Thinking skills in realistic mathematics

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Introduction

One of the most enduring ideas concerning mathematics instruction is the following: mathematics consists of a set of indisputable rules and knowledge; this knowledge has a fixed structure and can be acquired by frequent repetition and memorization. In the past twenty-five years, far-reaching changes have taken place in mathematics instruction. More than in any other field, such changes were influenced by mathematicians who had come to view their discipline in a different light. Their observations went a long way towards stimulating a process of renewal in mathematics instruction. New consideration was given to such fundamental questions as: how might mathematics best be taught, how might children be encouraged to show more interest for mathematics, how do children actually learn mathematics, and what is the value of mathematics?

According to Goffree, Freudenthal, and Schoemaker (1981), the subject of mathematics is itself an essential element in 'thinking' through didactical considerations in mathematics instruction. Moreover, the notion is emphasized that knowledge is the result of a learner's activity and efforts, rather than of the more or less passive reception of information. Mathematics is learned, so to say, on one's own authority. From a teacher's point of view there is a sharp distinction made between teaching and training. To know mathematics is to know why one operates in specific ways and not in others. This view on mathematics education is the basic philosophy in this chapter (Von Glazersfeld, 1991) In order to understand current trends in mathematics education, we must consider briefly the changing views on this subject.

The philosophy of science distinguishes three theories of knowledge. Confrey (1981) calls these absolutism, progressive absolutism and conceptual change. In absolutism, the growth of knowledge is seen as an accumulation, a cumulation of objective and empirically determined factual material. According to progressive absolutism a new theory may correct, absorb, and even surpass an older one. Proponents of the idea of conceptual change have defended the point of view that the growth of knowledge is characterized by fundamental (paradigmatical) changes and not by the attempt to discover absolute truths. One theory may have greater force and present a more powerful argument than another, but there are no objective, ultimate criteria for deciding that one theory is incontrovertibly more valid than another (Lakatos, 1976). Mathematics has long been considered an absolutist science. According to Confrey (1981), it is seen as the epitome of certainty, immutable truths and irrefutable methods. Once gained, mathematical knowledge lasts unto eternity; it is discovered by bright scholars who never seem to disagree, and once discovered, becomes part of the existing knowledge base.

Leading mathematicians however have now abandoned the static and absolutist theory of mathematics (Whitney, 1985). Russell (in Bishop, 1988) once explained that mathematics is the subject in which we never know what we are talking about, nor whether what we are saying is true. Today mathematics is more likely to be seen as a fluctuating product of human activity and not as a type of finished structure (Freudenthal, 1983). Mathematics instruction should reveal how historical discoveries were made. It was not (and indeed is still not) the case that the practice of mathematics consists of detecting an existing system, but rather of creating and discovering new ones. This evolving theory of mathematics also led to new ideas concerning mathematics instruction. If the essence of mathematics were irrefutable knowledge and ready-made procedures, then the primary goal of education would naturally be that children mastered this knowledge and these procedures as thoroughly as possible. In this view, the practice of mathematics consists merely of carefully and correctly applying the acquired knowledge If, however, mathematicians are seen as investigators and detectives, who analyse their own and others' work critically, who formulate hypotheses, and who are human and therefore fallible, then mathematics instruction is placed in an entirely different light. Mathematics instruction means more than acquainting children with mathematical content, but also teaching them how mathematicians work, which methods they use and how they think. For this reason, children are allowed to think for themselves and perform their own detective work, are allowed to make errors because they can learn by their mistakes, are allowed to develop their own approach, and learn how to defend it but also to improve it whenever necessary. This all means that students learn to think about their own mathematical thinking, their strategies, their mental operations and their solutions.

Mathematics is often seen as a school subject concerned exclusively with abstract and formal knowledge. According to this view, mathematical abstractions must be taught by making them more concrete. This view has been opposed by Freudenthal (1983) among others. In his opinion, we discover mathematics by observing the concrete phenomena all around us. That is why we should base teaching on the concrete phenomena in a world familiar to children. These phenomena require the use of certain classification techniques, such as diagrams and models (for example, the number line or the abacus). We should therefore avoid confronting children with formal mathematical formulas which will only serve to discourage them, but rather base instruction on rich mathematical structures, as Freudenthal calls them, which the child will be able to recognize from its own environment. In this way mathematics becomes meaningful for children and also makes clear that children learn mathematics not by training formulas but by reflecting on their own experiences.

In the 1970s, the new view of mathematics, often referred to as mathematics as human activity, led to the rise of a new theory of mathematics instruction, usually given the designation: realistic. As it now appears, this theory is promising, but it is not the only theoretical approach in mathematics instruction; three others can be distinguished: the mechanistic, the structuralist and the empirical (Treffers, 1991). In the following we just give a brief characteristic of each approach, because it is beyond the scope of this chapter to discuss the three schools in extension:

- The mechanistic approach reflects many of the principles of the behaviouristic theory of learning; the use of repetition, exercises, mnemonics, and association comes to mind. The teacher plays a strong, central role and interaction is not seen as an essential element of the learning process. On the contrary, mathematics class focuses on conclusive standard procedure.
- According to the second approach − the structuralist − thinking is not based on the children's experiences or on contexts, but rather on given mathematical structures. The structuralist tends to emphasize strongly the teacher's role in the process of learning.
- The outstanding feature of the third trend − the empirical − is the idea that instruction should relate to a child's experiences and interests. Instruction must be child-oriented. Empiricists believe that environmental factors form the most important impetus for cognitive development (Papert, 1980). Empiricists emphasize spontaneous actions.

Realistic mathematics instruction as progressive mathematization

In this section we present five features which characterize realistic mathematics. At first we are dealing with learning in a context and second with the use of models. The third point (the mathematical subjects are not atomized but interwoven) is not of so much relevance for this book, while the three characteristics of the process of mathematization (construction, reflection, and interaction) are analysed in the following sections.

