stress the signification process in which the student develops his own symbolisations, and improves and uses them in relation to the meaning that he attaches to them (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Gravemeijer et al., 2000). Our study did not show much of this bottom-up process, because the computer algebra environment already offers a symbol system and is not very flexible concerning notation and syntax. However, the confrontation with a ready-made symbol system also involves the process of developing meaning while using the system. This links up with the technical and conceptual aspects of instrumentation. In that sense, theories on symbolisation provided a background for understanding the growing sense for the symbols within the computer algebra environment. Furthermore, the use of realistic problem situations such as the cylinder task provided a reference to a meaningful situation for the students. The use of the technological tool did not make such a reference redundant and in the meantime did not hinder generalisation and abstraction.

4. Discussion

In the statistics project, different instances of diagrammatic reasoning have been analysed. It proved useful to let students construct their own diagrams once they had some experience with specially designed software, and reflect on the diagrams they had made in relation to the problem that had to be solved. The alternation of providing carefully designed software and letting students reinvent their own graphs and strategies proved useful. Looking back we notice that the growing samples activity shows a subtle mix of providing and designing (cf. van Dijk et al., 2003). The students had already experience with minitool graphs there were provided, but in the fourth lesson they had to design their own graphs for new situations (very large samples). After the first cycle of growing samples, students received feedback with dot plots. In other words, there was opportunity for a reinvention process, but this was guided carefully. This means that the answer that our design research gives to such questions is subtle and situated, but not easy to generalise.
As intended in the HLT, students did indeed learn to reason about distributions and sampling in a meaningful way in relation to statistical diagrams. This learning process was analysed semiotically and, as expected, the relationship between symbolising and the development of meaning and mathematical objects could be characterised as reflexive.
Also, in the calculus project it appeared that symbols and reasoning could develop in a reflexive process. The IT-tools afforded students many situations with presented representations to analyse. During these analyses the character of the representations changed from situation-close to organising tools for modelling motion. In this way, through emerging models construction and interpretation of graphs and scientific concepts became rooted in the activities of the students. Moreover, we found that classroom discussions where students discuss their solutions and pose new problems to be solved are essential for a learning process during which symbolisations and knowledge of motion co-evolve.

The project on learning algebra in a computer algebra environment showed that the technical skills using the computer algebra device required, were closely related to a conceptual understanding of the mathematics behind the technique. Developing computer algebra techniques and instrumentation schemes proved to be difficult for the students. Overcoming these difficulties often involved a conceptual development that was relevant for the targeted insight. The joint development of technical skills and conceptual understanding is seen as support for the instrumental approach to using IT tools.

5. Conclusions

On the issue of the research question from paragraph 1, we notice an agreement on the tension between the bottom-up reinvention process and the top-down character of IT use. Moreover, we notice an agreement on the importance of the perspective of symbolising, though in a different way for each of the studies. On the issue of the methodology, we conclude that the notion of Hypothetical Learning Trajectory was helpful in the preliminary phase of each of the research cycles, and that gathering data by means of dedicated mini-interviews with students was a fruitful strategy. During the teaching experiments and retrospective analyses the HLT also served in guiding the observations and analysis. We now briefly elaborate on each of these conclusions.

In each of the studies we noticed a *tension* between the targeted bottom-up reinvention process and the top-down character of IT. The term guided reinvention is used for a process in which students experience learning mathematics as if they could have invented it themselves, while being supported by the teacher and instructional materials (including the IT-tools). The IT-tools intrinsically have a top-down character because they offer a restricted repertoire of representations and. For computer algebra this top-down character is the strongest, because the large variety of possibilities is combined with a restricted flexibility in symbols, notation and syntax. Somehow this disadvantage of IT-tools has to be compensated by instruction; reflection, interaction, carefully designed problems, and so on. It is important that the teacher not just compares different diagrams and solutions with the students, but also stimulates students to formulate which problems they solve, why certain solutions work and in which direction future activities have to take place.

The focus on *symbolising* proved viable. In the projects on statistics and calculus, using semiotic and perception theories turned out useful for analysing the relationship between symbolising and development of meaning. We assume on the basis of these projects and prior research that a carefully designed trajectory of symbol and meaning development is necessary to give students the opportunities to learn mathematics. In that process, students need to get ample opportunities for their own constructions and reflection on them. In the algebra project, this issue was different, because the computer algebra systems are ready-made symbol systems that are hard to change. As a consequence, the focus was not on the development of increasingly sophisticated symbols, but on the development of meaning within an existing symbol system. Realistic contexts proved important in that. The approach of instrumentation, which considers dealing with a ready-made symbol system in relation to conceptual development, can be seen as a specification of the symbolising issue in the case of ready-made systems.

With respect to the methodology of design research, we consider the *Hypothetical Learning Trajectory* a useful instrument in all phases of design research. During the design phase it is the theoretically grounded vision of the learning process, which is specified for concrete instructional activities. During the teaching experiments, the HLT guides observations, data collections, which questions have to be asked during mini-interviews, and offers a framework for educational decisions during the teaching experiment. The mini-interviews proved useful for getting detailed information about what symbols meant for students and how they reasoned with these symbols. In the retrospective analysis phase, the HLT served as a guideline for data selection and offered conjectures that could be tested during the analysis.

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