Diagrammatic reasoning and hypostatic abstraction in statistics education*

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Abstract

Peirce's notions of diagrammatic reasoning and hypostatic abstraction are relevant to educational research in areas where diagrams and abstraction play an important role. In this paper, I analyze an example from statistics education in which diagrammatic reasoning created opportunities for hypostatic abstraction. For instance, where students initially characterized data points as being 'spread out,' they later said, 'the spread is large.' This is a prototypical example of hypostatic abstraction — taking a predicate as a new object that can have predicates itself. More generally, the notion of diagrammatic reasoning proved helpful to identify the key learning processes involved in learning to reason about statistical concepts.

1. Learning paradox in mathematics education

Over the past years, semiotics is increasingly used in mathematics education research to study processes of meaning-making in relation to signs (Cobb et al. 2000; Gravemeijer et al. 2002; Hoffmann et al. 2005). Signs are crucial in mathematics, statistics, and science, for instance, because learning these subjects and communicating about their invisible objects is impossible without signs. At the same time, and this makes mathematics and statistics so hard for students, the visible signs represent invisible mathematical objects or relations that students still need to learn about. For example, a drawing of a triangle is just a token, but in mathematics it is supposed to represent a triangle as a type. Consequently, learning mathematics is a complex semiotic activity that requires both the construction of mathematical meaning and the interpretation and development of mathematical notation (Sáenz-Ludlow 2003). In other words, learning can be described as a process of developing conditions of understanding and appropriately using signs.

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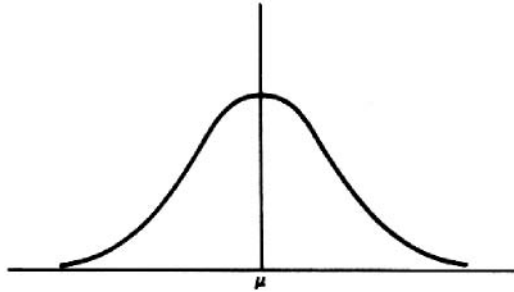

Figure 1. *Bell shape of the normal distribution*

A central question for mathematics education, which can be seen as a form of the learning paradox (Bereiter 1985), is therefore the following: How can students learn that sign S refers to mathematical object O if they do not already know that S refers to O? Hoffmann (2003) argues that Peirce's semiotics can help us overcome this learning paradox if we employ the notion of diagrammatic reasoning, and the purpose of this paper is to substantiate this philosophical claim with empirical data. Thus the paper aims to show how Peirce's notions of diagrammatic reasoning and hypostatic abstraction are useful in analysing students' learning, in particular in areas where diagrams and abstraction play an important role, such as in statistics education.

The empirical data stem from a teaching experiment in grade eight (age thirteen) in the Netherlands using educational statistics software. The main question was how students with little statistical background could learn to reason about distribution, which is one of the key concepts in statistics. Distribution is a sophisticated concept that is typically represented by continuous shapes such as the famous bell shape (figure 1) for the normal distribution, and that is typically introduced in grade ten or higher in a rather technical way that students generally find hard to appreciate. (As a historical aside, it is worth mentioning that Peirce was among the early scientists using the normal distribution for modeling phenomena such as reaction times of a young man on several consecutive days, see figure 2). The learning paradox for this particular topic would be: How can students adequately reason about 'distribution' using conventional signs of distributions if they do not already know that these signs refer to distributions?

2. An instructional sequence

Addressing this question, my PhD research (Bakker 2004) was a sequel to design-based research carried out by Cobb and his team (Cobb 2002;

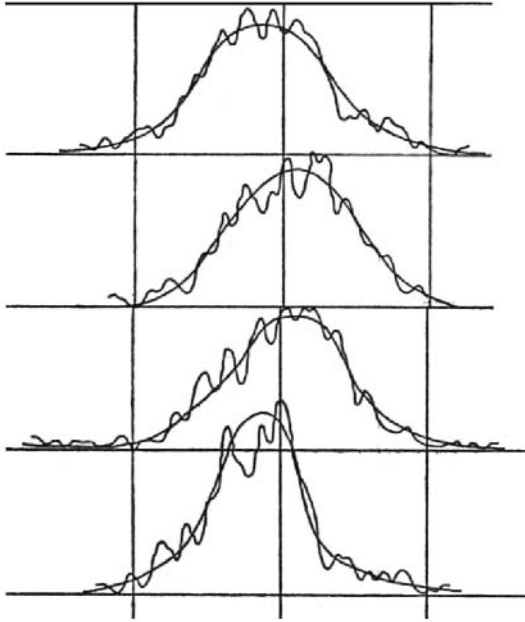

Figure 2. *Reaction times of a young man on consecutive days with smoothed distribution curves (NEM 3: 676)*

Cobb et al. 2003). They designed a series of instructional activities that supported mathematical practices leading to an understanding of distribution and this development was supported by means of a series of small software applications (Java applets) that were called Minitools (Cobb et al. 1997).

