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Spatial structuring and the development of number sense: A case study of young children working with blocks

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ABSTRACT

This case study discusses an activity that makes up one of five lessons in an ongoing classroom teaching experiment. The goal of the teaching experiment is (a) to gain insight into kindergartners' spatial structuring abilities, and (b) to design an educational setting that can support kindergartners in becoming aware of spatial structures and in learning to apply spatial structuring as a means to abbreviate and ultimately elucidate numerical procedures. This paper documents children's spatial structuring of three-dimensional block constructions and the teacher's role in guiding the children's learning processes. The episodes have contributed to developing the activity into a lesson that could foster children's use of spatial structure for determining the number of blocks. The observations complement existing research that relates spatial structuring to mathematical performance, with additional insight into the development of number sense of particularly young children in a regular classroom setting.

1. Introduction

What repeatedly stands out from studies on early childhood development is the apparently natural drive of young children to go out and explore the world. In fact, researchers have drawn parallels between children, scientists and poets with regard to their sense of wonder and the intense way in which they experience the world (Gopnik, 2004; Gopnik, Meltzoff, & Kuhl, 1999). What is disconcerting, then, is that "early childhood education, in both formal and informal settings, may not be helping all children maximize their cognitive capacities" (National Research Council, 2005). Moreover, several researchers have warned about the apparent gap between children's informal, intuitive knowledge and interests, and the formal learning opportunities at the start of their schooling (Clements & Sarama, 2007; Griffin & Case, 1997; Hughes, 1986). This is reflected in many early elementary mathematics curricula that recognize the importance of number sense (Casey, 2004; Clements & Battista, 1992) without accrediting the spatial sense that children typically develop at a very early age (Ness & Farenga, 2007; Newcombe & Huttenlocher, 2000).

An overwhelming body of research has discussed the development of mathematical thinking in terms of either spatial sense or number sense. Relatively few studies, however, have considered what role early spatial sense may play in supporting the development of number sense. Such an association seems viable in light of studies that have related elementary students' spatial structuring abilities to their counting skills (Battista & Clements, 1996; Battista, Clements, Arnoff, Battista, & Van Auken Borrow, 1998) and early school mathematical performance (Mulligan, Prescott, & Mitchelmore, 2004; Mulligan,

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Mitchelmore, & Prescott, 2005; Mulligan, Mitchelmore, & Prescott, 2006). Research on how to support the development of number sense of particularly kindergarten children (aged 4–6 years) is important because around that age (a) children are building a more solid foundation for their number sense (Griffin, 2004a), while, at the same time, (b) they are expected to bridge their relatively intuitive and informal mathematical knowledge with the more complex mathematics of a formal school setting (Clements & Sarama, 2007).

This "revolutionary" period in the development of mathematical thinking (Griffin, 2004a) has inspired the present ongoing investigation into the development of and association between early spatial sense and emerging number sense of kindergarten children. The aim of this research is to gain a more in-depth understanding of how the development of young children's number sense may best be supported in an educational setting (Van Benthem, Dijkgraaf, & De Lange, 2005; Van Nes & De Lange, 2007; Van Nes & Doorman, 2006). For this purpose, classroom activities were developed that are meant to encourage young children's use of flexible numerical procedures using their early spatial sense, and help them prepare for formal school mathematics.

In this paper, we present a case study in the form of one particular classroom activity that involves the counting of blocks in a three-dimensional construction. This activity is one of five lessons that were conducted in a classroom teaching experiment to investigate the development of kindergartners' spatial structuring abilities. The purpose of this paper, then, is to explore the lesson with regard to (a) kindergartners' spatial structuring ability and (b) the characteristics of an educational setting (i.e., the teacher and the instruction activity) that may support the development of spatial structuring abilities in three-dimensional settings. These abilities are expected to help the children to eventually gain insight into numerical relations that may help them to learn to abbreviate numerical procedures such as determining, comparing and operating with quantities.

After investigating the theoretical association between spatial and number sense, we document the children's responses to the activity as well as the proactive role of the teachers. In light of current theories on the role of spatial structuring for supporting the development of number sense, this should encourage the implementation of such an activity in a kindergarten classroom for supporting the development of children's insight into numerical relations.

2. Theoretical background

2.1. Defining number sense

Number sense can broadly be defined as the ease and flexibility with which children operate with numbers (Gersten & Chard, 1999). In summarizing an extensive list of components, Berch (2005) stated that

> possessing number sense ostensibly permits one to achieve everything from understanding the meaning of numbers to developing strategies for solving complex math problems; from making simple magnitude comparisons to inventing procedures for conducting numerical operations; and from recognizing gross numerical errors to using quantitative methods for communicating, processing, and interpreting information (p. 334)

Early quantitative abilities include children's ability to *subitize* and compare quantities by making correspondences (Clements & Sarama, 2007; Van den Heuvel-Panhuizen, 2001). Subitizing may be defined as an automated perceptual process that all people can apply only to small collections of up to around four objects (see also *perceptual subitizing*, Clements, 1999). As children progress in their ability to count, they discover easier ways of operating with numbers. They come to understand that numbers can have different representations and can act as different points of reference (Berch, 2005; Griffin & Case, 1997; Van den Heuvel-Panhuizen, 2001). The present research specifically focuses on awareness of quantities, on giving meaning to quantities and on relating the different meanings to each other (Van den Heuvel-Panhuizen, 2001), because such knowledge is necessary for determining a quantity (i.e., counting), for comparing quantities and for preliminary arithmetic. This requires insight into *numerical relations* which can be achieved through the structuring (e.g., splitting or decomposing and composing) of quantities (Hunting, 2003; Steffe, Cobb, & Von Glasersfeld, 1988). Such insight, in turn, underlies a well-founded number sense and determines the ease with which children progress to an understanding of higher order mathematical skills and concepts (Griffin, 2004b; Van den Heuvel-Panhuizen, 2001).

2.2. Defining spatial sense

Spatial sense encompasses the ability to "grasp the external world" (Freudenthal, in National Council of Teachers of Mathematics [NCTM], 1989, p. 48). Within the vast body of research that exists on spatial sense, the three main components of spatial sense that appear most essential for "grasping the world" and for developing mathematical thinking are spatial visualization, spatial orientation, and shape (cf. Clements & Sarama, 2007; Owens, 1999).

Spatial visualization involves the ability to mentally picture the movements of two- and three-dimensional spatial objects. In spatial visualization tasks, all or part of a representation may be mentally moved or altered (Bishop, 1980; Clements, 2004; Tartre, 1990a), requiring object-based transformations where the frame of reference of the observer stays fixed (Zacks, Mires, Tversky, & Hazeltine, 2000). Young children already apply spatial visualization skills, for example, when they imagine where in the kitchen they can find a snack.

The second component of spatial sense that we study is spatial orientation. This is what Clements (2004, p. 284) refers to in describing how we "make our way" in space. In spatial orientation, the self-to-object representational system is at work because the viewer reorients the imagined self (Hegarty & Waller, 2004). An example is when one examines two photographs to compare the position of the camera. This involves changing the egocentric frame of reference with respect to the environment while the relation between object-based and environmental frames of reference stays fixed (Hegarty & Waller, 2004). One can navigate through space by operating on relationships between different positions in space with respect to one's own position (Clements, 2004; Tartre, 1990a, 1990b).

