Though I did not use the term explicitly, didactical phenomenology already played a part in my former work. In the present book I stress one feature more explicitly: *mental objects* versus *concept attainment*. Concepts are the backbone of our cognitive structures. But in everyday matters, concepts are not considered as a teaching subject. Though children learn what is a chair, what is food, what is health, they are not taught the *concepts* of chair, food, health. Mathematics is no different. Children learn what is number, what are circles, what is adding, what is plotting a graph. They grasp them as *mental objects* and carry them out as *mental activities*. It is a fact that the concepts of number and circle, of adding and graphing are susceptible to more precision and clarity than those of chair, food, and health. Is this the reason why the protagonists of concept attainment prefer to teach the number concept rather than number, and, in general, concepts rather than mental objects and activities? Whatever the reason may be, it is an example of what I called the anti-didactical inversion.

The didactical scope of mental objects and activities and of onset of conscious conceptualisation, if didactically possible, is the main theme of this phenomenology. It was written in the stimulating working atmosphere of the IOWO.* So it is dedicated to the memory of this institution that has been assassinated, and to all its collaborators, who continue to act and work in its spirit.

* The Netherlands 'Institute for Development of Mathematics Education'.

CHAPTER 1

AS AN EXAMPLE: LENGTH

1.1–1.11. PHENOMENOLOGICAL

1.1–3. *What is Length?*

1.1. "Length" has more than one meaning. "At length", "going to the utmost length", "length and width" include in their context "length" in different meanings. The one I am concerned with becomes clear if along side the question

what is length?

I put a few other questions:

what is weight?
what is duration?
what is content?

"Length", "weight", "duration", "content" are *magnitudes*, among which length has its special status.

If I use the word length in the sense, made more precise here, I mean length of something, of a "long" object. "Length" then is synonymous with "width", "height", "thickness", "distance", "latitude", "depth", which are related to other dimensions or situations. For the sides of a "lying" rectangle one prefers "length" and "width", for a "standing" one, "width" and "height".

1.2. Without stressing it, I have turned my question "what is length?" towards an answer such as "length of . . . is . . .". This is a typically mathematical turn: transforming apparently isolated terms into symbols of functions. The question

what is "mother"?
what is "brother"?
what is "neighbor"?

are more easily answered according to the pattern

mother of . . . is . . .
brother of . . . is . . .
neighbor of . . . is . . . .

More precisely:

mother of x is she who has born x,

brother of x is every y such that y is a male and x and y have the same parents,
neighbor of x is every y such that x and y live beside each other.

Afterwards "mother" can also be defined in an "isolated" way:

x is mother if there is a y such that x is mother of y.

Linguistically "man", "stone", "house" belong to the same category as "mother", "brother", "neighbor" – as nouns they enjoy a substantiality, though that of "mother", "brother", "neighbor" differs from that of "man", "stone", "house". "Being mother", "being brother", "being neighbor" get a meaning only by the – explicit or implicit – addition "of whom". In "they are brothers", "they are neighbors" the additional "of . . ." seems unnecessary but is not: they are brothers or neighbors of each other.

1.3. Back to "length", interpreted in "length of . . ." as a functional symbol: a function that talks about "long objects" how long they are, though not necessarily numerically specified, as in

the length of this bed is 1.90 m.

Functional value may be vague: long, very long, short, very short, and so on. The reason why I neglect these values now is that I will start by focusing on a phenomenology of *mathematical* structures. Are "long", "very long", "short", "very short" not mathematical concepts? Such questions will be answered later on; in order not to complicate things, I delay the answer.

1.4. *Magnitudes* *

Before continuing let me consider the terms mentioned earlier. All of them aim at functions:

weight: weight of (a heavy object),
duration: duration of (a time interval),
content: content of (a part of space).

Let me introduce abbreviations:

$l(x)$: length of x,
$p(x)$: weight of x,
$d(x)$: duration of x,
$v(x)$: content of x,

* *Directed* magnitudes will incidentally be considered in Sections 15.9–12. Otherwise "magnitude" is always understood in the classical way. In this context "rational" and "real" always means "positive rational" and "positive real".

where x is something that can properly be said to have a length, weight, duration, content.

We again pose the question of the possible values of the function l (and of p, d, v). Not "long short", "heavy light", "big small", respectively, but since we speak mathematics, more precise values. This does not oblige us to state something like 1.90 m, 75 kg, 7 sec, 3 m^3, expressed in the metric system, or in any *a priori* system of measures. This is a liberty we can profit from to get deeper insight. Indeed, it appears that we can go rather far without accepting any special system of measures.

Let us call the

values of l lengths,
values of p weights,
values of d durations,
values of v contents,

and the

system of lenths L,
system of weights W,
system of durations D,
system of contents V,

and look for their properties.

1.5. *Adding Lengths*

The first thing we notice is that we can *add* lengths even before conceiving them numerically. How is it done? Given two lengths α and β, we provide ourselves with two "long objects" x and y with lengths

$$l(x) = \alpha, \qquad l(y) = \beta,$$

respectively, and compose them (in a way that asks for detailed explanation) into a new "long object" $x \oplus y$. This object has a certain length, consequentially named $l(x \oplus y)$. It was our intention to define the sum of lengths α and β and by definition we put

$$\alpha + \beta = l(x \oplus y),$$

that is to say

1.5.1 $\quad l(x) + l(y) = l(x \oplus y).$

In other words

the length of the composite equals the sum of the composing parts.

As regards this kind of definition one has to pay attention to one point: For the lengths x and y we have chosen representative "long objects" x and y, respectively, with lengths as prescribed. Instead we could have chosen other representatives, say x' and y', thus such that again

$$l(x') = \alpha, \qquad l(y') = \beta,$$

which would lead to a composite $x' \oplus y'$. In order for the definition 1.5.1. to be meaningful, we must be sure that

1.5.2 $\quad l(x' \oplus y') = l(x \oplus y),$

in other words, that

the length of the composite does not depend on the choice of the representatives.