Starting with an intuitively clear representation, the first Minitool supplies a bar graph in which each bar has a length corresponding to the data value it represents (figure 3). The bars can be removed (figure 4) and if the endpoints are collapsed down onto the axis the result is a dot plot as provided in the second Minitool (figure 5). In the first Minitool, students can organize data, for instance by sorting or hiding subsets of data, and by sorting the data by size. In the second Minitool, students can organize data with different options, for instance making their own groups, two equal groups (precursor to the median), four equal groups (precursor to box plot), and fixed interval width (precursor to histogram).

The central idea of the design of the instructional sequence was that students learned to solve statistical problems using simple representations of data that made sense to them. These problem situations were designed such that students might experience the need of a more sophisticated

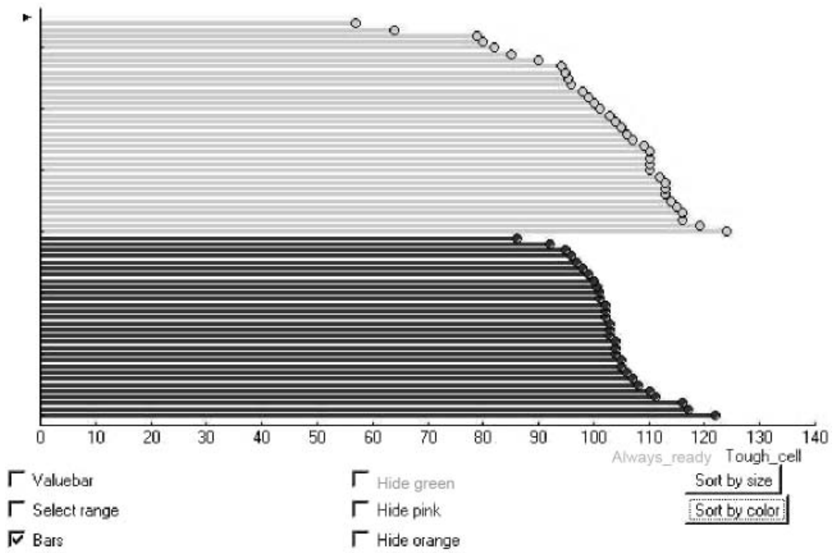

Figure 3. *A value-bar graph of Minitool 1, with options to sort value bars by size, by color, to hide subsets, hide bars, and select a range. The data are life spans of two battery brands in hours.*

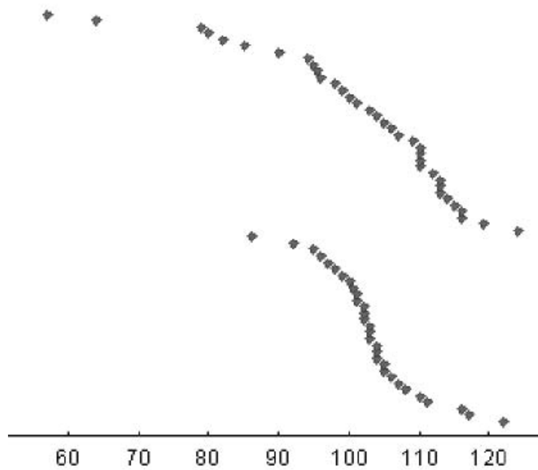

Figure 4. *Minitool 1 with the value bars removed as a transition to Minitool 2, in which dots are collapsed down onto the horizontal axis*

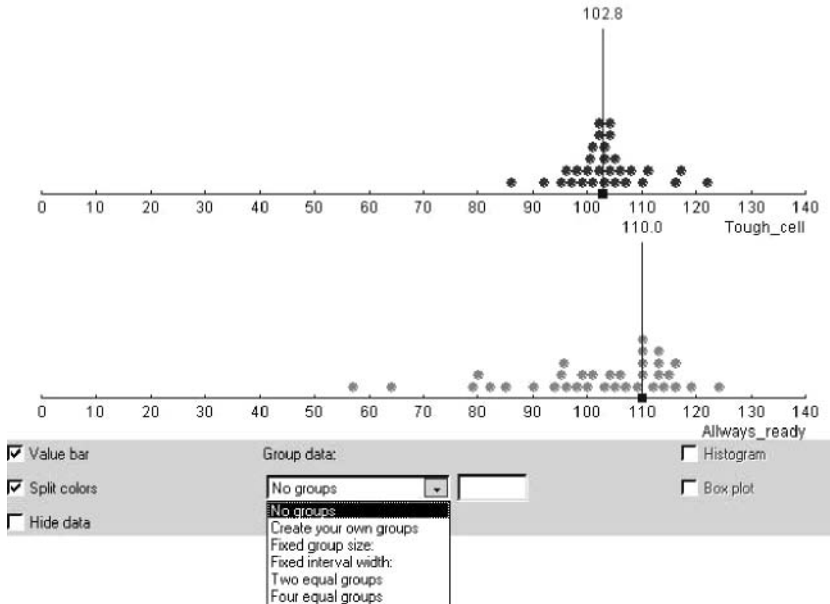

Figure 5. *A dot plot in Minitool 2, with more options to organize data points (this is the same data set as in Figures 3 and 4)*

representation that could help them to solve more advanced problems. Cobb (2002) describes how his American seventh-grade students came to reason about distributions as 'hills.' In retrospect, he analyzed this development as a chain of signification, referring to Walkerdine (1988), who in turn based her theory on Lacan (1968).