The new realistic approach to learning and thought process in children has far-reaching consequences. Mathematization is viewed as a constructive, interactive and reflective activity. To begin, the point of departure for education is not learning rules and formulas, but rather working with contexts. A context is a situation which appeals to children and which they can recognize in theory. This situation might be either fictional or real, and forces children to call upon the knowledge they have gained by experience − for example in the form of their own informal working methods − thereby making learning a meaningful activity for them, A well-chosen context can induce an active thought process in children, as the following example shows.

Let us start to give children of, say, 11 years the following formal and bare problem, not presented in a context: $6 \div 3/4$. Many of them will have a great deal of trouble finding a solution (Streefland, 1991). Some will answer, for example: $2/4$, $3/24$ or $4\ 1/2$. They manipulate at random with the given numbers, for instance $6 \div 3 = 2$, so $6 \div 3/4$ must be $2/4$. This child views fractions as whole numbers and so do other students (Lesh et al., 1987). But some students will calculate that $6 \times 4 = 24$ and that 24 divided by 3 equals 8. It is true that the latter answer is correct, but when these children are questioned more closely, it turns out that they understand almost nothing about the operation which they themselves have just performed. They just remembered a rule they learned by heart, they know that the given solution is correct however they don't know why.

Now, the same children are next given the following context problem which is accompanied by a picture: a patio is 6 metres long; you want to put down new bricks and the bricks you are going to use measure 75 centimetres in length ($3/4$ of a metre). How many bricks will you need for the length? This problem is the same as the previous one, but it has now been presented within a context, a picture of a patio and the bricks to put down. This presentation elicits a child's own, informal approach: measuring out. This approach provides insight into the problem, something which the symbolic form ($6 \div 3/4$) did not do. Some students even manipulated and took the measure in reality, this means they measured out step by step 75 centimetres and after 8 steps they counted 6 metres. So the answer must be 'eight', they concluded. This example demonstrates that working with contexts − which, if carefully constructed, can be considered paradigmatic examples − form the basis for subsequent abstractions and for conceptualization. That is because thinking must achieve a higher, abstract level and at that level these particular contexts no longer serve a purpose. That is not to say that a process of decontextualisation occurs, but rather recontextualisation. The children continue to work with contexts, but these contexts become increasingly formal in nature; they become mathematical contexts. Their connection with the original context, however, remains clear. The process by which mathematical thinking becomes increasingly formal is called the process of progressive mathematization. Contexts, thus, have various functions. They may refer to all kind of situations and to fantasy situations (Van den Heuvel-Panhuizen, 1996). It is important that the context offer support for motivation as well as reflection. A context should indicate certain relevant actions (to take measures in the example above), provide information which can be used to find a solution-strategy and/or a thinking-model.

Of course, leaving the construction to the students does not guarantee the development of successful strategies. However it guarantees that students get the opportunity to practice mathematician's thinking and problem solving processes. Strategies are tried, tested and elaborated in various situations.

In the previous discussion we have not argued that a student presented with 'bare' numerical tasks (like $6 \div 3/4$) will necessarily fail to solve the problem. Hence we were not suggesting either that students who are given context problems will necessarily produce the right solution. In recent research there is found a strong tendency of children to react to context problems ('word problems') with disregard for the reality of the situations of these problems. Let us give two examples of items used in research (Greer, 1997; Verschaffel et al., 1997):

- 'An athlete's best time to run a mile is 4 minutes and 7 seconds. About how long would it take him to run 3 miles?'
- 'Steve has bought 4 planks of 2.5 metre each. How many planks of 1 metre can he get out of these planks?'

In four studies, discussed by Greer (1997), the percentage of the number of students demonstrating any indication of taking account of realistic constraints is: 6%, 2%, 0% and 3%. The student's predominating tendency to apply rules clearly formed an impediment to thoroughly understanding the situation.

Verschaffel et al. (1997) confronted a group of 332 students (teachers in training) with word problems and found they produced 'realistic' responses in only 48% of cases. Moreover the pre-service teachers considered these 'complex and tricky word problems' as inappropriate for (fifth grade) children. The goal of teaching word problem solving in elementary school, after their opinion, was "...learning to find the correct numerical answer to such a problem by perforn1ing the formal-arithmetic operation(s) 'hidden' in the problem" (Verschaffel et al., 1997, p. 357).

When solving word problems students should go beyond rote learning and mechanical exercises to apply their knowledge (Wyndhamn & Säljö, 1997). Their research showed that students (10-12 years of age) gave in most cases logically inconsistent answers. The authors interprete these findings by claiming that the students focus on the syntax of the problem rather than on the meaning. That means that the well-known rule-based relationship between symbols results in less of attention being paid to the meaning. The students follow another 'rationality', that is, they consider word problems as mathematical exercises "… in which a algorithm is hidden and is supposed to be identified." (Wyndhamm & Säljö, p. 366). Hence they do not know or realize that they are expected to solve a real life problem.

Reusser and Stebler (1997) discuss another interesting research finding namely the fact that pupils 'solved' unsolvable problems without 'realistic reactions'. For example:

- 'There are 125 sheep and 5 dogs in a flock. How old is the shephard?' (Greer, 1997).

A pupil questioned by the investigators gave as his opinion: 'It would never have crossed my mind to ask whether this task can be solved at all'. And another pupil said: 'Mathematical tasks can always be solved'. One of the author's conclusions is that a change is needed from stereotyped and semantically poor, disguised equations to the design of intellectually more challenging 'thinking stories'. What we need are better problems and better contexts. Finally, Reusser and Stebler (1997) − following Gravemeijer (1997) − give as their interpretation of the research findings that the children are acting in accordance with a typical school mathematics classroom culture.