We refer to the third component of spatial sense as shape. Similar to spatial visualization, it has to do with mentally manipulating spatial forms from a fixed perspective (McGee, 1979; Owens & Clements, 1998). It involves making reference to shapes and figures as well as to familiar structures such as one's own body. Communicating about such structures increases children's vocabulary and enriches their imagination. Young children can separate forms from the figure in which they are embedded using gestalt principles. Insight into shapes and their relations enables children to make reference to familiar figures such as one's own body, to geometrical figures such as mosaics, and to geometric patterns such as dot configurations on dice or dominoes (cf. Clements & Sarama, 2007).

In agreement with early factor analytic studies that have shown the positive effects of spatial skills on mathematics achievement in later school years (Bishop, 1980; Clements, 2004; Guay & McDaniel, 1977; Tartre, 1990a, 1990b), the NCTM Standards (1989, 2000) have strongly recommended increasing the emphasis in early mathematics curricula on the development of spatial skills through the teaching of geometry and spatial sense. This acknowledges that both spatial and number sense entail the core of mathematics in the early years. The aim of the present research is to investigate whether and, if so, how the development of these two constructs, which are both essential to young children's mathematical development, may be related. It is expected that the development of children's number sense, as defined above, could benefit from a greater curricular focus on children's spatial sense, specifically in terms of their spatial structuring ability.

2.3. Spatial structuring as a common factor

In defining number sense, we described how the ability to (de)compose quantities is essential for the development of insight into numerical relations. Regarding spatial sense, we now zoom into a relationship that we see between each of the three components of spatial sense, and part-whole relations as well as the (de)composition of spatial objects (cf. Clements & Sarama, 2007). First, in spatial visualization, the ability to manipulate mental images can support children in rearranging objects to explore their composition. Second, the spatial structuring factor in spatial orientation involves integrating previously abstracted items to form new structures. Third, insight into shapes helps children to perceive parts and wholes of geometric patterns, congruence, symmetry, and transformations. We have taken this to suggest that the three components share a *spatial structuring ability*. Therefore, in this research we focus on spatial structuring to see how it may influence young children's ability to (de)compose quantities for gaining insight into numerical relations.

2.3.1. Defining spatial structuring

Spatial structuring may be defined as

> the mental operation of constructing an organization or form for an object or set of objects. Spatially structuring an object determines its nature or shape by identifying its spatial components, combining components into spatial composites, and establishing interrelationships between and among components and composites. (Battista & Clements, 1996, p. 503)

The focus in this research on spatial structuring as one particularly important factor in the development of number sense, particularly in terms of numerical relations, is inspired by how children typically begin to formalize their understanding of quantities by associating a certain quantity with spatial structures. Examples of such spatial structures are finger patterning and recognizing a quantity in the configuration of dots on dice. In order to organize and make sense out of such visual information, the process of perceiving structure requires a child to spatial visualize.

The mental extraction of structures from spatial configurations (i.e., identifying a "gestalt") is also what Arcavi (2003) found to aid his students' counting processes. For Arcavi's students, the "gestalt" could involve "breaking and rearranging the original whole" or "imposing an 'auxiliary construction' whose role consisted of providing visual "crutches," which in themselves were not counted, but which supported and facilitated the visualization of a pattern that suggested a counting strategy" (Arcavi, 2003, p. 229). One can see how young children can also use "gestalts" to, for example, rearrange objects that are to be counted. The spatial structure that subsequently arises can help the child to read off the quantity and hence abbreviate the counting procedure (e.g., creating two rows of three to see six).

When young children are asked to determine the quantity of a randomly arranged set of objects, they initially tend to count each object. As the set grows, this procedure eventually confronts them with the difficulties of keeping track of which objects have already been counted, and with the time-consuming process that accompanies the counting of larger sets. Hence, the benefits of applying spatial structure to mathematical problems are evident when reading off a quantity (i.e., seeing the quantity of six as three and three), when comparing a number of objects (i.e., one dot in each of four corners is less dots than the same configuration with another dot in the center), when extending a pattern (i.e., generalizing the structure) and when building with blocks (i.e., relating the characteristics and orientations of the constituent shapes and figures). As such, children's early ability to grasp spatial structure appears essential for developing general mathematical abilities such as insight into numerical relations, ordering, comparing, generalizing and classifying (NCTM, 2000; Papic & Mulligan, 2005).

2.3.2. Spatial structuring ability and mathematical performance

This paper builds especially on research by Battista and Clements (1996) and by Mulligan et al. (2004, 2005, 2006). In studying how third, fourth, and fifth grade students count cubes in a 3D array, Battista and Clements (1996) and Battista et al. (1998) found the spatial structuring abilities of students to provide the input and organization for the numerical procedures that the students used to count an array of cubes. Spatial structuring is also influenced by the students' attempts at counting, and it is based on how students physically and mentally act on a spatial configuration. Spatial structuring as a type of organization is considered to contribute to insight into important mathematical concepts such as patterning, algebra, and the recognition of basic shapes and figures (Mulligan et al., 2006; Waters, 2004). From this, Battista and Clements concluded that spatial structuring fundaments later algebraic and spatial thinking.

More support for associating young children's spatial sense, in terms of their spatial structuring abilities, with their emerging number sense, comes from research on children with mathematical learning difficulties. Several studies in this area have shown how low-achieving children tend to not use any form of spatial or mathematical structure and instead continue to rely on superficial features (Butterworth, 1999; Mulligan et al., 2005; Pitta-Pantazi, Gray, & Christou, 2004; Van Eerde, 1996). Further, in an analysis of structure present in 103 first graders' representations in various mathematical tasks, Mulligan et al. (2004) were able to code the individual profiles as one of the following four stages of structural development:

(1) *Pre-structural stage*: representations lack any evidence of mathematical or spatial structure; most examples show idiosyncratic features.
(2) *Emergent (inventive-semiotic) stage*: representations show some elements of structure such as use of units; characters or configurations are first given meaning in relation to previously constructed representations.
(3) *Partial structural stage*: some aspects of mathematical notation or symbolism and/or spatial features such as grids or arrays are found.
(4) *Stage of structural development*: representations clearly integrate mathematical and spatial features. (Mulligan et al., 2004, pp. 395–396).

Mulligan et al. (2004, 2006) found that mathematical structure in children's representations generalizes across various mathematical domains, and that children with a more sophisticated awareness of patterns and structures excelled in mathematical thinking and reasoning in comparison to their peers and vice versa. Although the correlations could not reveal causal effects, the researchers concluded that young children are capable of understanding more than unitary counting and additive structures, and they suggested that instruction in mathematical patterns and structures may stimulate children's learning and understanding of mathematical concepts and procedures. This coincides with the conclusions of Battista et al. (1998), that it is the student who has to learn to construct a meaningful structure, as a spatial structure is not embedded in an object. As such, both Battista and Clements (1996), Battista et al. (1998) and Mulligan et al. (2004, 2005, 2006) highlight the role of children's ability to spatially structure in the development of their mathematical competencies.