I have to take care that my way of combining "long objects" fulfills this condition.

In a similar way this ought to be true of weights. Given two weights α and β the sum of which I propose to define, I take two "heavy objects" x and y with weights α and β, respectively, compose them into a new "heavy object" $x \oplus y$ and define

$$p(x) + p(y) = p(x \oplus y).$$

Again, replacing x and y by x' and y' with the same weights, respectively, must not change the weight of the composite. This requirement looks self-evident, and it is so for a good reason, indeed: we would never have focused on length, weight, and so on if this condition were not fulfilled.

A second remark: If composing is meant to lead to defining the sum, it must be carried out in such a way that the components do not overlap. Suppose I want to add a length α to itself in order to define the length $\alpha + \alpha$. Then for each of the summands I need *another* representative, thus

$$l(x) = \alpha, \qquad l(y) = \alpha,$$

in order to get

$$\alpha + \alpha = l(x) + l(y) = l(x \oplus y).$$

So I cannot manage with one representative for each length. Fortunately with lengths it is rather easy to provide oneself with two, three, or more representatives of the same length; instruments like a ruler can repeatedly be applied. In the case of weights and so on, the difficulty of obtaining enough representatives looks greater, but this is a point we are not concerned with here.

In carrying out the operation $\oplus$ as imagined in the various cases, order does not play a role and, as a consequence, addition of lengths, weights, etc., obeys the laws of commutativity and associativity:

$$\alpha + \beta = \beta + \alpha,$$
$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma).$$

The first property stated in the systems L, P, D, and V of lengths, etc., therefore is:

I. A commutative and associative operation (+) of addition in L, and so on.

1.6. *Order of Lengths*

Adding will later be joined by subtracting; that is, "the smaller from the bigger". But "smaller" and "bigger" are ideas we have not yet come across. They will now be considered.

Relations like "smaller bigger" belong to the so-called *order relations*: any pair of elements α, β of L is in exactly one of the situations

1.6.1 $\quad \alpha < \beta, \qquad \alpha = \beta, \qquad \beta < \alpha$

and for three of them, $\alpha, \beta, \gamma \in L$,

1.6.2 $\quad$ if $\alpha < \beta$ and $\beta < \gamma$ then $\alpha < \gamma$

holds (the so-called *transitivity*).

Such an order relation can now be defined in L by means of the addition. We express the property that

by adding, something can become only larger,

in a formula

1.6.3 $\quad \alpha < \alpha + \kappa$

for any lengths α and κ. This immediately ensures transitivity 1.6.2. Indeed, if $\alpha < \beta$ and $\beta < \gamma$, then there is a κ and a λ such that

$$\beta = \alpha + \kappa, \qquad \gamma = \beta + \lambda,$$

so

$$\gamma = \beta + \lambda = (\alpha + \kappa) + \lambda = \alpha + (\kappa + \lambda),$$

and thus

$$\alpha < \gamma.$$

The first requirement, 1.6.1, on an order relation is a bit trying. It means

1.6.4 $\quad$ If $\alpha \neq \beta$, then there is
either a κ with $\beta = \alpha + \kappa$,
or a λ with $\alpha = \beta + \lambda$,

though not both together.

For a moment I call two "long objects" x, y directly comparable if either x can be considered as a composing part of y or y as a composing part of x. Then 1.6.4 can be translated as follows into the language of "long objects":

1.6.5 Given two "long objects" x, y, then I can find directly comparable "long objects" x', y' such that $l(x) = l(x')$, $l(y) = l(y')$, and however I choose x', y' in this way, one thing is true:
either x' is a composing part of y',
or y' is a composing part of x'.

The second property we have stated for the systems L, W, etc. is:

II. The definition $\alpha < \alpha + \kappa$ for all lengths α, κ determines a total order in L, etc.

1.7. *Multiplying Lengths*

If we repeatedly add the same length, then the resulting lengths can be denoted as

$$1\alpha = \alpha,$$
$$2\alpha = \alpha + \alpha,$$
$$3\alpha = \alpha + \alpha + \alpha,$$

and so on; in general

$$n\alpha = \alpha + \ldots + \alpha \text{ with } n \text{ summands.}$$

Laws like

1.7.1
$$(m+n)\alpha = m\alpha + n\alpha,$$
$$(mn)\alpha = m(n\alpha),$$
$$n(\alpha + \beta) = n\alpha + n\beta,$$
$$\text{if } \alpha < \beta \text{ then } n\alpha < n\beta,$$

are obvious.

From adding we have now derived multiplying elements of L, etc., by positive integers, that is, elements of $\mathbf{N}^+$. As an inverse of this operation one has dividing, which means:

Given a length α and an $n \in \mathbf{N}^+$, then the equation

1.7.2 $n\beta = \alpha$

has a solution β. There is only one such β, since if

$$n\beta' = \alpha,$$

then

$$\beta < \beta' \quad \text{or} \quad \beta = \beta' \quad \text{or} \quad \beta' < \beta.$$

In the first and third case this would result in

$$\alpha = n\beta < n\beta' = \alpha, \qquad \alpha = n\beta' < n\beta = \alpha,$$

respectively, which is impossible, and leaves us with

$$\beta = \beta'.$$

The solution β of 1.7.2 gets the name

$$\beta = \frac{1}{n}\alpha.$$

Thus $\frac{1}{n}\alpha$ is defined by

1.7.3 $n\left(\frac{1}{n}\right)\alpha = \alpha.$

This then is our third property of lengths, etc.:

III. For every $\alpha \in L$, etc., and every $n \in \mathbf{N}^+$ there is one $\frac{1}{n}\alpha \in L$, etc., such that

$$n\left(\frac{1}{n}\alpha\right) = \alpha.$$

The following laws for dividing are easily verified:

1.7.4
$$\frac{1}{m}\left(\frac{1}{n}\alpha\right) = \frac{1}{mn}\alpha,$$
$$\frac{1}{n}(\alpha + \beta) = \frac{1}{n}\alpha + \frac{1}{n}\beta,$$
$$\text{if } \alpha < \beta, \text{ then } \frac{1}{n}\alpha < \frac{1}{n}\beta.$$

1.8. *Rational Multiples of Lengths*

Multiplying and dividing elements of L, etc., by elements of $\mathbf{N}^+$ can be combined. One puts

1.8.1 $m\left(\frac{1}{n}\alpha\right) = \frac{m}{n}\alpha,$

which results in multiplying of lengths by positive rational numbers. Since, however, a rational number can be denoted in various ways,

$$\frac{m}{n} = \frac{km}{kn},$$

we have to make sure that the definition 1.8.1 is valid; that is, we have to prove that

$$m\left(\frac{1}{n}\alpha\right) = km\left(\frac{1}{kn}\alpha\right).$$

This is indeed true. According to 1.7.4

$$\frac{1}{kn}\alpha = \frac{1}{k}\left(\frac{1}{n}\alpha\right),$$

thus

$$km\left(\frac{1}{kn}\alpha\right) = mk\left(\frac{1}{k}\left(\frac{1}{n}\alpha\right)\right) = m\left(\frac{1}{n}\alpha\right).$$

Thus we can multiply every length, etc., by every positive rational number $r \in \mathbf{Q}^+$. We easily find the laws, for $r, s \in \mathbf{Q}^+, \alpha, \beta \in L$, etc.:

1.8.2
$$(r+s)\alpha = r\alpha + s\alpha$$
$$r(s\alpha) = (rs)\alpha$$
$$r(\alpha+\beta) = r\alpha + r\beta$$
if $\alpha < \beta$, then $r\alpha < r\beta$.

1.9. *Real Multiples of Lengths*

Starting from one length, etc. say α, we can form all its rational multiples. They constitute a set $\mathbf{Q}^+\alpha$. In $\mathbf{Q}^+\alpha$ two arbitrary elements are rational multiples of each other. So $\mathbf{Q}^+\alpha$ cannot possibly exhaust what we imagine the system of lengths to be. Indeed, the diagonal and side of a square are not rational multiples of each other. However, $\mathbf{Q}^+\alpha$ does exhaust the system of lengths, etc. "aproximately". One knows about a property, the so-called Archimedean axiom:

IV. Given an $\alpha \in L$, etc., then there is no element of L, etc., bigger than all of $\mathbf{Q}^+\alpha$, and no element of L, etc., smaller than all of $\mathbf{Q}^+\alpha$.

I now take an arbitrary $\beta \in L$, etc. It does not necessarily belong to $\mathbf{Q}^+\alpha$, but according to IV it must lie "in between". For each $r \in \mathbf{Q}^+$

1.9.1 $\quad r\alpha < \beta \quad$ or $\quad r\alpha = \beta \quad$ or $\quad \beta < r\alpha$

holds. I now want to represent β as a real multiple of α,

$$\beta = u\alpha, \qquad u \in \mathbf{R}^+$$

in such a way that the order fits, that is

if $u < v$ then $u\alpha < v\alpha$ for $u, v \in \mathbf{R}^+$,

In particular for $r \in \mathbf{Q}^+$,

1.9.2
if $r < u$ then $r\alpha < u\alpha$,
if $u < r$ then $u\alpha < r\alpha$.

How to find such a u? Well, 1.9.1 causes a partition of $r \in \mathbf{Q}^+$ into three

classes (the second can be empty, or can consist of one element if β is a rational multiple of α). Such a partition is called a Dedekind cut:

the lower class: the $r \in \mathbf{Q}^+$ with $r\alpha < \beta$
the upper class: the $r \in \mathbf{Q}^+$ with $\beta < r\alpha$,

where at most one $r \in \mathbf{Q}^+$ can escape this division. Now there is a real $u \in \mathbf{R}^+$ that "causes" the cut, that is to say

if $r\alpha < \beta$ then $r < u$.
if $\beta < r\alpha$ then $u < r$.

If now we put

$$\beta = u\alpha,$$

we fulfill the requirements of 1.9.2.

It has been shown that

of two given elements of L, etc., each is a positive real multiple of the other.

We can now conclude with the property

V. For each $\alpha \in L$, etc., and each $u \in \mathbf{R}^+$, there is an $u\alpha \in L$, etc. Similarly to those of $\mathbf{Q}^+$ one can formulate laws for $u, v \in \mathbf{R}^+$ and $\alpha, \beta \in L$ etc.:

1.9.3
$$(u+v)\alpha = u\alpha + v\alpha$$
$$u(v\alpha) = (uv)\alpha$$
$$u(\alpha+\beta) = u\alpha + u\beta$$
if $u < v$ then $u\alpha < v\alpha$.

1.10. *Length Measure*

Let us break off the exposition and not insist on a systematic approach to magnitudes.

For instance we could continue with a numerical treatment of magnitudes: A measuring unit (m, kg, sec, m^3, or suchlike) is chosen in order to express each length, and so on, as a positive real multiple of the unit. Then each length, etc., is represented by a measuring number and according to its generation we find the fundamental rule

under the composition $\oplus$ the measuring numbers are added,

from which follows among others

the longer, heavier, . . . object has the bigger measuring number.