According to his interpretation of chains of signification, once students have developed a signified belonging to one signifier (e.g., value-bar graph), this signified can 'slide under' the next signifier in the sequence, thus forming a new sign. In this way, complex meanings were developed and students were guided in 'reinventing' conventional plots such as dot plots.

I tried to use the chain of signification notion as a design heuristic and an instrument of analysis, but during teaching experiments in grade seven, carried out before the eighth-grade experiment that this paper gives an example from, it turned out that this notion of a chain of signification was too linear in both the design and analysis phase; a similar observation is made by Sfard (2000). One reason for the difference in experienced applicability might be that Cobb and his team were interested in the macro-level of emerging mathematical practices and managed to foster common

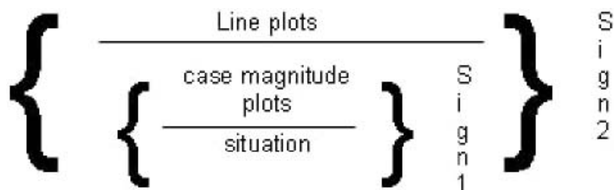

Figure 6. *Cobb's (2002: 188) summary of the chain of signification as used in the analysis of a seventh-grade teaching experiment (case magnitude plot is another name for value-bar graph; line plot is another name for dot plot)*

socio-mathematical norms (Yackel and Cobb 1996) for their classroom, whereas I was interested in conceptual development at the level of individuals within the classroom and I had evidence that *comparing* different signs such as bar graphs and dot plots was a useful instructional activity. The following example from Bakker and Hoffmann (2005) may illustrate this.

In the tenth lesson of a teaching experiment in grade seven, students were comparing different graphs they had made themselves of a weight data set. This comparison led to a discussion of a 'bump' (as a distribution shape) that was visible in one but not the other graph and that turned out to be crucial in the development of a notion of distribution. Though it was possible to reconstruct a chain of signification leading to both graphs separately, the chain theory did not seem to provide a solution to describe a *comparison* of the graphs. I therefore searched for a theory that would be viable in network-like situations and I am grateful to Michael Hoffmann for drawing my attention to Peirce's semiotics (e.g., Peirce *NEM*, *CP*).

3. Peirce's semiotics

The two aspects of Peirce's semiotics that lent themselves most to my purpose of analyzing students' learning were *non-linearity* and the possibility of stressing the *dynamic character* of interpreting and making signs within the theory itself. But Peirce's semiotic framework also offers a differentiated notion of sign and has a consistent epistemological basis (Hoffmann 2003). It has furthermore been developed in the context of mathematics, logic, philosophy, the history of science, and other disciplines Peirce studied, and it is therefore not surprising that his semiotic concepts apply more easily to mathematical signs and symbols than those of Saussure (1974 [1916]), whose sign notion has a textual origin (Ducrot and Todorov 1983). One thing that also bothered me in the use of 'signifier' and

'signified' in the chains of signification was that signifiers referred to external representations, but for Saussure, as mentioned by Mortensen and Roberts (1997), 'signifier' and 'signified' did *not* refer to an external representation and its referent: *both* concepts were defined as purely psychological entities.

Peirce's semiotics is by no means a ready-made theory that can be applied to mathematics education and he only slightly touched on educational issues (Eisele 1976). I stuck as closely as possible to Peirce's original definitions to avoid confusion and eclecticism, but sometimes needed to choose from different definitions, because Peirce's views developed throughout his lifetime (Ducrot and Todorov 1983). In the following, Peirce's notions of sign, collateral knowledge, diagram, diagrammatic reasoning, and hypostatic abstraction are elaborated insofar as needed to analyze the classroom episodes in later sections.

3.1. *Sign and collateral knowledge*

In Peirce's semiotics, a sign stands in a triadic relation to an object and what he called an 'interpretant' as represented in figure 7. In later writings, he defined this 'interpretant' generally as 'the proper significant outcome of a sign,' or as its 'effect' (*CP* 5.473, 5.475). Thus, the interpretant can be a reaction to a sign or the effect in acting, feeling and thinking or, in other words, the sign's 'meaning.' A sign-mediated effect need not be a response to one single sign, but could be the response to several signs. Conversely, the effect of interpreting a sign can also be multiple actions or the production of multiple signs. Sign-activity therefore occurs within series, webs, and networks of signs in which interpretants are responses to objects through the mediation of signs. Peirce's conception of a triadic sign relation therefore permits a representation of 'non-linear' processes of interpretation that are characteristic of the complexity of classroom

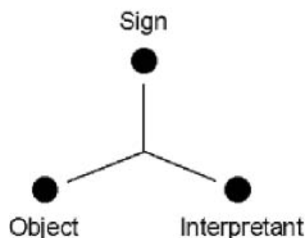

Figure 7. *Peirce's sign in a triadic relationship to object and interpretant*

communication, because an interpretant can well be the response to different signs, and interpreting a sign can lead to different interpretants.