2.3.3. Kindergartners' spatial structuring ability and number sense

The research above gives reason to believe that spatial structuring ability is an important factor in the development of number sense, particularly with regard to insight into numerical relations. This paper contributes to this research with observations from a teaching experiment with kindergartners, children who are younger than the students in Clements and Battista's studies. Studying and stimulating the development of number sense of kindergartners is important for two main reasons. First, kindergartners' progress in mathematics strongly depends on the extent to which the school's instruction succeeds at relating to the child's level of understanding, and at bridging the child's initially informal learning methods with relatively formal teaching procedures (Clements & Sarama, 2007). This is crucial for preventing learning difficulties that may arise at a later stage in formal schooling (Allsopp, Lovin, Green, & Savage-Davis, 2003; Clements & Sarama, 2007; Starkey, Klein, & Wakeley, 2004). Hence, a greater understanding of the association between kindergartners' spatial structuring ability and number sense may help to design instruction that can build on children's early spatial structuring abilities and support the development of number sense.

The second importance of studying the development of kindergartners' number sense has to do with the changes that children of this age are said to experience in their understanding of numbers and quantities. In the Central Conceptual Theory (Griffin, 2004b; Griffin & Case, 1997), young children's ability to compare quantities and their ability to count are described as two initially separate competencies. Children at the age of four have difficulty coordinating these competencies, but an essential developmental step occurs by the age of five or six when these two competencies merge into "a single, superordinate conceptual structure for number" (Griffin, 2004a, p. 40). Such a conceptual structure is said to integrate children's intuitive understanding of quantity with knowledge about number, enabling children to use numbers without having to rely on objects that are physically present. This new conceptual structure provides children with the conceptual foundation for number sense that is believed to fundament formal mathematics.

2.3.4. The present study

The present study extends Mulligan et al.'s (2004, 2006) research in two ways. First, whereas Mulligan et al. analyzed structure in first graders' representations as they solved tasks across a range of mathematical domains, the focus of the study in this paper is specifically on kindergartner's procedures for determining quantities. Secondly, Mulligan et al. administered tasks to the children individually, while the activities for the teaching experiment in this paper are designed for interactive classroom teaching in accordance with the principles of realistic mathematics educational (Freudenthal, 1971, 1973, 1978; Treffers, 1987). In realistic mathematics education, mathematics is seen as "a human activity" which is to be "reinvented" by the students. The instruction is intended to "guide" students towards an understanding of mathematics that derives from what is meaningful to the students. This requires bottom-up reasoning from the perspective of the student rather than a mere top-down imposition of the curricular learning goals.

Mathematical learning in the present research is placed in a social context with a focus on both the classroom community as well as the individual perspective. To warrant ecological validity, the classroom implementation of the activity was meant to create a concentrated and therefore more powerful educational setting in which the children could interact with each other and with the teacher and learn about how to make use of structure to abbreviate numerical procedures (cf. Cobb & Yackel, 1996). Observations of this, in turn, contributed to understanding how the children's spatial structuring abilities could best be supported in class.

In agreement with the Central Conceptual Theory, the central hypothesis of this study is that one way children may learn to relate the understanding of quantities with numerical procedures and fundament their number sense, is through discovering how use of spatial structures can abbreviate and elucidate numerical procedures such as determining, comparing and operating with quantities. The lesson discussed in this paper was part of a teaching experiment that was intended to support children in achieving such a crucial developmental step.

The study in this paper addresses two main questions:

1. What characterizes kindergartners' spatial structuring abilities of three-dimensional block constructions?
2. How may teachers promote the use of spatial structure for ultimately abbreviating and improving kindergartners' numerical procedures?

3. Data collection and method of analysis

The case study in this paper concerns one of five instruction activities that were conducted with a class of kindergartners in the second cycle of an exploratory teaching experiment. The teaching experiment followed the guidelines of design research in a realistic mathematics education setting. This entails innovative curriculum development and understanding of teaching and learning processes (Gravemeijer & Cobb, 2006). In essence, the interest lay more in finding out *how* an activity worked than in merely establishing *that* it worked (Freudenthal, 1991). This means that a sequence of activities was developed to implement in class, and data was gathered on the learning processes of the children and on the effect of the activity. The data, in turn, provided the input for a second run through the teaching experiment with a set of revised activities. This iterative procedure contributed to further insight into how the children were learning to make use of spatial structures as a means to abbreviate their numerical procedures.

Data for this activity was collected in the spring of 2007 during an hour-long lesson in a kindergarten class at a local elementary school. The class consisted of 23 children ranging in age from 4 to 6 years. The researcher discussed the activity with the teacher in the half hour before the lesson to prepare her for teaching the class independently. With the teacher conducting the lesson, the role of the researcher was to stand by and ask the children additional questions, to help coordinate the activity, to take field notes, to videotape the lesson and to make last-minute changes to the activity in case that seemed necessary for providing relevant information for the research.

The purpose of the activity was to study whether and how children would use the spatial structure of a construction of five or ten plastic blocks to abbreviate the procedure for determining and comparing the number of blocks in the construction. Since realistic mathematics education takes into account the reflexive role between symbolizing and understanding concepts, the particular use of manipulatives in this study concerns mathematical activity that is intertwined with children's experiences and current levels of understanding (Gravemeijer, 1998). Hence, instead of merely "teaching by showing," the children themselves were meant to participate in the development of their insight into the use of spatial structure for supporting numerical procedures.

The first part of the activity started with a classroom discussion with the purpose of introducing the children to structured and unstructured three-dimensional constructions. The teacher first discussed with the children that one "block" in a "construction" stood for one "room" in a "house." Then she explained that an "easy house" was a house that was built in such a way that made it more accessible to count its number of blocks (or rooms) than a so-called "difficult house." Finally, she checked that the children understood her terminology by having them "build a house with three rooms" and seeing how they construct it using three blocks.

The children and teacher were sitting around two constructions that were made up of uniformly colored Duplo® blocks. Each child was familiar with Duplo® blocks. These blocks are more convenient for the activity than solid wooden blocks because they can be fixed together. Children could therefore examine their constructions by, for example, lifting them off the table. One of the constructions was built symmetrically and the other was built asymmetrically (see (a) and (b) in Fig. 1). The term 'symmetry' is used to highlight the organized way in which the blocks are arranged, which can support the abbreviation of a numerical procedure such as counting. The teacher used the phrase "easy house" to communicate such symmetry to the children.

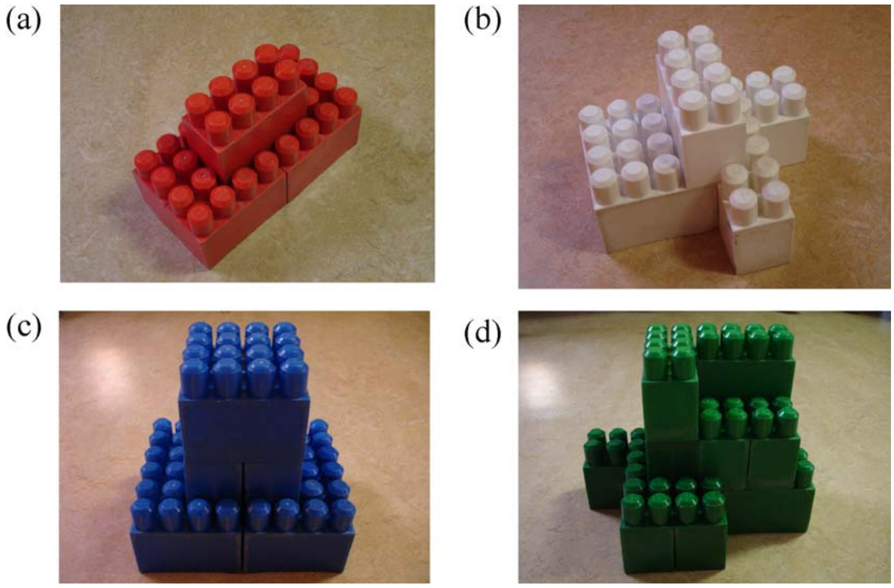

Fig. 1. The structured (i.e., symmetrically built) and unstructured (i.e., asymmetrically built) block constructions that were examined during the classroom discussion: (a) The structured construction of five blocks, (b) the unstructured construction of five blocks, (c) the structured construction of ten blocks, and (d) the unstructured construction of ten blocks.