1.11. *What is Lacking Here*

The preceding was an example of phenomenology; namely, for the mathematical structure "magnitude". Or rather, it was a fragment of such a phenomenology. No attention was paid to measuring; connections between different magnitudes should have been considered; and finally, what has not been mentioned at all is that length is ascribed not only to "long objects" but also to broken and curved lines. How broken lines, say the perimeter of a triangle, should be dealt with is easy to guess. Curved lines are a different case. The classical way is approximation by broken lines but I shall skip it here in order to resume it later on.

1.12–1.29. DIDACTICALLY PHENOMENOLOGICAL

The preceding was not *didactical* phenomenology. In order to stress the difference I started with phenomenology as such. But also in the sequel didactical phenomenology will often be preceded by phenomenology as such, to create a frame of concepts and terms on which the didactical phenomenology can rest.

The difference between phenomenology and *didactical* phenomenology will soon become apparent. In the first case a mathematical structure will be dealt with as a cognitive product in the way it describes its – possibly non-mathematical – objects; in the second case, it will be dealt with as a learning and teaching matter, that is as a cognitive process. One could think about one step backwards: towards a genetic phenomenology of mathematical structures, which studies them in the cognitive process of mental growth.

One might think that a didactical phenomenology should be based on a genetic one. Indeed I would have been happy if, while developing the present didactical phenomenology, I could have leaned upon a genetic one. This, however, was not the case, and the longer I think about the question, the more I become convinced that the inverse order is more promising. In the sequence "phenomenology, didactical phenomenology, genetic phenomenology" each member serves as a basis for the next. In order to write a phenomenology of mathematical structures, a knowledge of mathematics and its applications suffices; a *didactical* phenomenology asks in addition for a knowledge of instruction; a *genetic* phenomenology is a piece of psychology.

All the psychological investigations of this kind which I know about suffer from one fundamental deficiency: investigations on mathematical acquisitions (at certain ages) have involved the related mathematical structures in a naive way – that is, they lack any preceding phenomenological analysis – and as a consequence, are full of superficial and even wrong interpretations. The lack of a preceding *didactical* phenomenology, on the other hand, is the reason why such investigations are designed in almost all cases as isolated snapshots rather than as stages in a developmental process.

1.13–1.25. COMPARISON OF LENGTHS

1.13–14. *Length Expressed by Adjectives*

1.13. Many mathematical concepts are announced by adjectives. Adjectives belonging to length are: "long, short", but also "broad, tight", "thick, thin", "high, low", "deep, shallow", "far, near", "wide, narrow", and finally also "tall, sturdy, diminutive, insignificant". Of course the ability to distinguish such properties precedes the ability to express them linguistically. For the adult it is – at least unconsciously – clear how these expressions are related to the same magnitude, length, and he often presupposes children to be well acquainted with this relation. Researchers in this field are often not aware of this difficulty. It is not farfetched to ask oneself how the child manages to develop a knowledge of these connections. A disturbing factor is the overarching of this complex of adjectives by "big and little", which can serve so many aims (up to "big boy" and "little girl").

Bastiaan (5; 3) asks how big is a mole. When I show with my hands a mole's length, he insists "no, I mean how high". He is compelled to differentiate "big". Clearly he is conscious of the fact that both cases mean a length.

The insight that both expressions mean a length is not at all trivial, for instance, that a *high* tree, if cut, is *long*. As a matter of fact, even adults may have problems with the equivalence of distances in the horizontal and the vertical dimensions, at least with regard to quantitative specification.

How does the connection within this complex of adjectives come into being? How is the common element constituted? If I may guess, I would attribute a decisive role to the hand and finger movements that accompany such statements as *that* long, *that* wide, *that* thick, and so on (likewise *that* short, and so on) – movements that can turn in different directions and possess different intensities but always show the same linear character. (Compare this with the mimic expressions of embracing, which may accompany "that much", but also "that thick", and with the mimic and acoustic expression of lifting belonging to "that heavy".)

The common element in this complex of adjectives for length is possibly not yet operational in all young school children; as a matter of consciousness it may even be absent in many older ones. Acquiring it and becoming conscious of it are an indispensable condition for mathematical activities.

1.14. Around such adjectives as "long" there is a complex of relational expressions like:

longer, longest, as long as, less long, not as long as, too long, very long.

Here again the ability of distinguishing precedes that of linguistic expression (for instance, something cannot pass through a hole because it is *too thick*; the smaller cube is placed upon the bigger). Inhibitions work against using comparatives and superlatives – "big" is used where "bigger" and "biggest" are meant.

The adjectives of the last list aim at comparing objects with respect to length. This activity develops long before what mathematicians call the order relation of lengths is constituted, not to mention becoming conscious of the order relation. The constitution of an order relation in whatever system includes at least the operational functioning of transitivity, that is, drawing factual conclusions according to patterns like

> *a* as long as *b*,
> *b* as long as *c*,
> thus *a* as long as *c*

and

> *a* shorter than *b*
> *b* shorter than *c*
> thus *a* shorter than *c*,

which of course does not mean the ability of verbalising or even formalising transitivity.

Contradictory Piaget, P. Bryant* showed that young children (from the age of four onwards) possess an operational knowledge of transitivity. On the other hand, I reported** on third graders who could apply the transitivity of weights in seesaw contexts but were not able to understand a formulation of transitivity.

Little if any information on the development of the concept of length can be drawn from traditional research. Thought on this subject is obscured by such terms as "conservation" and "reversibility", which are supposed to cover the most divergent ideas, and by a faulty phenomenology.

1.15. *Congruence Mappings*

One of the mathematical notions that have been absorbed by "conservation" in order to be mixed together with quite different ones is

> invariance under a set of transformations.