Second, the process of mathematization is characterized by the use of models. Some examples are schemata, tables, diagrams, and visualizations. Searching for models − initially simple ones − and working with them produces the first abstractions. Children furthermore learn to apply reduction and schematization, leading to a higher level of formalization. We will demonstrate, once again this using the previous example. To begin, children are able to solve the brick problem by manipulating concrete materials. For instance, they might attempt to see how often a strip of paper measuring 3/4 of a metre fits in a 6-metre-long space. At the schematic level, they visualize the 6-metre-long patio and draw lines which mark out each 3/4 of a metre or 75 centimetres. The child adds 75 + 75 + 75... until the 6 metres have been filled The visualization looks as follows:

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3/4 3/4 3/4 3/4 3/4 3/4 3/4 3/4

An example of reasoning on a formal-symbolic level is as follows: 75 centimetres fits into 3 metres 4 times. We have 6 metres, so we need 2 × 4 = 8 bricks. The formula initially tested can also be applied, but this time with insight: 1/4 metre fits 4 times into 1 metre, so it fits 24 times into 6 metres. But I only have 3/4 of a metre, so I have to divide 24 by 3, and that makes 8. At this formal level, moreover, the teacher can also explore the advantages and disadvantages of the two methods with the children.

Third, an important element of realistic mathematics instruction is that subjects and curricula (such as fractions, measurement and proportion) are interwoven and connected, whereas in the past, the subject matter was divided − and so atomized.

Fourth, two other important characteristics of the process of mathematization are that it is brought about both by a child's own constructive action and by the child's reflections upon this action.

Finally, learning mathematics is not an individual, solitary activity, but rather an interactive one.

Construction

Learning mathematics is a constructive activity, an aspect which has been emphasized by many authors (including Bruner, 1986, 1996; Cobb, 1994; Cobb et al., 1997; Resnick & Klopfer, 1989; Steffe, Cobb, & Von Glazersfield, 1988). Children construct internal, mental representations. These might be concrete images, schemata, procedures, working methods at the abstract-symbolic level, intuitions, contexts, schemata of solutions, or thought experiments. To make clear how individuals construct different kinds of representations we present now a example (this example is above the cognitive level of school children).

Suppose we were able to tighten a rope around the equator so that it is taught and lies flat on the surface. We cut the rope and insert a metre-long piece of rope between the two cut ends. Once again we tighten the rope so that it is taught all around. The question is: How far above the ground is the rope now? The mental representations of many adults will contain various elements. To start, they will ask themselves what the circumference of the earth is: 44,000 kilometres. The rope − they can picture it before their very eyes − must therefore measure the same in length. Another metre − they reason to themselves − scarcely matters in proportion to that enormous distance. Probably the rope barely lifts off the ground. These mental representations actually consist of concrete images which form a basis for solving the problem, conceived of as the relationship between that one metre and the entire circumference of the earth. A mathematically trained problem-solver will construct entirely different representations. He or she will immediately dismiss any concrete facts and reduce the earth to a circle, focusing in on the relationship between the radius and the circumference. This relationship is then converted into a formula, $2\pi R$. The representation is created by converting a concrete problem into an adequate mathematical formula. The rope lies about 16 centimetres above the surface of the earth.

Learning mathematics as a constructive activity means that a child's own discoveries are taken seriously. This does not mean that their discoveries are always on the mark, but they do give the teacher a recognizable handle from which he or she can begin to teach. The teacher learns the general outlines of the representations of children and can adjust his approach accordingly. But what is the function of representations in mathematics instruction? The representational point of departure and the 'representational view of mind' seems to require some constructivist comment. What is criticized and rejected is the metaphor of the mind as a mirror that reflects a mathematically prestructured environment unaffected by individual and collective human activity (Von Glazersfeld, 1991). Correct, internal representations are constructed by confronting them with external representations. These are socially and culturally determined (Cobb, 199). Children do indeed actively construct their own mathematical knowledge, but their purpose is to participate increasingly in taken-as-shared mathematical practices. These practices are played out both in the classroom as in society and science.

If rules and procedures are prescribed prematurely and one-sided, blocking a child's own representations, problems will ensue. The following recorded fragment of conversation serves as a concrete illustration. Henry, a good pupil of 9 years of age, is busy working out subtraction problems in his mathematics workbook. The book gives the following formula to complete the problems:

94 - 52 = ... - ... - ... = ... - ... = ...

Researcher:	How do you do that?
Henry:	First you subtract 50, that's 44, and then you subtract 2, making 42.
Researcher:	Do you always have to do it that way?
Henry:	Yes.
Researcher:	Can't you subtract 2 first?
Henry:	That's not allowed.
Researcher:	But why not?
Henry:	Because the book says. (He points out the following example: 54 - 31 = ., 54 - 30 - 1=., 24 – 1 = 23)
Researcher:	What if I subtract 2 first anyway?
Henry:	But that's against the rules.
Researcher:	Will the answer be different if you subtract 2 first?
Henry:	Maybe.