After studying the two five-block constructions, the class studied and compared two ten-block constructions, again one built symmetrically and the other built asymmetrically (see (c) and (d) in Fig. 1). The teacher tried to take on a proactive role in initiating discussion and reflection on different spatial structures and on the way that spatial structures may affect numerical problem solving. She asked the children to determine the number of blocks in each of the houses and guided discussions about what seemed to make one particular house more accessible for being counted than another.

To gain more insight into the children's own constructions and understanding of spatial structuring, the second part of the study was meant to investigate whether and how the children would go about applying spatial structures on their own. An as representative focus group as possible was formed in close consultation with the teacher. The eight children in the focus group were each provided with a set of either ten, eleven, or twelve uniformly colored Duplo® blocks on the table in front of them. The teacher first asked the children to build a house which they thought was "easy" to count. After comparing and discussing the constructions, the children then built a house that was "difficult" to count. Finally, they were asked to each build both an "easy" and a "difficult" house with their blocks and show what they understood to be the difference of the constructions in supporting the counting of the blocks.

During the classroom discussion, the researcher videotaped the class. While the focus group was working on the activity and the rest of the class was working on other tasks, only the children in the focus group were videotaped. All the videos were supplemented with audiotapes that were recorded by the voice-recorder that the teacher carried around during the lesson. Additional sources of data included the block constructions that the children constructed in the focus group, the interviews that were conducted with each child before and after the teaching experiment, the children's scores on national school tests, the field notes that the researcher took during the lesson and the questionnaire that the teacher completed during the debriefing.

The videos were analyzed using the data analysis software program ATLAS.ti. Each video was segmented into clips based on sequences of observed interactions, negotiations and activities that appeared relevant to each didactical episode in the activity (cf. Andrews, 2004; Powell, Francisco, & Maher, 2003). Next, codes were linked to the clips to facilitate the organization of the raw data (Jacobs, Kawanaka, & Stigler, 1999). The analyses of the clips were guided by methods that derive from the constant comparative method for developing theory that is grounded in data (Strauss & Corbin, 1998). Through working with these clips, particular episodes in the activity were identified that appear to convey the children's insight into the spatial structures of the block constructions and that illustrate the role of the teacher in supporting the development of this insight.

4. Results and discussion

4.1. Exploring 3D spatial structures in a classroom setting

The purpose of the classroom discussion was for the children to explore differences between a structured and an unstructured 3D construction and the influence of structure on determining the number of blocks in the construction. The following episodes document the children's spatial structuring ability and the teacher's proactive role in the classroom activity. This should highlight how an educational setting could appeal to children's spatial structuring ability and inspire them to make use of this ability for abbreviating numerical procedures such as determining, comparing and operating with quantities.

4.1.1. Gauging the children's initial spatial structuring ability

To start off the activity, the teacher asked the children how many rooms they thought the red (symmetrically built) construction had. One by one, the children kneeled by the construction and confidently pointed to each block as they counted five rooms. The teacher then asked the class how many rooms were in the white (asymmetrically built) house, and whether they also thought it was "easy to see the number of rooms" in that construction. As more children were kneeling by the construction to count the blocks, some said it had four blocks while others said it had five or six. Most children had difficulty determining the number of blocks in the asymmetrical construction because they tended to make errors such as mistagging, miscounting, or confusing blocks. Two children tried to explain that it was difficult to count the blocks because of "how they were put together." Apparently, counting the number of blocks in the asymmetrical construction was not as easy for the children as it had first seemed to them. This is illustrated by Sam.

Sam:	[confidently answers from where he is seated] There are six!
Teacher:	You're saying that there are six. Well, go and have a look. How many are there really?
Sam:	[moves to the structure and points to the blocks as he counts] Five. [counts again starting on another side] But, if you look at it from this side, then there are six.

This episode illustrates how the unstructured arrangement of the blocks was a source of confusion for Sam because it hampered his ability to keep track of which blocks he had already counted. This coincides with what Battista and Clements' (1996) said about how spatial structuring provides input for counting. Indeed, although most of the children under- or over-counted the asymmetrical construction, they usually did count the structured constructions more confidently and accurately than they did the unstructured constructions. Finally, the teacher asked Kate to explain why she thought the rooms in the white house were not so easy to count.

Kate:	Because there are little openings.
Teacher:	Yes, little openings. . .that makes it a little different, doesn't it?

Kate and the teacher use the word "openings" to denote the gaps that exist in a structure that is not entirely solid. This shows how Kate was already trying to put into her own words how she interpreted the structure of the construction and what difficulties she had with counting the blocks.

After most of the children had had a chance to share with the class the number of blocks that they thought were in the constructions, James changed his mind and said that actually there were six (rather than five) blocks in the symmetrical construction. He thought that one block was covered by other blocks in the middle of the construction. It seems that James referred to what he had discovered to be "hidden" blocks in an asymmetrical construction, to now also question the number of blocks in a symmetrical construction. James appeared to be generalizing the difficulties that he had encountered in counting the blocks in the asymmetrical construction. It is a striking example of James' progress in spatial structuring because he was trying to visualize the structure of the construction and he used it to reason about the number of blocks. Indeed, during his next turn, he decided to lift the construction up so that he could determine the number of blocks from the bottom rather than only the side. Apparently, he realized that the side view can be confusing and that examining the lower layers of blocks from the bottom can offer him a better perspective of the arrangement of the blocks in the construction. This method indeed can help to abbreviate the counting procedure.

Another level of spatial structuring ability was illustrated by Anne who called out that she had thought of a "reken maniertje" (i.e., mathematical procedure) for determining the number of rooms in the symmetrically built construction.

Anne:	Four plus four is eight. And because there are four on the bottom and the large block on top is four, because it is also a square. And so four plus four is eight.

Despite her error in counting the number of blocks in the bottom of the construction, Anne had made more reference to arithmetic than the rest of the class had so far. Anne, in fact, had scored exceptionally high on national school tests and had used relatively sophisticated strategies to solve the problems in the pre-interview. Her approach to this activity further supports a developmental trajectory in children's spatial structuring ability, where high-achieving children are ultimately expected to reach an arithmetic level of thinking that reflects their insight into spatial structures (cf. Mulligan et al., 2005). This agrees with the development in children's counting ability that has been described as requiring perceptual or figural units (i.e., requiring the physical presence of the blocks to be able to touch or see them in order to count them properly) towards being able to manipulate abstract unit items (Steffe et al., 1988).