According to Peirce:

> All our thinking is performed upon signs of some kind or other, either imagined or actually perceived. The best thinking, especially on mathematical subjects, is done by experimenting in the imagination upon a diagram or other *scheme*, and it facilitates the thought to have it before one's eyes. (*NEM* 1: 122)

An advantage of Peirce's semiotics is that it is based on an *epistemological* reflection on the role of signs as mediators between *objects* of knowledge on the one hand, and the *meaning* (interpretants) of these objects on the other. Signs therefore fulfill both a representational and epistemological, knowledge constituting function (Hoffmann and Roth, this issue).

As Hoffmann and Roth (2005) point out, *collateral knowledge* (*CP* 8.183) as a network of knowledge forms, mediates between signs and interpretants, where 'knowledge' includes the ability to do something. This notion of collateral knowledge or experience is crucial in solving the learning paradox for mathematics education: it is by enhancing the collateral knowledge in relation to signs that students learn to interpret more advanced statistical signs and to use these as means for statistical reasoning. Classroom communication using these signs for solving problems is then essential to build partially shared (or 'taken as shared') collateral knowledge or experience (see also Hoffmann and Roth, this issue; Smith-Shank, this issue).

A further important aspect of Peirce's signs concerns both the possibilities that signs can be composed of other signs and that they can be components of more complex signs. This is important for his concept of a diagram, which we need to address the concept of diagrammatic reasoning.

3.2. *Diagram*

Peirce defines a diagram as a sign 'which is predominantly an icon of relations and is aided to be so by conventions. Indices are also more or less used' (*CP* 4.418). Thus, a diagram is a complex sign that may include icons, indices and symbols (as indicated by the hint at conventions). Most important, however, is its iconic character, which results from the fact that a diagram is supposed to represent *relations* (cf. Stjernfelt 2000). One reason of why diagrams were so important for Peirce is that one can experiment with them according to a certain syntax, since in contrast to

simple icons, diagrams ‘should be carried out upon a ... system of representation’ (*CP* 4.418). According to this syntax, any experiment performed on diagrams will lead to certain experiences (*CP* 5.9) that can be incorporated into the experimenters’ collateral experience. This does not mean that all students have the same experiences, nor that they need to know all conventions and hidden rules of the diagrams they make.

Is a diagram a thing on paper or a computer screen, or is it a general type? A diagram on paper is a token, a visible particular. However, if the relations of a diagram are interpreted as ideal, the diagram is a type. For example, if we prove that the angles of a triangle in Euclidean geometry sum up to 180 degrees, we should use a geometrical diagram as a type because we cannot prove any general or ideal relations from just the token of one particular drawing if it is not interpreted as standing for a triangle as a general mathematical concept.

With this distinction, we can clarify the close link between statistical diagrams and concepts. If diagrams are just taught as tokens (how do you draw a histogram?), students are not so likely to conclude any general information from them. To develop *concepts*, students need to learn to reason with diagrams as types. For example, a hill or bump should become more than an icon (image or metaphor) of the physical objects ‘hill’ or ‘bump’; they should become diagrams of distributions. This issue leads to Peirce’s concepts of diagrammatic reasoning and hypostatic abstraction.

3.3. *Diagrammatic reasoning*

For Peirce, diagrammatic reasoning involves three steps:

1. The first step is to *construct* a diagram (or diagrams) in a representational system, which in my view includes diagrams in computer software and even informal student sketches of a statistical distribution. Such a construction of diagrams is motivated by the need to represent the relations that students consider significant in a problem.
2. The second step of diagrammatic reasoning is to *experiment* with the diagram(s). Any experimenting with a diagram is being executed within a representational system and is a rule or habit-driven activity, situated within a practice — thus contributing to an expansion of collateral experience. What makes experimenting with diagrams important is the rationality that is immanent in them by their cultural-historical origin. The rules define the possible transformations and actions, but also the constraints of operations on diagrams. Peirce

stresses the importance of *doing* something when thinking or reasoning with diagrams:

> Thinking in general terms is not enough. It is necessary that something should be DONE. In geometry, subsidiary lines are drawn. In algebra, permissible transformations are made. Thereupon the faculty of observation is called into play. (*CP* 4.233)

3. The third step is to *observe* the results of experimenting, which I interpret as a *reflection* step. As Peirce wrote, the diagram constructed by a mathematician 'puts before him an icon by the observation of which he detects relations between the parts of the diagram other than those which were used in its construction' (*NEM* 3: 749). In this way, he can 'discover unnoticed and hidden relations among the parts' (*CP* 3.363). The power of diagrammatic reasoning is that 'we are continually bumping up against hard fact. We expected one thing, or passively took it for granted, and had the image of it in our minds, but experience forces that idea into the background, and compels us to think quite differently' (*CP* 1.324).