Let us now look at an example in which children are given a change to develop their own constructions. A group of 10 to 11-year old students was asked how they would go about solving the following problem. There are a number of bottles on a table, and each bottle has a different shape. None of them has labels, so no one scan tell just how much each bottle can hold. How would you figure out which bottle can hold the most water? The children were asked to present their ideas and talk about one another's ideas. One child suggests weighing the bottles. No, another says, hold them under water and see how much the water rises. A third suggests dumping the contents on the floor and seeing which puddle is the biggest. This is good example of a practical situation in which children are constructing knowledge, taken-as-shared (Cobb, 1994). Note that here is qualified the word ‘shared’. The children’s solutions do not match precisely, but they are considered ‘compatible’ and are therefore worth discussing. And this is what happened. The children were criticizing and commenting each other solutions until one child proposed to use a glass as a measure. All the children insisted with this idea but now rose the question of how big the glass should be. At the end of the discussion they decided to choose a small glass but not too small. The idea that children’s own constructions form the point of departure for the teaching-learning process in mathematics instruction is one of the fundaments of the realistic school. Confrey (1985) argues that a person’s knowledge is necessarily the product of his/her own constructions or mental acts. Thus s/he can have no direct or unmediated knowledge of any objective reality. Knowledge is created by means of images or representations and these are products of our mental actions (Gardner, 1987). But if their own constructions are so very important, children should be allowed to nurture their own constructions (whatever their quality may be). It is not necessarily that this would lead to anarchy and blocked communication during mathematics class, because constructions arise through interaction with other children and with the teacher. Bruner (1986) too asserts that constructivism is not a sort of cultural relativism or an homage to the proposition that ‘anything goes’. Neither should constructions be understood in Piagetian terms. Piaget (1976) was concerned with individual constructions which arise from the subject’s own position and which are the result of intrinsic and autonomous processes; ‘mathematical practices’ have relatively little influence on them.

Lo, Grayson, Wheatly, and Smith (1990) discuss the close relationship between construction and interaction in the following fashion: “From a constructivist’s perspective learning occurs when a child tries to adapt her functioning schemes to neutralize perturbations that arise through interactions with our worlds” (p. 116). Two important aspects, constructions and interactions, are important in the above statement. Although construction of knowledge is a personal act, it is by no means an isolated activity as many people’s interpretations of constructivism imply. Constructivists recognize the importance of social actions as ‘the most frequent source of perturbations’ (Salomon, 1989). Interactions, thus, lead to construction, because the process of interacting often causes perturbations in the normal pattern of behaviour − particularly when unexpected problems arise − which the person involved will try to resolve by seeking his or her own solutions (constructions). A study conducted by Saxe (1988) investigated how young Brazilian candy sellers, who generally had little or no schooling, had learned mathematics. These children had learned to fix cost prices, to calculate skilfully in cash amounts, and to think in ratios (3 pieces of candy is 500 cruzeiros, 1 piece is 200), but when confronted with classroom problems − for example, reading and comparing double-digit numbers − they were at a complete loss. They had created their own constructions in the process of solving problems encountered in daily social interaction. Constructions, in turn, may once again lead to interactions, in the sense that constructions are 'tested' in interactions: do my ideas make sense, are they valid?

The construction of internal mental representations is one of the features of the process of learning mathematics. We conceive the development of internal representations as a process of signification (Kirshner & Whitson, 1997; Walkerdine, 1997). So we do not make a distinction between an externally represented world and an internally representing world. Representation is looked upon as a process in which new signs in a cyclic process of signification constantly emerge. An internal representation (signifier) transforms and is the basis (signified) for the construction of a new internal representation (signifier). Hence a person constructs internal representations on the basis of internal representations.

The process of learning mathematics distinguishes itself from the process of learning other school subjects to the extent that in mathematics, constructions – in the sense of internal representations which children formulate based on knowledge gained through experience – consistently show a closer correspondence with external representations than in other school subjects (Cobb et al., 1987; Freudenthal, 1983). Children gain experience in the use of measurements, numbers, ratios, and fractions and construct (intuitive) representations. In theory these representations form a basis upon which the teacher can build, although this is not always the case. Particularly this is not the case when children learn to operate with mathematical symbols. Many errors are based on the default nature of natural language encoding processes, as Kaput (1987) bas stated. Kaput discusses the well-known Student-Professors problem (Clement, 1982). At a certain university, for every 6 students there is one professor. Write an algebraic equation that expresses the relation between the number of students and professors. Consistently the natural language overrides the algebraic rules as is shown by the high error rates (40-80%) across age and the predominance of the '6s = p' error, typified by Kaput as the 'reversal error'.

The representations which the children construct concerning physical phenomena – also known as preconceptions, misconceptions or intuitive ideas – generally deviate so far from actual physical reality that they are useless as a basis for conceptualization. For example, children associate energy with eating a Mars bar; in their minds, evaporation is the same as disappearing, heat means feeling nice and warm, and light is a ray which goes from the eye to an object, rather than the other way around (Van der Valk, 1989). These representations are useless in instruction, but the teacher must be familiar with them in order to understand the problems that arise in conceptualization. This applies as well to representation in other school subjects. For example, when studying history, children have a great deal of trouble forming representations based on their own experience. One child, for instance, regularly confused ‘Enlightenment’ with ‘more light’.

Interaction

Realistic instruction in mathematics is not only constructive but also interactive. Several authors have pointed out the importance of this interactive, or social, dimension of learning. Bishop (1988) has argued to replace ‘impersonal learning’ and ‘text teaching’ with ‘mathematical enculturation’, thereby emphasizing the relationship between education and culture. Pimm (1990) uses the term ‘mathematical discourse’, while Salomon (1989) speaks of ‘cognitive partnership’.

Granott and Gardner (1994) constructed a theoretical framework of interaction, based on the view of multiple intelligence approach. After their opinion the effect of interaction depends on two dimensions. The first dimension is the relative expertise: non symmetric (‘parallel activity’ for instance) till asymmetric (‘apprenticeship’). The second dimension is the degree of collaboration. ‘Scaffolding’ is for instance an example of collaboration of a high degree, while ‘imitation’ is in fact an independent activity (no collaboration).