As an aside I will illustrate this notion by a number of examples:

> The *number of elements* of a set ("cardinality") is invariant under one-to-one mappings.
>
> *Convexity* of a plane figure is invariant under affine mappings.***
>
> *Parallelism* of lines is invariant under affine mappings.***

* P. Bryant, *Perception and Understanding in Young Children*, London, 1974, Chapter 3.

** *Weeding and Sowing*, p. 255.

*** A reader not acquainted with the concept of "affine mapping", may read instead: parallel projection.

> The difference between the *surfaces* of a *sphere* and of a *ring* is invariant under arbitrary deformations (the surface of a ring cannot be deformed into that of a sphere).
>
> *Lengths* of line segments and *measures of angles* of pairs of lines are invariant under congruence mappings of the plane or the space (movements and reflections, with glide reflections included).
>
> *The length ratio* of line segments and *measures* of angles of pairs of lines are invariant under similarities.
>
> The property of being a *regular pentagon* is invariant under similarities.
>
> The property of being a *cube with side 1* is invariant under congruence mappings.
>
> The property of a plane figure to represent the *digit 2* is invariant under movements (though not under reflections).
>
> The *shape* of a figure is invariant under similarities.
>
> Both the *shape and size* of a figure are invariant under congruence mappings.

The expression *congruent* is well-known; congruent figures are, as it were, the same figure laid down in various places. In mathematics this concept is made more precise by that of a congruence mapping, which extends to the whole plane or space; then figures are called congruent if they can be carried into each other by congruence mappings.

The simplest figure is the line segment. "Equal" rather than "congruent" line segments is an older terminology. The now prevailing terminology reserves "equality" to coincidence; that is, actual identity. Yet congruent line segments are equal in a sense; that is, with respect to length. And conversely: line segments with the same length can be carried into each other by congruence mappings.

1.16. *Rigid Bodies*

Line segments are mathematical abstractions. They are connected with the former "long objects" via the phenomenon of the rigid body. A rigid body can be displaced, and provided it is not badly belaboured, it remains congruent with itself under this operation. Rigidity is the physical realisation of the property we called invariance of shape and size under movements. The fact that in geometry we consider by preference properties that are invariant under movements is related to the dominance of rigid bodies in our own environment – molluscs would prefer another kind of geometry.

I am pretty sure that rigidity is experienced at an earlier stage of development than length and that length and invariance of length are constituted from rigidity rather than the other way round. Rigidity is a property that covers all dimensions while length requires objects where one dimension is privileged or stressed. However, stressing this one dimension may not lead to restricting the preserving transformations. If lengths are to be *compared*, the free mobility of rigid bodies must play its part. The mobility must be fully exploited; all movements must be allowed, not only the most conspicuous translations, but also rotations, in order to compare "long objects" in all positions. The shape of a body or the stressing of one dimension as its length may not result in restricting the mappings under which rigidity expresses itself as invariance. Adjectives like "high, low" within the complex of terms that indicate length can exert an influence to restrict the set of transformations; "high, low", stressing one direction in space, can lead to restricting the set of transformations to those that leave invariant the vertical direction – displacements along and rotations around the vertical – a restriction that would impede the overall comparison of line segments and "long objects".

1.17. *Similarities*

Side by side with the congruence mappings I repeatedly mentioned the similarities. The latter play a part in interpreting visual perception. "What is farther away, looks smaller" (at least at big distances); this is a feature unconsciously taken into account by a perceiver and sometimes made conscious to himself – a curious interplay which has been studied many times. If a rigid body moves away, its shape as understood by us remains invariant; visually conceived the rigid bodies are invariant even under similarities, while the similarity ratio depends on the distance between object and perceiver.

Nevertheless just this fact can contribute a great deal to the mental constitution of rigidity. The invariance suggested by the continuous behaviour of some striking characteristics might provoke the attribution of more invariances, in particular those of size and length.

1.18–20. *Flexions*

1.18. The rigidity of rigid bodies has to be understood with a grain of salt. Though its wheels and doors can turn independently, a car can globally and under certain circumstances be considered as a rigid body. Another extreme case is clay, which by mild force can be kneaded and deformed. In defining rigidity all depends on what you call "not badly belaboured". A liquid or a gas can be given some other shape without using any force, but according to the degree of rigidity more or less strong forces are needed to deform a rigid body. More or less rigid parts can be movable with respect to each other, such as in the case of animal bodies, while certain arrangements of the parts with respect to each other may be privileged, such as the state of rest, which can be congruently copied ad lib. It is that privileged state in which length measures of animal bodies are defined. The heights of, say, two people are compared while they are standing; we are convinced that they do not change when they sit down, and we know that they will show anew the former relation if they

rise again. We also judge that if they sit down and the taller person looks smaller, the difference must be ascribed to longer legs – something we can reconsider under the viewpoint of addition of lengths.

1.19. What comes about here is another principle of invariance of length, that is to say, invariance under a kind of transformations other than planar or spatial congruence mappings. It is transforming "long objects" by plying or bending them with a negligible effort: two objects to be compared are laid side by side or one on top of the other while certain deformations are allowed. Typical examples of this are measuring instruments other than the ruler and the measuring stick – for instance, the measuring tape, the folding or coiled pocket-rule – but a more primitive device used to measure lengths, the piece of string, should not be forgotten. It shows marvellously two ways of comparing lengths: in the tight state it measures a straight length, and fitted around a curvilinear shape it measures a circumference.