At the end of the activity, the teacher once again asked the children why they thought that it was easier to count the blocks of the blue (i.e., large symmetrical) construction compared to the green (i.e., large asymmetrical) construction. Ella started by saying that she could "see" that, but she could not tell the teacher what it was that she was seeing. This resembles how Ron answered that one is "flat" and the other is "very tall." Kate came closest to an explicit comparison of the two structures in saying that "the blue construction has no openings while the green one does." This suggests that she was aware of a difference in symmetry between the two constructions. In light of the previous observations, these three suggestions broadly summarize the different levels of thinking about spatial structure that the children seemed to be at by the end of this classroom discussion. This resembles Mulligan et al.'s (2004) first three of the four general stages of young children's structural development, which range from representations that lack any mathematical or spatial structure (like Ella) to representations that show some elements of structure (like Ron) to representations that integrate mathematical and spatial structural aspects (like Kate).

The extent to which these children used spatial structure to count the blocks in the construction is also analogous to the strategies that Battista and Clements (1996) classified in their research. These strategies range from counting all the cubes in the array individually to conceptualizing the cubes as if they filled space, and organizing the cubes into layers. The students in Battista and Clements' study who spatially structured the cubes into layers, often skip-counted or multiplied to find the total, whereas the students who did not coordinate the cubes into a structure counted and often recounted the cubes one by one. The episodes in this activity support Battista and Clements' findings in that they cautiously suggest a relatively foregoing (i.e., at kindergarten level) developmental sequence that involves initially becoming aware of spatial structure (like Ella) and then using that awareness to abbreviate counting procedures (like Anne).

With regard to gauging children's insight into the structure of the constructions, it appears that the children could confidently determine the number of blocks in the symmetrical constructions, but they struggled with the asymmetrical constructions. Still, some children seemed to implicitly understand why the blocks in the asymmetrical constructions were difficult to count, because they could explain it as having something to do with "how the blocks were fit together." What makes the variety of children's explanations significant is that, in reference to the first research question, their different levels of thinking carefully suggest a sequence of abilities that illustrates how children differ in the ways that they use their rather implicit insight into spatial structure for counting blocks in a construction. The challenge for teachers is to support the children in advancing their initial insight into spatial structure towards using the constructions to abbreviate their counting procedures.

4.1.2. The role of the teacher in advancing children's spatial structuring ability

In this section, several didactical issues are highlighted that determine the effect of an educational setting on the development of children's spatial structuring ability. The teacher played a proactive role from the start of the activity in the way that she instantly took on an inquisitive and stimulating stance. For example, she repeated the children's informal wording rather than introducing her own description of the construction, and she encouraged the children to question the situation. Her attitude welcomed the children's enthusiasm and curiosity, setting the stage for the rest of the activity. It is this attitude that fits well with teachers' guiding role in realistic mathematics education and, more specifically, with Battista et al.'s (1998) suggestion that children must be guided in learning to use and reap the benefits of spatial structuring. This is illustrated by how, rather than simply telling the children, the teacher tried to guide Sam into analyzing the arrangement of the blocks in the construction.

Teacher:	[Sam kneels closer to the construction and counts again] But is that block on the side a separate one?
Sam:	[shakes his head and looks around to see what the others are suggesting]

Since Sam was still confused, the teacher recapitulated the situation.

Teacher:	So, how many blocks are there? [the children have no suggestions and some join Sam to inspect the construction]. Every block is one room [Sam still looks around for input from others. Since no one has suggestions, he puts his hand on two of the blocks and looks at the teacher for support]. That's two, and then there's one on top. . .that's. . .
Sam:	Three.
Teacher:	And then one at the front [Sam points to a block]. No, not that one [Sam hesitantly tries another block]. What do you think Sam?
Sam:	[quietly answers] Five. . .six.
Teacher:	Five or six? You're not very sure?
Sam:	[shakes his head and walks back to his chair]

This last episode illustrates the challenges that coincide with guiding children towards looking for and deciding to make use of spatial structure in the construction without imposing this on them. Judging from Sam's relatively hesitant responses, it seems as though he was mainly trying to follow the teacher's instructions without really understanding what the questions meant. Perhaps the teachers' questions were barely appealing to Sam's level of understanding and that, as a result, Sam was trying to "reach" to the answer that the teacher was aiming for.

Another example of the teacher's proactive role in supporting children's spatial structuring ability, occurred when the teacher let James take the construction apart to check the number of blocks. In this way, she was encouraging James to explicate the spatial structure of the construction. It gave James more insight into the structure of the construction by allowing him to actively take part in his own learning process. Further, the different effects of the structures became more apparent to him when he rebuilt the symmetrical construction correctly, while, after also taking apart and counting the number of blocks in the asymmetrical construction, he rebuilt that construction relatively hesitantly and incorrectly. The significance of this is reiterated in Battista and Clements' (1996) statements about how children need to act with manipulatives in order to understand spatial structuring.

According to the definition of spatial structuring as stated at the beginning of this paper, spatial structuring implies the need for perceptual and motor actions towards what is to be structured. This relates to a counting ability that develops from requiring perceptual unit items or figural unit items, to using motor unit items (i.e., using motor acts as a substitute for a perceptual or figural item) and eventually to using abstract unit items (Steffe et al., 1988). As such, the decomposition of the structure in the previous episode, simplified the question from a rather abstract level to a level that those children who require the immediate or figural presence of each block for counting, could relate better to.

At the same time, however, Anne's insight into the arithmetic structure of the construction ("there are eight because I know four plus four is eight"), gave the teacher an important opportunity to introduce the class to more formal procedures for counting the blocks. She did this by asking Anne to recount the blocks from a different angle. In this way, the teacher exposed the class to two different methods of counting the blocks. One was the perceptual counting approach, which most of the class was applying, and the other was the more abbreviated and relatively abstracted method of determining the whole by referring to separate components of the spatial structure.

Kate seemed to catch on to Anne's method quickly as she explained to the teacher that she had already seen six blocks on the bottom because "three and three is six." Although this would have been a good opportunity for the children to explicitly reflect on why this more arithmetic procedure could abbreviate the counting procedure, the teacher did not ask Kate to elaborate on her reasoning so that the class could come to understand her thinking. Instead, the teacher tried to involve the whole class more arithmetically by asking them "and four and six is. . .?" This exemplifies a teaching style that focuses on a future learning goal (formal addition) while neglecting the children's own level of understanding. The problem with such a top-down focus is that the children may learn to reproduce what the teacher asks (counting and basic addition), yet still have trouble applying the skills to other practical situations that require more insight and understanding (i.e., higher order operations, Allsopp et al., 2003; Freudenthal, 1991).

Up to this point in the activity, the children stayed focused on the two constructions independently from one another. They neglected to compare the structures and that is what seemed to prevent them from relating the structures to their counting strategies. Hence, in an effort to advance the children's understanding, the teacher tried to guide them further towards attending to and comparing both constructions:

Teacher:	So why is it that, for one house it sometimes looks as if there are six blocks, and for the other one, I immediately see that there are five? How can that be, do you think?
Sam:	It looks like there are more.

To stimulate the discussion beyond merely a perceptual comparison between the sizes of the two houses, the teacher introduced the children to the two ten-block constructions. The children spontaneously started to count the blocks in the construction, but none of them counted correctly. This provided the teacher with a stimulating starting point for discussing why they may have counted incorrectly, and how the spatial structures of the constructions could be of use to them. To further support children's spatial structuring and their insight into numerical relations, the teacher turned to the children's own constructions in a focus group setting.