Interactive teaching has also been called ‘cooperative learning’ (Slavin, 1986), ‘classroom discourse’ (Cazden, 1988), ‘mutual instruction’ (Glaser, 1991) ‘guided construction of knowledge’ (Mercer, 1995) and ‘interactive instruction’ (Treffers & Goffree, 1985). Bruner (1986) a proponent of ‘discovery learning’ − learning on one’s own − in the 1960s, revised his ideas several years ago: “My model of the child in those days was very much in the tradition of the solo child mastering the world by presenting it to himself in his own terms. In the intervening years I have come increasingly to recognize that most learning in most settings is a communal activity, a sharing of the culture. It is this that leads me to emphasize not only discovery and invention but the importance of negotiation and sharing.” (Bruner, 1986, p. 127). There is no contradiction however between invention and sharing, the contrary is true: both activities influence each other. If a person makes his own invention, it is worthwhile and even in many cases necessary to discuss this invention. And this discussion is the basis for new inventions.

Nowadays, this view of (cognitive) development and learning is classified as social-constructivism, a classification which meshes with the realistic approach to mathematics instruction. In some studies (Driver, Asoko, Leach, Mortimer, & Scott, 1994; Roazzi & Bryant, 1994) is defended the point of view that learning and thinking always take place in a social situation. Learning, they say, is situated learning (Kirshuer & Whitson, 1997), cognition is social cognition. Bruner’s (1986) designation for the acquisition of knowledge is ‘negotiation of meaning’. Not only words, concepts, gestures, and rituals, but also numbers, symbols, images, visual and graphic representations, etc. have a whole range of meanings. In the case of children, these meanings are frequently highly subjective. In response to the question “How old are you?”, one child was heard to answer “I’m four, but when I ride in the bus I’m three” (in the Nether1ands children under the age of four can ride public transportation free of charge). Another child believes that when teachers roll the dice, they get double sixes more often than children do. Teaching, says Bruner, means negotiating meaning. You say that zero is “nothing”, but what then does zero degrees mean on the thermometer? A child does not believe that her face has a surface. “Why not?” asks the teacher. “Because it isn’t length times width,” the child responds. By applying the Socratic method, the teacher was gradually able to convince the child that the ‘surface’ was not exclusively linked to the algorithm l × w. Bruner’s tribute to Vygotsky (1977) is not at all surprising. According to the latter, a child’s higher psychic functions (such as language and thought) first take shape as a social (interactive) activity and only later as an individual activity. Language first functions as a means of communication; afterwards it becomes internalized and serves an individual, self-regulatory function. One of the key concepts in Vygotsky’s theory is that education should anticipate actual development. He refers in this connection to the ‘zone of proximal development’, and it is this idea which inspired Bruner’s ‘negotiation of learning’. Both are concerned with interactive instruction, which Freudenthal (1984) typifies as ‘anticipatory learning’ and Van Parreren (1988) as ‘developing education’.

Realistic mathematics instruction is interactive, even though children must naturally be given the chance to work independently. As demonstrated in studies carried out by Doise and Mugny (1984), however, the point is that allowing children to experience various perspectives − in other words, showing them that there are other children with other ideas about how to solve a mathematics problem − will stimulate their thinking. Mechanistic (and individualistic) mathematics instruction can exclude such experiences because children are required to comply with the procedures given in their textbook. Discussion is restricted because the essence of instruction lies in teaching irrefutable procedures. Realistic instruction, on the other hand, is based on the exchange of ideas, not only, as in the past, between teacher and pupils, but also between the pupils themselves. Interaction stimulates reasoning, using and analysing arguments, thinking about own solutions and the solutions of others, so interaction reinforces the thinking ability. Currently, social interaction in the classroom is receiving much attention where an important line of research focuses on effects of small group work (Hiebert, 1992). It goes without saying that there should be a ‘genuine’ occasion for discussion. That is why the point of departure for realistic instruction is frequently a problem in context; again, this emphasizes how tightly interwoven context and interaction are.

A simple example is the following: a teacher asks his class (6 and 7-year-olds) to think up as many ways as they can of doing the sum 5 + 6. The children are allowed to discuss this among themselves and together they came up with several methods: counting from 6 on up; adding 5 + 5 to reach 10 and then adding 1; counting the fingers on both hands and then adding 1; adding 6 + 6 and subtracting1, etc. Teacher and pupils then discuss which method is the handiest and why. This process leads the children to reflect spontaneously on their own actions: they are forced to compare their methods with those of the other children and consider which is the best (this example makes clear that in realistic mathematics instruction the students not only are confronted with context problems, but with bare problem as well).

Reflection

According to Hiebert (1992), reflection or metacognition can be defined as the conscious consideration of one’s experiences, often in the interests of establishing relationships between ideas or actions. It involves thinking back on one’s experiences and taking the experiences as objects of thought. With respect to terminology, reflection is seen most frequently in Russian research reports (Davydov, Lompscher, & Markova, 1982; Nelissen & Tomic, 1996; Stepanov & Semenov, 1985; Zak, 1984). The terms self-monitoring or self-regulation are also applied (Glaser, 1991). It would be most tempting to spend a great deal of time discussing the many questions, controversies and dilemmas which have arisen in the literature concerning the concept of reflection − consider, for example, the discussion concerning the extent to which reflective skills are general or contingent on context (Perkins & Salomon, 1989).

We generally do not reflect while performing a routine task, for the simple reason that there is no cause to do so then. There is, however, reason for reflection whenever we are confronted with a problem for which there is no immediate solution at hand. Reflection begins when we ask ourselves how best to approach the problem: 'Should I do it this way or that way?' (planning). Once we have set to work, other questions arise: "Is this working?" (self-monitoring), perhaps even "Can I do it?" (self-evaluation). Other obvious questions are "Will this succeed?" (anticipation) and, finally, "Am I happy with this?" (evaluation). If the solution turns out to be a dead end, then we are forced to ask ourselves "Shouldn't I try something else?" (consider switching methods). These are, in brief, the most important elements of reflection during the process of problem-solving. Reflection plays a significant role in learning to solve mathematical problems, and indeed in human action in general. Through reflection students learn to analyse their own actions critically and also become less dependent on their teacher. Their thinking becomes more systematic, however this is not the case with all students. Some students must be stimulated frequently. Reflection also allows them to investigate problem-solving methods and procedures for general applicability, and increases the flexibility of their thinking. The most important aspect, however, is that reflection builds self-confidence by allowing pupils to discover what they really think and why they think it. Without this knowledge, every result might seem – and in fact might very well be − serendipitous, an awareness that does little to build up confidence: the pupil might not be so lucky the next time around.