Diagrammatic reasoning creates conditions for learning: in the process of making a diagram, experimenting on it and communicating about it collateral knowledge about the use of a new sign can be developed. Possibly the need is felt to construct a new diagram that better serves a purpose. If new concepts are created in this step, we are potentially dealing with 'hypostatic abstraction.'

3.4. *Hypostatic abstraction*

Peirce distinguished different types of abstraction, but this paper focuses on 'hypostatic' abstraction because '[Hypostatic] abstraction is an essential part of almost every really helpful step in mathematics' (*NEM* 4: 160). It is regarding a certain characteristic of a set of objects as a new object. For Peirce, 'an "object" means that which one speaks or thinks of' (*NEM* 1: 124) and hypostatic abstraction puts 'an abstract noun in place of a concrete predicate' (*NEM* 4: 160) — not as a linguistic trick, but as a genuinely creative act. For example, if we change 'honey is sweet' into 'honey possesses sweetness' (*CP* 4.235) and consider 'sweetness' as an object that we can talk about, we have a simple example of hypostatic abstraction. And in mathematics, cardinal numbers are hypostatic abstractions derived from a predicate of a collection (*CP* 5.534).

The central point of diagrammatic reasoning is that it creates opportunities for forming hypostatic abstractions (Bakker and Hoffmann 2005); that is, new signs such as the terms 'dots,' 'shape,' or 'spread.' Hypostatic abstraction also takes place when part of a diagram becomes perceived as an entity on its own, a *new object*. The central point of hypostatic abstraction, in turn, is that the new objects formed by this process can be used as *means* for further diagrammatization, and for further steps of enhancing collateral knowledge and thus of learning.

4. Background information on the teaching experiment

In focusing on distribution as the key concept of their research, Cobb et al. (2003) underemphasized the issue of sampling (Gravemeijer 2000). Part of my initial teaching experiments in grade seven were therefore meant to design instructional activities that would involve students in reasoning about sampling, and the conjecture emerged that students could learn to reason about distribution by reasoning about bigger and bigger samples.

The purpose of the teaching experiment in grade eight, presented here, was to test and elaborate this conjecture by designing and testing concrete instructional materials around sampling activities and reasoning about distribution. The first three lessons were meant to foster a practice of wondering about sample size and the reliability of conclusions in relation to distribution aspects such as mean and variation. In the first lesson, in the context of battery life spans, all students initially wanted to test only one or two batteries to find out which of two battery brands had a longer life span; in other words, they did not expect any variation in one type of battery. After discussion, they thought testing ten to fifteen was sufficient. In the third lesson, students reasoned about the problem of how many students of their age could go on a balloon ride if normally eight adults are allowed — a question that stimulated them to think about averages in relation to sampling. Building on these three lessons, the fourth lesson was explicitly about larger and larger samples. The remaining six lessons were, by and large, a matter of refining and developing issues that emerged during the fourth lesson. The idea was to start with students' own ideas and guide them toward more conventional notions and representations, a process that Freudenthal (1991) labeled as 'guided reinvention.'

This paper focuses on the fourth of the series of ten lessons (fifty minutes each). Half of the lessons were carried out in a computer lab. In parts of these lessons, students used two Minitools, with which they analyzed data sets, for instance, on battery life span (figures 3–5), car colors,

and salaries. As a researcher I was responsible for the design of instructional materials and the teacher was responsible for the teaching, though we discussed in advance on a weekly basis both the materials and appropriate teaching style. Three pre-service teachers served as assistants and helped with videotaping and interviewing students and with analyzing the data.

The teaching experiment was carried out in an eighth-grade class with thirty students in a state school in the center of a Dutch city. The thirteen-year-old students in this study were being prepared for pre-university or higher vocational education (forty percent of Dutch students attend these types of education. The remaining sixty percent of students are prepared for regular vocational education). These eighth-grade students had no prior instruction in statistics; they were acquainted with mean, bar charts, and line graphs, but not with dot plots, histograms, box plots, or distribution.

5. Example: Diagrammatic reasoning about bigger and bigger samples

The activity of larger samples consisted of three cycles of making sketches of a hypothetical situation and comparing those sketches with graphs displaying real data sets in order to stimulate students to use more sophisticated representations of data distributions. In the first cycle, students had to make a graph of their own choice of a predicted weight data set with sample size ten — a sample size they thought was reasonable. The results were discussed by the teacher to challenge this small sample size, and in the subsequent cycles students had to predict larger data sets (one class, three classes, and all eighth graders in the province).