As opposed to the rigid bodies considered earlier, I will call these objects *flexible* the admissible deformations of these objects being called *flexions*. Flexions are reversible – this is an important feature. Moreover, flexible objects possess one or more privileged states. Among the privileged states there might be one in which the object is straightened and used as a measuring instrument: the measuring tape, the folding pocket-rule, the coiled-rule, and again the piece of string that can be stretched with a little force and that in this state resists further stretching. One's own body is of the same kind; in order to have it measured, one jumps to one's feet (though not to one's toes). Similarly, one measures the length of a stalk or reed or a stair-carpet: by stretching. Or of a car antenna: by pulling it out. A sheet of paper is flexible, though there is a well-defined state of maximal stretching. Plastically deformable substances such as clay are again different, a "long object" made of clay, if carefully handled, can be considered as flexible, though a kneading transformation is no flexion.

1.20. Where can we put the flexions mathematically? The mathematical counterparts of the rigid bodies (which may be moved without being badly belaboured) were the geometrical figures subjected to movements in the plane or in space, transformations that map everything congruently; in particular every line segment whatever its length or direction might be. If our objective is *measuring lengths*, this requirement is exaggerated; in order to serve for measuring, the "long objects" need display this invariance in the length direction only. Only in the length direction should the object be rigid; there is no need for rigidity in the other dimensions. This kind of object is mathematically idealised by what is called curves – curves which are described by a moving point or appear as boundaries of a plane figure. Of course curves which are – entirely or partially – straight are also admitted: straight lines and broken lines. It is these *mathematical curves* that are subjected to *mathematical flexions*. What does this term mean? If it refers to curves, I am concerned with one dimension only

– no width and no thickness – and in this one dimension they shall be rigid. The arc length, which as a measure replaces straight length, should be invariant under flexions. Mathematically, flexions are defined as mappings of curves that leave the arc length invariant.

But what do we mean by the arc length of a curve? The answer looks obvious: straighten the curve while not stretching it and read the arc length on the final straight line segment. Well, isn't it a vicious circle? What do we mean by straighten without stretching? No stretch – this just means that the arc length must be preserved, but arc length still has to be defined. As a matter of fact, it is curious that I prohibited stretching only, and kept silent about shrinking, but of course the mistake you can make when straightening the object, is pulling too hard and stretching. This shows once more that the alleged clarity of the straighten-out definition of arc length rests not on visual but on kinesthetic intuition.

Yet another definition of arc length deserves to be considered. In order to be compared, curves are rolled upon each other. In particular, in order to measure the length of a curve, it is rolled upon a straight line. Rolling yes, but of course skidding is forbidden. But what does it mean mathematically: no skidding? That the pieces rolling along each other have the same (arc) length. This again closes the vicious circle.

There is no escape: In order to define flexions mathematically, we must first know what arc length is, and arc length must be defined independently with no appeal to mechanics.

How this is to be done, I have already mentioned. First, one defines the length of a polygon – that is, a curve composed of straight pieces – as the sum of the lengths of those pieces. Given a curve, it is approximated by "inscribed" polygons, that is, with their vertices on the curve. The smaller the composing straight pieces, the better the curve is approached. In this approximation process one pays attention to the respective lengths: as the curve is approached by the polygons, the lengths converge to what is considered as the length of the given curve. Not only should the total curve get an arc length by this definition, but also each partial curve, and it is plain (though the proof requires some attention) that these lengths behave additively: if a curve is split into two partial curves, the length of the whole equals the sum of the lengths of the parts. It is now clear what we have to understand by a mapping that preserves the arc length (by a flexion): not only should the total arc length be left invariant. but also that of each part.

It is strange that an intuitive idea like *invariance of arc length* and *straightening without stretching* requires such a cumbersome procedure in order to be explained mathematically. The reason is now obvious: when trying attempts at explaining arc length mathematically, we are compelled to renounce our *mechanical* experiences. It is particularly intriguing that *physically* I can compare two flexible objects by flexion or the borders of two plane shapes by rolling the one upon the other, before I start *measuring* length, whereas our *mathematical* definition of flexion presupposes arc length, which includes the whole measuring procedure and even the addition of lengths.

1.21. *Rigidity and Flexibility*

We have been concerned with two kinds of mappings:

congruence mappings in plane of space, and
flexions of curves.

Both are mathematically defined by the invariance of length, though the first requirement cuts deeper than the second if the view is fixed on curves and arc length.

The fact that congruence mappings and flexions leave length invariant is implicit in their definition. In physics the counterpart of mathematical congruence mappings and flexions is the movement of rigid bodies and the bending of flexible bodies, but whether in physical practice something is (at least approximately) a rigid body or a flexible body and which physical operations are allowed if length should (at least approximately) be preserved are physical facts, depending on experiences we have somehow acquired. This acquisition of experience starts rather early, certainly as early as in the cradle. It is empirical and experimental, and though this experimenting starts, as Bruner asserts, in an *enactive* way, in the course of development it is supported more and more by representative images of what is recollected or pursued (the *ikonic* phase), and it becomes more and more conscious in order to be verbalised (the *symbolic* phase). In the context of the phenomenon of "length" a phenomenological analysis is required to state and to distinguish invariance under congruence mappings and flexions, but anyway it is clear that the related learning process starts in the enactive phase (with no representative images and unconsciously, that is in the most effective way) and that pieces of it can be made conscious in the learning process.

Bastiaan (3; 9) finds a glass marble on the foot path: "If I push hard, it would roll into the street". It does happen. The marble rolls under the tyre of a car parked at the curb. Bastiaan cannot reach it. I show him a little stick. By sight he judges: "It is not hard enough." It is a soft stick, but nevertheless he succeeds.

This example does not concern using rigid or flexible objects to compare lengths. What matters here is experiences with mechanical properties of things. At a certain moment in his development a child judges at sight whether something is "hard" enough to be applied as a lever to exert a certain power (ikonic phase), and he even finds words – hard enough – to express this fact (symbolic phase).