4.2. *Creating structure in a focus group setting*

Battista and Clements (1996, 1998; see also Cobb, Yackel, & Wood, 1992) found that students must act on manipulatives in order to learn to create spatial structures. Students do not "read off" structures from objects since structure is not "in" them. Instead, proper instruction should encourage students to spatially structure by themselves, because the very act of counting and reflecting on these structures can stimulate spatial restructuring and meaning making. This coincides with the first levels of counting using perceptual and figurative units as a step up towards more abstract numerical procedures (Steffe et al., 1988). As such, for the children to understand the use of spatial structuring for counting, the second part of the activity directed the children in the focus group specifically towards acting and reflecting on their own constructions.

4.2.1. *Exploring symmetry in constructions that are "easy" to count*

In building constructions that they thought would be "easy" to count, most of the children made a construction that resembled the structured construction that was discussed in class earlier. They associated the question directly with the classroom discussion that featured this construction as having a structure that was "easy" to count. Ron, however, built a solid, four-layered structure, Fiona built a solid construction that resembled a house with a pointed roof, and James built a construction that looked to the other children like "an apartment building." He had piled up six layers of two blocks, each layer arranged orthogonally to the next (see Fig. 2).

James said that his construction was easiest to count because he had "only made a pile of blocks." He showed the teacher how he could easily and correctly count the number of blocks by pointing to each of the blocks separately while turning the structure on the table. When the teacher asked James why he thought it was easier to count the blocks in his structure than in, for instance, Ron's construction, James responded with an explanation that illustrates his understanding of structure:

James:	[pointing to the blocks in his construction] Because I, with four blocks like this, piled them up, piled up, piled up. Look, I'll do it again [takes the construction apart]. I just do it like this [starts rebuilding the structure], and those like this, and those again like this.

From the confident way in which James was explaining his structure to the group, it seems that he may have observed a type of organization in how the layers were repeating and in how straightforward it was for him to reconstruct the layers after taking them apart. When James said that he could just "see" the number of blocks, the teacher asked him how many blocks there were altogether:

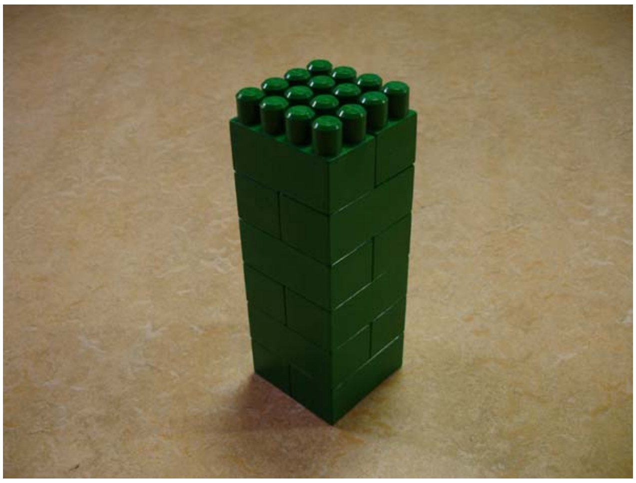

Fig. 2. The structured, "easy" construction that James built out of his twelve blocks. Although James referred to the layers in the construction as the reason for why "the house was easy to count," he had trouble making use of the layers to actually abbreviate the counting procedure.

Teacher:	Can you see how many there are without turning the house?
James:	[while looking at the structure and examining its sides] Two, six [. . .] two, six [. . .] two, four, six, and then eight, eleven, thirteen! [looks back up at the teacher]
Teacher:	Wrong, what comes after eight?
James:	Ten!
Teacher:	Yes, if you keep adding two, right? Yes, so it's two, four, six, [James joins in] eight, ten. Yes, that's really good!

This episode could suggest that James presupposed the advantage of layering his blocks in order to count them in an abbreviated way. With help from the teacher, he was able to make reference to this structure as he counted by twos. The errors and subsequent corrections in his counting could coincide with what Battista and Clements (1998) consider to be part of learning, since in their research "students' progression to more sophisticated structurings often resulted from perturbations caused by recognition of inadequacies in current structurings" (p. 529).

A second interpretation of the episode, however, is that although James may have presupposed the advantage of layering his blocks, he may not have been aiming for counting by twos in the way that the teacher expected him to. In fact, the teacher did not take the opportunity to try to understand why James made the mistake of counting by twos, or rather calling out numbers by skipping one, and then unexpectedly saying eleven after correctly counting "two, four, six, eight." This begs the question as to the extent to which James understood the counting by twos procedure if he made such a mistake. Although he succeeded in counting by twos all the way up to eight, he may not have mastered counting by twos for larger numbers yet.

Battista and Clements (1998) also note that it is practically impossible for children to understand what it means to use area formulas for multiplying objects in arrays if they do not "see" the row-by-column structure. Analogously, James may probably not understand the counting by twos procedure, much less the counting of many objects in a structure, if he has not yet grasped the nature of the structure in the layers of his construction. As such, the teacher could have asked James to rebuild the layers and explain *why* he said that eleven came after eight. In this way she would have created a more bottom-up educational setting that could help James merge his insight into structure and counting and master the counting by twos procedure.

Following the discussion about James' construction, the teacher tried to show each of the children how counting by twos (and threes in the case of Ron's construction) can be more effective for abbreviating the counting of the blocks in their constructions than one by one counting. Although the children still followed the teacher's procedures hesitantly, the important outcome of this part of the activity is that the children had all built symmetrical constructions in the first place. This points to children's early understanding of an "easy" construction as one that is structured and relatively symmetrical. Although the teacher wanted to help the children by showing them how they could abbreviate the counting of the blocks, more work is needed to introduce a more bottom-up perspective to this rather top-down manner of teaching. It also shows that the children need to develop the proficiency to count by twos (or threes) in order to be able to use a structure of twos for abbreviating their counting procedures.

4.2.2. Perceiving asymmetry in constructions that are "difficult" to count

In the next part of the activity, the children were asked to build constructions with structures that made it very difficult to count the blocks. This focus group setting encouraged children to compete with each other to build constructions that were the "easiest" or most "difficult" to count. Ron, for instance, called out that Dora's construction was "still too easy to count," and James repeatedly wanted to rebuild his construction into an "even easier house." Yet, none of the children could explain what it was that made the "difficult" house different from the "easy" one. The teacher therefore challenged the children by telling them that she did not find the houses that they were building very difficult to count. The children also commented