5.1. *First cycle of predicting a sample*

Figure 8 shows examples of three different types of diagrams the students in the class made to show their predictions. The work of three students in particular (here called Russ, Chris and Sandra) will be used for extended discussion. Despite a clear hint offered by the teacher to use a dot plot as in Minitool 2, there were three value-bar graphs (such as in Minitool 1, e.g., Ruud's diagram) and eight with only the endpoints (such as with the option of Minitool 1 to 'hide bars,' like Chris's diagram); in addition to nineteen dot plots (such as in Minitool 2, like Sandra's diagram). This means that their diagrams were heavily influenced by their experiences with the Minitools, and it could well be that several students found

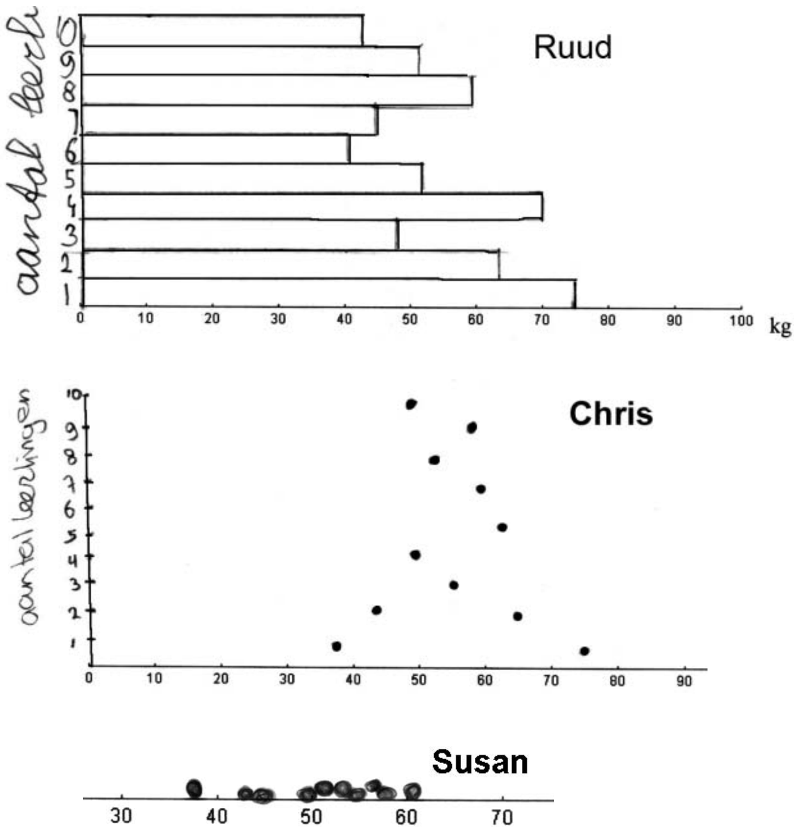

Figure 8. *Student predictions (Ruud, Chris, and Sandra) for ten data points (weight in kilograms)*

the diagram of Minitool 1 easier to understand than of Minitool 2. For the remainder of this section, I will use these three students as a focus group, because their work gives an impression of the variety of the whole class. The learning abilities of these students varied considerably: Ruud and Chris's report grades were in the bottom third of the class, whereas Sandra had the best overall report score of the class across all subjects.

After this first step of diagrammatic reasoning, students making diagrams, the teacher showed three samples of ten data points on the blackboard and asked students to compare their own diagrams (figure 8) with the diagrams of the real data sets (figure 9) to stimulate reflection on the graphs — the third step of diagrammatic reasoning. Experimenting with

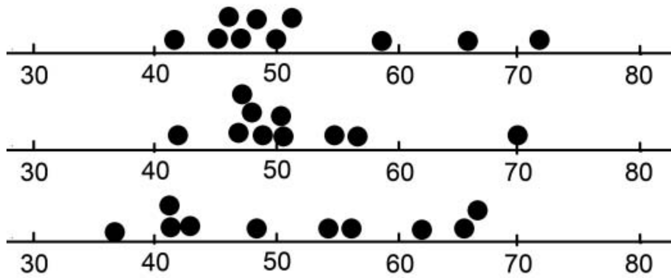

Figure 9. *Three real data sets in Minitool 2 (weight in kilograms)*

similar diagrams was done in the first and third lesson, when students worked with the Minitools.

There was a short class discussion about the graphs with real data before students worked for themselves again. The point relevant to the analysis is that students started using predicates to describe statistically relevant features of the diagrams.

Teacher: We're going to look at these three different ones [samples in Figure 9]. Can anyone say anything yet? Give it a try.
Jacob: In the middle [diagram], there are more together.
Teacher: Here [pointing to the middle diagram of Figure 9] there are many more together, clumped or something like that. Who can mention other differences?
Jacob: Well, the lowest, I think it's all the furthest apart.
Rick: The mean is usually somewhere around fifty.

The written answers of the focus group were the following:

Ruud: Mine looks very much like what is on the blackboard.
Chris: The middle-most [diagram on the blackboard] best resembles mine because the weights are close together and that is also the case in my graph. It lies between thirty-five and seventy-five kilograms.
Sandra: The other [real data] are more weights together and mine are further apart.