I do not have the slightest idea how this complex of mechanical properties becomes mentally constituted; an able physicist, observing children, could discover a lot of things in this field. There is one conjecture which I dare pronounce: that rigidity precedes flexibility. The environment strongly suggests the rigid body as a model. Surprising experiments show that under conditions of incomplete information about kinematical phenomena there is a strong tendency to interpret them as movements of rigid bodies.

As a consequence I think that length is first constituted in the invariance context of congruence mappings – that is, connected to rigid bodies – and only at a later stage gets into that of flexions – that is, of flexible objects. This can happen if the child sees lengths compared or even measured by flexible instruments – fitting ("is the sleeve long enough?") and measuring with a tape.

In any case it is crucial to pay attention to the double invariance context of length.

1.22. *Make and Break*

I hesitated – unjustly as it will shortly appear – as to whether I should augment the two kinds of transformations that show invariance of length (that is, congruence transformations and flexions) with a third, which I would call

break–make transformations:

a "long object" is broken into pieces and remade.

The "long object" may be a stick that is factually broken, or a string that is cut, or a train of blocks that is split into two or more partial trains. In the first two examples remaking will not yield a complete restoration of length even if carried out carefully, with some loss in the second case if the partial strings are *tied* together. In the third case the restoration can be complete though it need not be: the parts can be put together in another order, and this can even be visible if the particular blocks are distinguished by length, colour, or other characteristics.

It is a meaningful and non-trivial statement that under break–make transformations length is invariant. It is meaningful if it is the original and final state that are compared, disregarding the intermediate ones. Indeed, how should we formulate the question if the intermediate states are to be admitted? "Do they remain as long together?" If "together" means adding lengths, this question is premature at the stage of simply comparing lengths, and if "together" means "taken together" the question aims at comparing the initial with the – now also mental – final state, which is no news.

Whenever the break–make transformation reproduces the initial state, the question "are they of the same length?" is trivial. Or rather, the answer reveals only whether the child that was questioned has remembered the initial state and is able to compare an actual and a mentally realised state with each other. If the final state is not wholly identical with the initial one, the answer also reveals whether the child knows which characteristic matters if length is meant. These two abilities will be reconsidered later.

Insight into length invariance under break–make transformations can be split into two components:

first, that under breaking (partitioning) and making (composing) "long objects" are transformed into long objects, and

second, that in composing "long objects" length is not influenced by the order of the composing parts.

Actually, these two insights form the basis for measuring lengths and will reappear in that context. If the second insight is to be placed into the context of invariance of length with respect to certain transformations on "long objects", instead of break–make transformations we could better use the term

permutation of composing parts.

I can now explain the hesitation I felt before writing this section. Break–make transformations or permutations of composing parts as a third kind of transformations look logically and phenomenologically superfluous. Within a phenomenology of magnitudes, and particularly length, as sketched in the beginning, the break–make transformations (permutations of composing parts) and the associated invariance of length can be derived from the congruence mappings, flexions and their invariance properties. But this derivability is a consequence of coupling the comparison of lengths with measuring, which is genetically and didactically premature. It is true that composing "long objects" occurs in that phenomenology as a special operation, indicated by $\oplus$, but the context in which it occurs is length rather than comparing length; namely the formula

$$l(x \oplus y) = l(x) + l(y).$$

$\oplus$ occurs there as a logical rather than geometrical and mechanical operation. $x \oplus y$ appears as something that is uniquely determined by x and y, whereas for break–make transformations it is essential that x and y can be put together in various ways and however composed, yield objects of the same length.

1.23–24. *Distance*

1.23. Up to now in our didactical phenomenological analysis we have considered length as a function of concrete objects (possibly replaced by their mental images). This, however, does not cover all cases of length. Length as distance between A and B answers the question "how far is B from A?" In a purely formal sense "how far?" is quite another interrogative than "how long?" In "how far . . . ?" two points occur as variables, whereas in "how long is this object?", the object is the only variable. Length is a function of whole objects, whereas distance is a function of two points "here" and "there". We are so accustomed to the procedure which connects both of them that we can hardly imagine the early stage where we must have acquired it by a learning process and ask ourselves whether this connection is as obvious for children as it is for us.

If "how far?" is to be reduced to "how long?", a "long object" must be placed between A and B, between here and there. So, if A and B are railway stations or stopping places along a highway, the rail connection or the stretch of highway may be considered as concrete "long objects" whose distance is asked for. In general, if there exists a concrete path between A and B, their distance is the length of the path; if there are more such paths, it should be stipulated which one is meant. But how far is it from the front room of my ground floor to the rear room on the first floor? From here to across the canal, if no bridge is visible? From here to the sky? Only from the context can it be understood what is meant. In the context of geometry, mechanics, and optics the distance is measured along a straight line; in the context of spherical trigonometry and in the context of (surface of air) navigation, along arcs of great circles, "geodesics" or shortest paths as determined by straightened strings on curved surfaces. Of course, with this remark I do not mean spherical trigonometry or navigation should have been studied or exercised in order to decide that lengths should be measured along geodesics; contexts like this develop long before they are made conscious. The value of rectilinearity is suggested to the young child, enactively, if he is called to come straight in your open arms, the ikonically by all the horizontal and vertical straight lines in his environment, and symbolically by straight lines in schemas and by the word "straight line". The part played by rectilinearity in the constitution of "length" remains unconscious until it is explicitly discussed. Straightening flexible objects if lengths are to be compared may still be an automatic act – for instance, automatic imitation – and there might be children who as automatically put between two unrelated points a mental "long object", an imagined ruler, or a string in order to interpret distance as length. Well-known experiments where children get disoriented as soon as a screen is placed between the two points may prove how important this act of inserting a "long object" can be for reducing "how far?" to "how long?". But whatever these experiments mean, if some judgment about the distance of unrelated points must be motivated, one cannot but make explicit the necessity of rectilinear connections. From this moment onwards the significance of rectilinearity for the concept of length becomes more and more conscious – another connection between length and rectilinearity will be indicated later on.