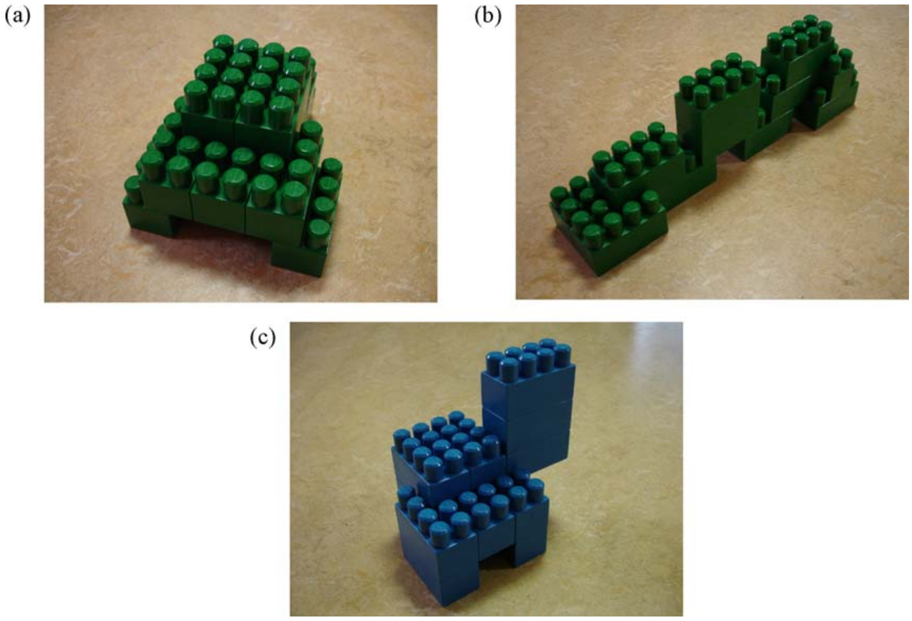

Fig. 3. The unstructured, "difficult" constructions that were built in the focus group by (a) James, (b) Ron, and (c) Sara.

on each other's constructions. This elicited more discussion and stimulated the children to show and tell what they thought a "difficult" construction would look like. Sara, for example, tried to convince the teacher that the house she was making was "difficult" because she "always has to turn her house around to count everything." This is an intriguing explanation in that it illustrates how Sara related the structure of her house to the tedious and potentially confusing procedure of having to count each of the blocks. Hence, the social setting of the activity supported the children to learn to give meaning to spatial structure since, in this way, they were responding to the challenges that they encountered as they built, compared and evaluated their constructions.

Once all the children were satisfied with their "difficult" constructions, the teacher made one more attempt to see what the children understood about structure and its use for counting the blocks. James' "difficult" construction is depicted in Fig. 3a.

James: It's just too hard to tell you. I can't count them. It's too hard.
Teacher: It's too hard. How come it's too difficult?
James: Because, because it is. Yes. [gesturing with his hands as if he is filling a mixing bowl] I just made it, like this, bla, bla, bla, and then this came out of it.
Teacher: Did you make the other house, the easy house in the same way or did you make it in a different way?
James: Yes.

In comparing the episode in which James described his "easy" house with this episode, it is clear that James experienced a difference in how he built the "easy" and "difficult" houses. Yet, his responses to the teacher's questions reveal that he could not elaborate on the nature of the difference. His "bla, bla, bla" response may be a clever indication of an unstructured approach of just haphazardly putting blocks together. The teacher, in turn, was having difficulty with asking effective questions that would help the children explicate the structures. Ron was able to reveal a little more about how he perceived structure in his "difficult" construction (see Fig. 3b):

Teacher: Can you count them easily Ron? [Ron shakes his head] Why not?
Ron: [looks at his elongated construction] Because they're all on top of each other and next to each other. This one can move.
Teacher: And the other one, the one you built first, that one was easier? [Ron nods] How can that be? What did you do differently with that one?
Ron: [gestures with his hands] Because that was piled up and here I made a fence.

Ron's "easy" construction was a block that consisted of four layers of three rectangular blocks. His rather informal choice of words was sufficient to communicate how he reflected on his insight into different spatial structures and the corresponding counting procedures, because he explicitly compared the "easy" construction to the "difficult" construction. This brings under the attention how easily young children's insights can be under-appreciated if instruction focuses too much on the quality of children's verbal explanations (see also Hughes, 1986). Moreover, the role of the teacher's language use in young children's understanding also became apparent in the way that the teacher was encouraging the children throughout the lesson as they were building and rebuilding their constructions. She told them to "make strange rooms," referring to asymmetries in the building, and, in response, Sara showed the "strange rooms" in her construction. It was indeed asymmetrically built with various overhanging layers (see Fig. 3c). Her "easy" construction, in contrast, was a 3-layered symmetrical block with two rectangular blocks in each layer. The teacher's comment also inspired the other children to build increasingly different and complex structures which the children then compared to one another.

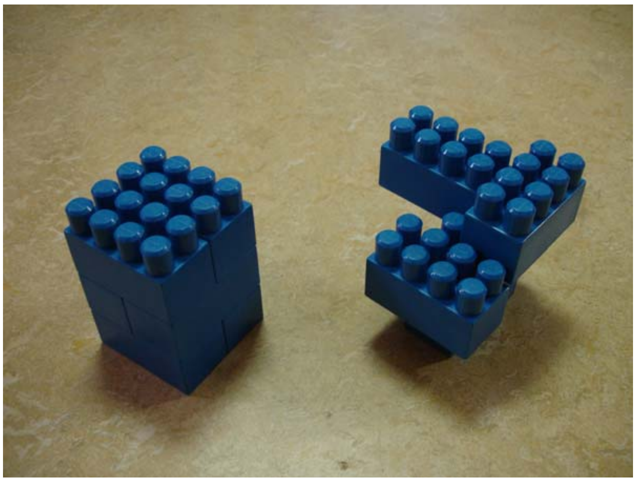

Fig. 4. The two constructions that Lisa built with her blocks. The left construction is structured and the right construction is unstructured. Lisa used six blocks for each of the two constructions.

In summary, interpreted in light of both the children's and the teacher's type of language use, it seems that the children not only appeared to have implicit insight into structure, but that they were also able to apply this insight to building their own constructions. Even the youngest children built surprisingly unstructured, asymmetrical constructions compared to the structured, symmetrical constructions that they built in the first part of the activity.

The episodes described in the paper thus far, have helped to gain insight into the various initial levels of spatial structuring ability of the children during the classroom activity and in the focus group. The role of the teacher throughout this activity was to encourage the children to make use of the structures for abbreviating and elucidating their numerical procedures. In the focus group, she supported the children in constructing their own "easy" and "difficult" houses and in helping them become aware of the difference between them the two types. Considering the difficulties that the children had with explicitly making use of the structures and explaining their strategies, the next challenge for the teacher was to encourage the children to further reflect on the difference between structures and their effects on numerical procedures. In this way, she could support children's insight into the influence of various types of structures on counting procedures. This may be an important step in giving meaning to how spatial structure can abbreviate numerical procedures.

4.2.3. Wrapping it altogether: comparing structures simultaneously

In a final attempt to guide the children towards a better understanding of how spatial structures can support the counting of the blocks, the teacher asked the children to each build two constructions out of their set of blocks; one construction with a structure that made it "difficult" to count the blocks and another with a structure that made it "easy" to count the blocks. The significance of this task is that the children, even the youngest, each built a clearly symmetrical and asymmetrical construction to physically show how the structure could make the numerical procedure either "easy" or "difficult." The children also enjoyed identifying which of the other children's constructions they thought was meant to be the "difficult" or the "easy" house.

A twist to this outcome, however, is that several of the relatively "difficult" houses were made up of more blocks than the "easy" houses. Dora, for example, built a bridge-like construction with many blocks to denote a "difficult" house. While it was difficult for her to hold the blocks in place, it was relatively easy to count each of them because the blocks were arranged in one single layer. Moreover, although the sequence of blocks was easy to count, this did not contribute to an abbreviated procedure for determining the number of blocks in the construction. Hence, what the children had not yet understood is that building a "difficult" construction is not always equally related to the number of blocks and to the difficulty of determining the number of blocks that it is made up of. It may have been more effective for the teacher to explicitly ask the children to make one "easy" (structured) and one "difficult" (unstructured) construction using an equal number of blocks in both constructions. Nevertheless, it was interesting to note that Lisa, for example, did build two differently structured constructions, each made up of six blocks (see Fig. 4).