Ruud's answer is not very specific, which was typical for most written answers in the first cycle. Chris used the predicate 'close together' and added numbers to indicate the range, probably as an indication of spread. Like Sandra many students in the class used predicates such as 'together,' 'spread out,' and 'further apart' to describe features of the data set or the graph. For the analysis it is important to note that the students used predicates (together, apart) and hardly any nouns (except for 'mean') in

this first cycle. The instructional activity focused on creating a language to describe and compare relevant features of diagrams. At this stage, students mainly used indices, predicates, and metaphors, but this was to change in the second cycle.

5.2. *Second cycle of predicting bigger samples*

The students generally appreciated that larger samples would be more 're-liable.' With the feedback students had received after discussing the samples of ten data points in dot plots, they had to predict the weight graph of a whole class of twenty-seven students and of three classes with sixty-seven students (twenty-seven and sixty-seven being the sample sizes of two real data sets from another school).

During this second cycle, all of the students made dot plots, probably because the teacher had shown dot plots on the blackboard, and because dot plots are less laborious to draw than value bars (only one student started with a value-bar graph for the sample of twenty-seven, but switched to a dot plot for the sample of sixty-seven). This would be an example of the need to use another type of diagram, and one that comes closer to a conventional way of representing distributions. In this second cycle, the written answers from the focus group of three students were:

Ruud: My spread is different.

Chris: Mine resembles the sample, but I have more people around a certain weight and I do not really have outliers, because I have ten about the seventy and eighty and the real sample has only six around the seventy and eighty.

Sandra: With the twenty-seven, there are outliers and there is spread; with the sixty-seven there are more together and more around the average.

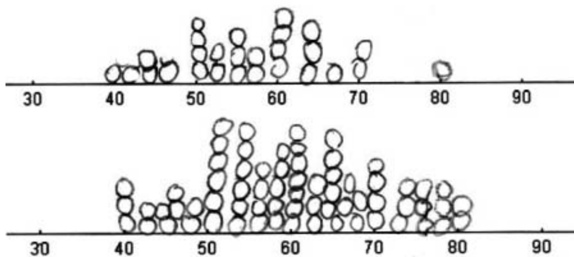

Figure 10. *Chris's diagram for one and for three classes (most diagrams looked like hers)*

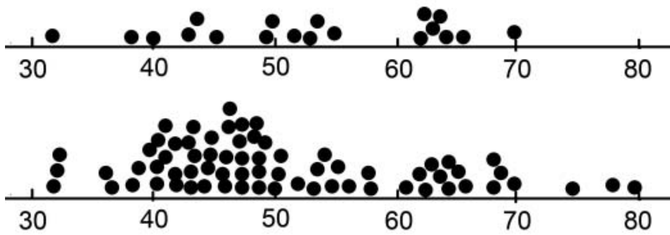

Figure 11. *Real data sets of size twenty-seven and sixty-seven for students from another school in Minitool 2*

In contrast to the first cycle, students used nouns instead of just predicates for comparing the diagrams. For example, many used the noun 'spread,' a hypostatic abstraction, whereas students earlier used only predicates such as 'spread out.' Of course, this does not always imply that if students use these nouns that they are thinking of the right concept. Statistically, however, it makes a difference whether we say, 'the dots are spread out' or 'the spread is large.' In the latter case, spread is an object-like entity that can have particular aggregate characteristics that can be measured (for instance by the range or the standard deviation). Other notions such as outliers, sample, and average are now used as nouns; that is, as hypostatic abstractions that can be talked about and reasoned with. It is not suggested that formulating and using these hypostatic abstractions are major learning steps. More important is the development of collateral knowledge 'in the background' allowing students to interpret and use signs.

5.3. *Third cycle of predicting a bigger sample*

In this last cycle of growing the sample, the task was to make a graph showing data of all students in the province, not necessarily with dots. The intention of asking this was to stimulate students to use continuous shapes and relate samples to populations, without making this distinction between sample and population explicit yet. The conjecture was that this transition from a discrete plurality of data values to a continuous entity of a distribution is important to foster reasoning about distribution as an object-like entity with which students could model data and describe aggregate properties of data sets.

The predictions of the focus group are presented in figure 12 and the written explanations were:

Ruud: Because the average [values are] roughly between fifty and sixty kilograms.

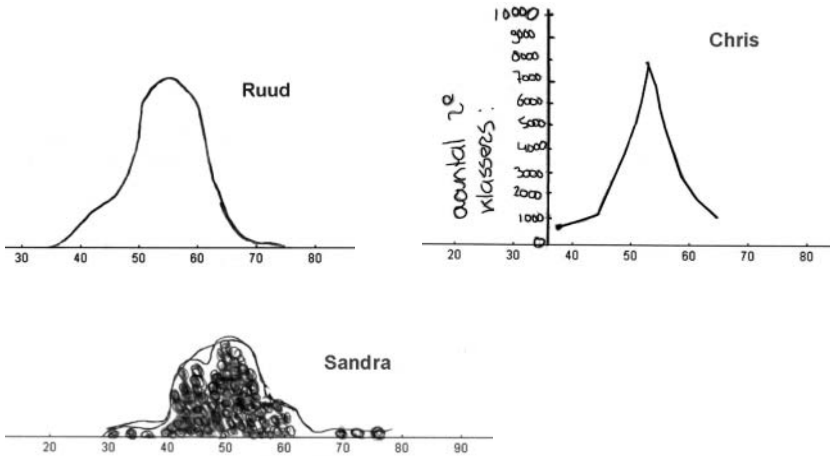

Figure 12. *Predicted diagrams by Ruud, Chris, and Sandra for weight data of all eighth graders in the province*