That reflection is closely tied to the mathematical learning process and to mathematical thinking can be deduced from the proposition, discussed above, that mathematization is a constructive activity. This activity, in turn, is permanently linked to interaction, as we have seen. We can imagine the connection between construction, interaction and reflection in the following manner: constructive thinking implies that interaction takes place concerning our own constructions (representations). We must naturally be able to test our own constructions and find out how valuable they are. By exploring − and anticipating − the ideas and criticisms expressed by others, we gain greater insight into our own ideas. Knowing what these ideas are and how we ultimately came up with them is called reflection. We internalize the dialogue which we originally conducted with others, turning it into a dialogue ‘with ourselves’. Reflection, thus, is nothing less than ‘internalized dialogue’: from primarily inter-individual to intra-individual activity. Through reflection, we continue to create new constructions, each time at a higher level. In short, reflection is development.

There is a relationship between reflection and the process by which pupils solve mathematical problems. In one study, the reflective thinking and mathematical problem-solving skills of two groups of students were compared (Nelissen, 1987). One group had been taught mathematics according to the realistic method (84 students) and the other according to the mechanistic method (60 students). One striking result was that the students in the first group were more flexible in their thinking than those in the second group, specifically because they were better able to switch strategies whenever necessary. They were less likely to concentrate purely on algorithmic solutions, and were able to develop strategies on the basis of their own experience. They tended to check their own approach without prompting and were aware of their own thought processes. In general, the children who were better able to solve problems were also better at reflection, in the ‘realistic’ group this was 43% of the students, while in the control-group this was 10% of the students.

A number of factors might serve to explain this close, positive relationship: (a) the school curriculum followed by children in the experimental group was based on problem-solving within a rich context. Instead of being given fixed, standard procedures to learn, pupils in these schools were allowed to think up their own constructions. Through interaction they were encouraged to reflect on their own approach. In this way, problem-solving and reflection were stimulated in relation to each other. Note that the children were not given direct, separate training in reflection. Research has shown that the training of functions in a separate programme has only a limited effect (Derry & Murphy, 1986); (b) by commenting regularly on each other’s actions, the ‘realistic’ children were able to generate a reflective attitude which may have had a positive impact on their problem-solving skills; (c) the children in the experimental group were taught the concepts, models and procedures they needed to solve problems and engage in reflection. To be able to reflect on a specific subject, they needed to acquire domain-specific knowledge; (d) reflection will only prove beneficial after children have come to view the actions they are reflecting on as meaningful. The children in the ‘realistic’ group found mathematics and problem-solving a meaningful activity. They were therefore more inclined to reflect on problem-solving than the children in the ‘mechanistic’ group. Children in the latter group saw little reason to apply their own reality to learning mathematics, because this reality was continually supplanted by prescribed standard algorithms. A study conducted by Stepanov and Semenov (1985) revealed that in order to be able to reflect on the process of problem solving, children must first see their own actions during this same process as meaningful. Meaning must therefore be given due attention during instruction, if children are to find reflection a meaningful activity. Reflection, in turn, is vital if the mathematization process is to run smoothly.

Solving mathematical problems

One important objective in mathematics instruction is that children be able to apply the concepts and skills they have acquired with reasonable success. Problem-solving is considered by many to be one of the most important areas of application. That is why so many researchers are interested in whether, and if so, how children solve mathematics problems. Where upon are processes of problem solving based, on declarative knowledge or on knowledge of procedures?

The respective opponents and proponents of ‘declarative representations’ and ‘procedural representations’ have been engaged in a vehement debate since the early 1970s (Gardner, 1987). Adherents of the first believe that the knowledge base is the most important factor in problem-solving, while adherents of the second relate success largely to the use of procedures and strategies. In the 1980s the dispute concerned whether such knowledge or such (reflective) procedures were general or domain-specific. This controversy led to yet another split in both camps. Although the debate rages on, its resolution seems to be in sight, specifically because human thought is increasingly being characterized as modular. Learning is therefore by no means a ‘content blind’ process; neither, according to Gardner (1987) are there such things as ‘general cognitive architectures’, as Piaget (1976) suggested. Several leading authors appear to share this opinion: Bonner (1990), Resnick and Klopfer (1989), and Schoenfeld (1989). All of these above mentioned researchers tend, albeit from different backgrounds, to maintain that problem-solving (in mathematics) will be most successful when based on a well-organized selection of domain-specific knowledge, but that the use of procedures which are (once again) domain-specific is also indispensable. Experts tend to use domain specific procedures and principles while novices are more likely to choose general strategies. For this reason novices often fail to solve the problems (Caillot, 1991).

Problem-solving in mathematics, then, is also characterized by a specific mathematical approach, involving the use of domain specific concepts, tools (procedures) and ways of thinking. A child who has not mastered this approach will have difficulty when solving mathematical problems. A few examples follow. A classroom of six- and seven-year-olds were given the following problem: 2 friends live next door to each other, one at number 3 and one at number 5. How many houses do they live in? The children answered: 8 houses. Numbers mean little to these children, except that they can be added up. Assigning a meaning to numbers is an important component of a mathematical approach. Many children (eleven or twelve years old) have difficulty solving the following type of problem: a walkman costs \$150 after the price has gone up by 20%. What was the original price? The answer most frequently given is: 20% of \$150 is \$30; the original price was therefore \$120. This answer is, of course, incorrect. Insight into this problem can, however, be provided by means of a diagram showing that \$150 does not equal 100% but 120%:

1/5	1/5	1/5	1/5	1/5	to be added:	1/5
20%	20%	20%	20%	20%		20%

5/5 is 100%; 6/5 then is 120%. So 6/5 × \$150 = \$125.