Since one construction was clearly structured and the other was clearly unstructured, Lisa's constructions exemplify how, in this final part of the activity, she appeared to be able to differentiate and construct spatial structures based on how easily she believed that the blocks of the construction could be counted. Such insight into the difference between structured and unstructured objects lays the foundation for learning to make use of spatial structures for abbreviating and integrating increasingly complex spatial and arithmetic procedures.

5. Conclusions

Before elaborating on the overall conclusions of this paper, it must be noted that this is but a case study which itself is part of a larger exploratory investigation. Hence, the conclusions are not only subject to the interpretative framework, but also to the relatively limited setting in which this lesson was conducted. This limited setting, for example, lead to short preparations times for the teacher. That may have been a reason for the confusion that the children experienced in understanding what the teacher meant with the terms "easy" and "difficult" structures.

Nevertheless, the outcomes are of interest because the observations (a) contribute to insight into kindergartners' spatial structuring abilities and their perception of the use of spatial structuring to support numerical procedures, and (b) illustrate how teachers may implement such an instruction activity to support kindergartners' spatial structuring ability and insight into numerical relations.

The limitations notwithstanding, this paper has explored and documented examples of children's developing insight into spatial structuring as they engaged in essential perceptual and physical activity with the blocks. This contributes to the ongoing investigation into how kindergartners' spatial sense, their spatial structuring abilities and their number sense may be related. In anticipation of further research, the observations could complement studies that support the stimulating role of spatial sense for the development of number sense, by highlighting the intertwinement of young children's early counting abilities, their spatial sense, their spatial structuring abilities and ultimately their number sense. Essentially, where traditional mathematics curricula focus mostly on children's counting abilities for developing number sense, the research in this paper illustrates mathematics instruction that highlights children's spatial sense. This can inspire teachers and encourage them to acknowledge the importance of spatial structuring in their classrooms, and answer to the need for instruction activities that promote spatial structuring strategies as a convenient alternative to unitary counting procedures.

5.1. Young children's spatial structuring ability and counting

The episodes in this paper appear to support Battista and Clement's (1996, 1998) strategies for counting cubes and Mulligan et al.'s (2004, 2006) stages in the development of young children's spatial structuring. The activity in the focus group, for example, exemplifies the three relationships between spatial structuring and counting that Battista and Clements outlined (1996) after studying how third, fourth and fifth graders count blocks in an array. First, as the children were building "easy" and "difficult" houses, they experienced how spatial structuring directs the counting procedure: in some cases the spatial structure guided the counting procedure and in other cases it led to counting errors that, in turn, inspired the children to restructure. Second, the children's attempts at counting helped them to understand the use of spatial structures in counting. In structuring and restructuring the constructions, the children were increasingly purposefully imposing either more or less spatial structure to come to constructions of which the blocks could be counted in abbreviated ways. Third, the manipulatives in this activity stimulated the children to analyze the spatial structure of the constructions and to reflect on their counting procedures.

Regarding the four developmental stages for structuring that Mulligan et al. (2004) outlined, what was presented in this paper provides additional support for the developmental trajectory of spatial structuring with a specific focus on early counting procedures of kindergarten children. The first stage that Mulligan et al. define, the pre-structural stage, may be recognized in the structure of the constructions that Dora built. She had copied the structured construction that was discussed in the classroom setting to represent an "easy house." Moreover, her bridge-like, single layered construction did not evidently represent a "difficult" house.

The second stage, the emergent stage, entails representations that show some elements of structure but focus on the aspect of the task that is most significant for the child (Mulligan et al., 2004). This can be recognized in James' constructions. Although his constructions showed some evidence of symmetry and asymmetry, he had difficulty with actually using the structure to abbreviate counting procedures and he tended to focus more on the size and creative appearance of his constructions than on their structure. Sara showed some evidence of the partial structural stage (stage 3) because both her "easy" and "difficult" constructions seemed to represent her awareness of symmetry and asymmetry as a factor for structure. Finally, Anne seemed to approach the fourth stage of structural development, because she regularly evidenced insight into structure that she also effectively applied to abbreviate her counting procedures.

5.2. Characteristics of the educational setting

Turning now to the teacher's role in supporting children's spatial structuring ability, two main issues came up that influence the effectiveness of the instruction activity. One of the issues concerns the type of language that is used both by the children and the teachers. In working with very young children, it is especially essential and at the same time challenging to relativize the children's ability to express their understanding both by nonverbal and verbal means. It may just be too early to posit that a child is lacking formal vocabulary rather than the insight or informal vocabulary per se (see also Freudenthal, 1984; Hughes, 1986). Similarly, the teacher must be able to properly formulate open-ended and follow-up questions, must be sure to elaborate on children's answers without filling in their thoughts or words, and must try to have the children work together as much as possible to promote interactive learning processes.

A related issue is the bottom-up, rather than top-down, design of the activity. A proper task should appeal to the children's existing body of knowledge, their interests, and their desire to explore, without focusing the evaluation exclusively on their verbal competencies. The teacher's role in this case study clearly illustrates the challenges that this type of instruction raises for a competent teacher. On the one hand, the way that the teacher led James into counting by twos suggests a predominantly top-down task. She commented afterwards on how difficult she thought it was to gauge what the children were thinking without steering them towards giving a particular response. Yet, her inquisitive attitude, the opportunities she gave the children to explain their reasoning to each other and to physically (rather than verbally) try and show what they were thinking, apparently supported the children's constructive learning processes because they were increasingly able to reflect on differences in structures and the influence on counting. Overall, then, the teacher tried to take on a proactive role in guiding the children through the activity, but her struggles illustrate the challenges that are inherent to teaching such young children.

5.3. Future research

Since this case study is but a start towards unraveling the association between spatial and number sense, it has raised many practical and theoretical questions that fuel the ongoing research. How do children's approaches to spatial structuring of three-dimensional constructions compare to their insight into, for example, patterns or dot configurations? What is the (longitudinal) relationship between children's familiarity with various spatial structures and whether and how they make use of spatial structures to approach numerical problems? How can teachers best uncover children's motivation for building the structured versus unstructured constructions in this activity?

The observations in this case study set the stage for further research into how children's spatial structuring ability may support the merging of their more intuitive understanding of quantity with knowledge of number. It is this convergence that should lay strong foundations for children's development and prepare them for the acquisition of more formal mathematical concepts and procedures that are typically introduced after kindergarten (Griffin, 2004a). Therefore, the final sequence of instruction activities should create an educational setting that may help to identify children who have difficulty recognizing spatial structures and who are susceptible to delays in developing numerical procedures, at a very early stage in their formal schooling. In addition, it may provide such children with instruction that is tailored to appeal to their strengths (e.g., early spatial sense) in order to advance from unitary to more abbreviated counting procedures. Ultimately, with this case study as an exemplary part of the theory building and instruction activity design process, the research may contribute to methods for discovering, remediating and preventing learning problems that concern the early, yet fundamental, mathematical competencies of very young children.

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