Chris: I think it is a pyramid shape. I have drawn my graph like that because I found it easy to make and easy to read.
Sandra: Because most are around the average and there are outliers at thirty and eighty [kilograms].

These three students were not alone in drawing continuous shapes: in total, twenty-three drew such shapes. During an interview after the fourth lesson, Ruud (like three others) literally called his graph a 'bell shape,' though he had probably not encountered that term in a school situation before. This could be seen as an example of guided reinvention. Chris's diagram was probably influenced by her collateral knowledge: during mathematics lessons students had learned to make and interpret line graphs. She introduced the vertical axis with frequency, though such graphs had not been used before in the statistics course. Sandra most likely started with the dots and then drew the continuous shape.

In the remaining lessons, we took the shapes proposed by students, supplemented by skewed distributions, to build upon their own experience and refine their notions of distributions and distribution aspects such as mean, median, spread, and skewness.

6. Discussion

In answer to the question of how students with little statistical background knowledge can be supported in reasoning about distribution, we

could say by progressive diagrammatic reasoning about bigger and bigger samples. The progressive sequence of activities of making diagrams of predicted situations and experimenting, in combination with well-guided reflection and observation encapsulates the key learning processes involved in diagrammatic reasoning. The three steps of diagrammatic reasoning clearly need not occur in a specific order. In more detail, these steps involve the following issues.

Making a diagram can be done on paper, in the head, and with a computer tool. In this case, students' paper diagrams were clearly influenced by the tools they had worked with. Constructing diagrams is an essential condition to get an object to refer to in communication; for instance, without objects to draw someone's attention to, we cannot use indices.

Experimenting with a diagram can also be done in different 'media,' but software has a clear advantage of being interactive and dynamic. Moreover, drawing hundreds of data points by hand is not something you ask students to do and yet some patterns in data only become visible if the data set is large enough. However, drawing diagrams by hand can also create the need for other, more aggregate diagrams such as dot plots or continuous sketches: when the predicted data set in the example became too big, students did not draw individual bars or dots anymore.

The interesting actions in the *reflection step* are predication and hypostatic abstraction, since these involve developing a language and concepts. This way of forming concepts can be stimulated in different ways. First, predicates should become topics of discussion so that they can be taken as entities in themselves — hypostatic abstractions (note that this term refers to both the process and its product). For example, talking about 'most' data can lead to talking about the 'majority'; or describing how dots are 'spread out' can lead to observing that 'the spread is large.' Second, students should be stimulated to be precise about what they refer to. For instance, if they use indexical words such as 'that' or 'it,' it is possible that they cannot express or do not know to which object they exactly refer. Precisely defining the topic of discussion is thus integral to conceptual development. Third, we should create situations in which students need conceptual objects as means of reasoning (cf. Sfard 1991), thus creating the need for shifting between representational and epistemological functions of signs. One way in which I tried this was by asking students to draw a diagram for an unreliable battery brand but with a high average life span, or to draw a diagram with little spread but a large range (Bakker 2004).

The analysis presented in this paper thus also illustrates what Hoffmann and Roth (this issue) call a dialectic relationship between the

representational and epistemological (knowledge constituting) functions of signs, and the importance of developing collateral knowledge on the other. In reasoning about concrete statistical problems (admittedly, the weight example is one of the least authentic) with a progression of diagrams, students enhance their collateral knowledge in relation to this series of diagrams, which are deliberately chosen to support their development of reasoning about distributions and which are both objects and means of communication. In this way, collateral knowledge coevolves with the activity of interpreting and producing diagrams.

More generally, this paper started with the question of how students can learn that a sign S (e.g., bell shape) refers to a mathematical object O (e.g., normal distribution) if they do not already know that S represents O. In brief, it is the activity of reasoning with a progression of signs in their dialectical functions and developing collateral experience that hints at a way to overcome the paradox. As Cunningham, Schreiber, and Moss stress, reasoning, according to Peirce, is 'to find out, from the consideration of what we already know, something else we do not know' (Cunningham et al. 2005: 178). The reasoning process as exemplified in this paper can be taken as a paradigmatic example of explaining learning as a process of diagrammatic reasoning in which opportunities for hypostatic abstraction occur. It is quite common in mathematics education research to study topics such as visualization, tool mediation, discourse analysis, abstraction, and reflection separately, but the concept of diagrammatic reasoning offers a way to capture all these major topics in one semiotic and epistemological framework.

Note

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