Schematization − or visualization − is an important mathematical strategy, a tool for solving problems. Other such tools are: estimating, simplifying problems, testing, changing perspectives and conducting a thought experiment. Gravemeijer (1988) has classified mathematization tools according to characteristics derived from mathematics itself. For example, structuring and generalizing are related to the category ‘generality’. Proofs and predictions are connected to ‘certainty’ because this is for a mathematician very important, symbolization and formalization belong to ‘precision’ and reduction and constructing algorithms to ‘conciseness’.

Resnick and Klopfer (1989) identify knowledge that plays an important role in problem-solving as ‘organizing schemata’, concepts which are ‘powerful’ and must be actively acquired. In other words, it is rich, flexible, ‘generative knowledge’. This knowledge, or specialist knowledge, forms the basis upon which we can construct the first representation of a problem which we must solve. This representation is the starting point for a successful problem-solving process. Bransford, Sherwood, Vye, and Rieser (1989) warn against the danger of rote knowledge or, as they put it, ‘inert knowledge’. Because children do not consider such formal knowledge ‘real’ or meaningful, they will not be able to apply it or only do so blindly, particularly in mathematics. Children acquire inert knowledge in mathematics when they are forced to learn formulas such as ‘To divide by a fraction, multiply by its opposite’ or to do plain problems involving meaningless numbers (leading to the type of problem discussed in the example above). In this connection, realistic mathematics instruction makes use of the models, schemata and concepts − called ‘conceptual models’ by Lesh (1985) − which form the core of the mathematical approach.

This approach is similar to Davydov’s (1977) idea of teaching children to work with theoretical concepts − which can be seen as concepts essential to a specific discipline − instead of concepts based on observation or empiricism. In this connection, Davydov has argued for the formation of theoretical thinking, similar in certain respects to Freudenthal’s (1984) development of mathematical thinking or attitude. In fact, the two are so similar that both have pointed out the same flaws in the empirical, inductive teaching approach. In this approach, knowledge comes into being through observation or empiricism. Here we can recognize the influence of the empiricist school, which, as we have seen, places very little emphasis on vertical mathematization − which is precisely what Freudenthal and Davydov do wish to emphasize. Mathematical or theoretical insights are required in order to be able to understand reality correctly (including learning tools such as the number line or abacus). Here, however, the similarity between Freudenthal and Davydov ends. According to Freudenthal, Davydov introduces theoretical concepts too early on in education; moreover, Freudenthal rejects the distinction made by Davydov between theoretical and empirical concepts. In Freudenthal’s view, theory is always inherent in empiricism; all action is implicitly theory bound.

Research into solving mathematics problems often focuses on initial mathematics problems and word problems. See for example De Corte, Verschaffel, and Greer (1996), Span, De Corte, and Van Hout-Wolters (1989) and Verschaffel and De Corte (1990) for a report on studies carried out on learning and problem-solving in mathematics. Following in the footsteps of Riley, Greeno and Heller (1983), Verschaffel and De Corte (1990) explored which internal representations children formed as a result of problems which they were given to solve, and which role the semantic features of these problems played in this. They also wished to discover how these representations formed the basis for actions, in particular problem-solving procedures. Their research revealed that semantic factors played a role in forming problem representations Many children did indeed tend to take their lead from the meaning of words (some of which were printed by chance) in the text (‘together’ or ‘with each other’) and to base their solution procedures on these words. It was also shown that many verbal, nonpropositional, grammatical characteristics influenced the choice of procedure and not only cognitive schemata, as authors such as Resnick (1983) and Riley et al (1983) have suggested. Finding the correct solution, these researchers believe, depends on the formation of adequate representations constructed from part-whole schemata.

There are, however, several objections which might be raised (Van Luit, 1994) to this emphasis upon the part-whole schema. To begin, there is no relationship with a child’s previous experiences, such as counting (Van Mulken, 1992). Second, this schema is based exclusively on the cardinal interpretation of numbers, whereas in truth numbers may appear in other forms: 9 is 6 + 3, but 9 is also 3 × 3 or the root of 81. Third, a child gets into trouble if he or she comes across a problem which does not contain a part-whole relationship; for example, John is 5 years old and his friend is 6.

A more general comment on research into solving word problems is that the influence exerted by semantic structural features has been given too much emphasis, at the expense of studying the influence which the nature and the size of numbers have on the choice of solution procedure (Van Mulken, 1992; Verschaffel & De Corte, 1990). This is known as ‘number sensitivity’ An example is: 62 - 18 = ? If we take account of the nature of the numbers in this problem, the obvious strategy is to subtract 20 first and then add 2 The problem 62 - 33 =?, however, requires a different approach, for example: 63 - 33 = 30, 30 - 1 = 29. For a lot of children this is not an easy problem, because it supposes much flexibility in thinking. In teaching children to solve word problems, one should emphasize neither mastery of procedures (such as the use of the part-whole schema) nor semantic features, but rather the structure of the problem, specifically the structure of the numbers. Proponents of realistic mathematics instruction argue for flexibility in choosing problem-solving methods or learning to choose them.

Some of the above mentioned researchers maintain that training in meta-cognitive skills – such as planning the course of a solution, setting up schemata, guessing the solution in advance – can simplify the approach taken to problems. However, in none of the studies was training conclusively shown to be successful, although progress was noted. Some researchers point out, no doubt correctly, that the weaker children lacked a certain knowledge background It was remarkable that the children were able to master a heuristic (for example, making an estimate), but that this did not automatically lead to their choosing the required, formal operation.

In problem solving the context can be important. The function of contexts is to elicit knowledge which children have gained through experience and which they can use once again in forming internal problem representations. This is not, however, a hard and fast rule. Sometimes meanings which have been acquired through experience are directly contrary to the mathematical meaning of concepts (and, as we remarked earlier, in physics this is the rule rather than the exception (Keil, 1989). Walkerdine (1988) drew attention to this in her series of carefully conducted observations. An association with the child's experience, she claims, means an association with contexts, and these are highly domain-specific. We cannot simply assume, therefore, that 'transfer' is affected simply by the insertion of mathematical relations into a 'meaningful context'. The author illustrates this by giving the following example. Children first learn the concept 'more' in a pedagogical situation: 'Just two mouthfuls more, mmm!' In mathematics, however, 'more' is the opposite of 'less', and that is an entirely different concept. There are, then, two entirely different 'discursive practices'; in other words, it is not always the case that children solve problems better by using knowledge based on their own experience.

Discussion and conclusions

There have been radical changes in the approach to mathematics instruction in the past few years. These changes came about because mathematicians began to view their own discipline differently, leading to new research on teaching methodology. This research was supported by new developments in educational psychology. Glaser (1991) analyses these developments. He points out the tendency to relate learning and thinking to specific domains. This approach has been defended by various authors. For example, in a series of studies (Keil, 1989) it is shown that indeed it is plausible that concepts develop largely in specific domains. Research has also demonstrated the importance of reflection.

Experts − adults or children − develop skills in order to plan and monitor their actions and predict what the results of their efforts will be. In an experiment, Glancey (1986) introduced the knowledge representation (including heuristic rules) of experts to a group of pupils. On the basis of these representations, the pupils learned to formulate hypotheses, recognize errors and, in particular, to better organize their knowledge base. Glancey's most original idea was to have the pupils observe experts and question them concerning their methods. The observation strategies and the questions were analyzed in advance, and an analysis was also done of how knowledge might be restructured during the learning process. Here interaction and reflection go hand in hand. The idea of learning as a process of continual restructuring, of increasing architectonics and deeper insight, has also been expounded by White and Frederiksen (1986). They furthermore emphasize establishing a link with the pupils' naive, intuitive models, as in mathematics instruction.

Constructivism is another tendency which Glaser (1991) has frequently come across in analyzing research reports. Although the emphasis in the past was on monitoring the pupils' learning processes, nowadays the pupils has more control over his own learning environment, and many studies have attempted to gain insight into how pupils construct their learning environment in order to be able to learn something, A new image of the pupil is coming into being; they are no longer 'good boys and girls' who learn everything by rote, but children motivated to explore and seek explanations.

It was not only among Anglo-Saxon cognitive psychology that is undergoing significant changes; the Russian Cultural Historical School has long been concentrating on research themes related to the new developments in mathematics instruction. The main focus is on reflective thinking (Stepanov & Semenov, 1985; Zak, 1984), interaction (Davydov et al., 1982), and the development of domain-specific concepts (Davydov, 1977), while education and instruction are in essence seen as the active interrelation of symbolic systems and meanings which a culture has brought forth (Leont'ev, 1980; Van Oers, 1987). For a more thorough comparison between the concepts presented by the Russian Cultural Historic School and realistic teaching methodology, readers are referred to Nelissen and Tomic (1995, 1996).

So, in the past few years a number of interesting new themes relevant to mathematics instruction have received a great deal of attention. We have argued that learning is a process which rests upon children's own constructive activity. Learning takes place in a social context (Bruner, 1996; Slavin, 1986). Mathematical ideas are not merely abstract; they are contained within language and concepts. Learning is a process in which the child masters its cultural heritage, by learning particular sets of symbols. If children are able to put their ideas into words, they will have a better grasp of their own way of thinking. We have also stated that when children are able to put their ideas into words, their teacher will gain greater insight into their thought processes. If their constructions prove to be unusable, the teacher can confront the children with alternative approaches, for example through problem-oriented questioning. When children discuss their ideas with one another (Mercer, 1995), they not only have to state their opinions more concisely, they also have to listen, think along with others and try to understand what the other children actually mean to say. Learning − and cognitive development − is increasingly viewed as a process in which metacognition (or reflection) fulfills a regulatory function.

Solving mathematics problems requires learning domain-specific rather than general knowledge. This knowledge is well-structured and flexible, and encompasses a knowledge of both content and procedures and reflective knowledge. In mathematics, tools and modes of thinking that typify mathematics should be maintained, and used to solve problems. The overriding concern is to maintain a flexible choice of tools, and that choice requires domain-specific reflection. To be able to reflect, however, knowledge of content is once again necessary; one can only reflect on the use of tools, strategies and concepts if one knows them. By now many of these new ideas have filtered through to the practice of teaching mathematics, in part because new programmes are being developed, implemented and supervised which are based on the realistic approach, and in part because teacher training now focuses on the new realistic didactic. It is, ultimately, the teachers themselves who must put these new ideas into practice.

Although there is a high degree of consensus among researchers in mathematics instruction, particularly on initial mathematics problems and word problems, up to the present theory has preceded carefully collected empirical data. If we also agree, on the basis of research findings, that construction, interaction and reflection are essential for learning mathematics, then the practice of mathematics instruction should be altered radically. Teachers are being asked to master a new approach to instruction and to their pupils. Among other things, this means a new approach to testing, to explaining, to cooperating and discussing, to working independently, to thinking intuitively, to understanding and developing concepts.

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