1.24. How does a child learn what matters if lengths are to be compared? Sets of objects of the same kind but of different length may play an important part: big and little spoons (and equal ones), long and short trains (and equal ones), high and low trees (and trees of equal height). The objects are compared at sight if they are lying parallel and side by side; in order to be compared they are brought into such a position, physically or mentally, as rigid bodies, by congruence mappings. This requires comparing physical with mental objects, and mental ones with each other. Memory for length initially functions in a rather rough way, it seems. Remembering length during long periods remains a difficult task. As for myself, I am often surprised that relations of length differ greatly from what I remember they should be. Comparing objects side by side gains precision in the course of development: the ruler is laid close to the line to be measured, while observing the prescription to aim perpendicularly to the line. The connection between "length" and "distance" is stressed, and the weight is shifted to "distance" if one of the objects to be compared, or both of them, bear marks by which the ends of the objects to be compared can be indicated. Comparing can be done indirectly, using the transitivity of the order relation; for instance, by taking distances between fingers of one hand, between two hands, between the points of a pair of compasses, or between two extant or intentionally placed marks on a long object, and carrying them from one place to another. With all these methods length as a function of long objects is replaced by distance as a function of a pair of points. It already starts with showing "that big" or "that small" with fingers or hands, although in its exaggerated appearance this gesture is more an emotional expression of "awfully big" or "miserably small" than a true means to compare lengths. More refined methods of comparing lengths are based on geometry and will be dealt with in that context.

1.25. *Conservation and Reversibility*

Before extending the analysis of *measuring* lengths I tackle the question already touched in Sections 1.12 and 1.15: how psychologists interested in the development of mathematical concepts deal with such concepts, in particular length. The investigations, started by Piaget, show the following pattern. The general problem is to acquire knowledge about the genesis of such fundamental concepts as number, length, area, shape, mass, weight, and volume. Subjects are shown groups of objects which agree with respect to one or more of these magnitudes (the same number of chips in a row, reeds of the same length, and so on) and are asked to state that they agree with respect to the characteristic A (number, length, or suchlike). Then one of the objects of the group is subjected to a transformation that according to adult insight does not change the characteristic A while other characteristics may be changed (for instance, changing the mutual distances of the chips in the row or bending the reed). After this operation the subject is asked whether the characteristic A has remained unchanged; if this is affirmed, one speaks of *conservation*, and the subject is classified as a "conserver". Psychologists are reasonably unanimous about the average age of conservation of the various characteristics, whereas people who have some didactical experience with children usually judge these ages absurdly high. The large percentages of non-conservers in psychological experiments are achieved by a particular strategy: The transformation that should be ascertained to conserve A is intentionally chosen so that it changes another characteristic B so drastically that the attention is diverted to B (for instance, if A is cardinal number or mass, a striking change of length, or if A is length, a striking change of position or shape). What is actually being investigated is whether the subject is able to separate these characteristics sharply from each other and how strongly he can resist attempts at misleading him. Built-in misleading is in general characteristic of the psychological, as opposite to the didactical, approach.

By no means should the question be rejected as to the stage of development at which children master invariances of certain magnitudes. On the contrary, it is a merit of Piaget's to have been the first to have formulated such problems. The problem, however, is obscured by the use of such terms as "conservation"; often the researchers themselves have no clear idea of the kind of transformations with respect to which the so-called conservation should be established. For each experiment designed to take in young children, one can contrive a more sophisticated version to embarrass adults. For instance, show a person two congruent paper clips and ask him whether they are equally long; the question is of course affirmatively answered. Then unfold one of them, straighten it out, and repeat the question. Whatever he answers can be wrong. It depends on what the experimenter meant. An adult subject would react to the question by asking, "What do you mean?" (In our terminology, length invariance under congruence mappings? Or under flexions?) Young children in the laboratory are not likely to ask counter questions. The fact that they do not ask proves that they are intimidated (in the terminology of the psychologist, "put at their ease") – their critical behaviour being eliminated by situational means.

For a good experimental design it is indispensable that experimenter and subject have a clear idea of the kind of transformations with respect to which invariance is to be established. Perhaps psychologists would answer that then the fun goes out of it, as the chance of getting wrong answers would be minimised. So much the better, I would say. Such a result would better agree with the opinions of children's capacities held by didacticians.

Of course this does not mean that all problems are disposed of. I could enumerate a lot of developmental problems that from the viewpoint of a sound phenomenology are interesting enough. For instance I would like to know whether constituting rigidity mentally precedes length, whether length invariance under congruence mapping and length invariance under flexions help or impede each other, what role is played by similarities in the mental constitution of length, and how the equivalence of "long" and "far" is acquired. So there are many more questions I would like to have answered. The most urgent question, I think, is about the significance of the break–make transformations for so-called conservation (not only of length). If I may trust my own unsystematic experience, I would consider them as crucial. Yet in order to answer such questions, a quite different design of experiments is required than that of snapshots, registering which percentage of subjects at a certain age do "conserve". Also required is a more positive mentality than that of tricking children into making mistakes.

Another vague term that is often used in that kind of research is "reversibility". Originally it was related to answers given by subjects when they motivate

pronouncements on conservation. For instance, one of two strings of equal length is made crooked while the other remains straight; the subject is asked whether they are still equally long. If it is affirmed, the subject is asked to give reasons. If he answers "If straightened out, it is again the same", he shows "reversibility"; that is, the capacity to mentally reverse the transformation, which is considered a good argument for equality of length. Of course, it is no argument at all, and though it is interpreted by the experimenter as such, it was probably not meant that way by the subject. From the equality of initial and final states nothing can be derived about intermediate ones. If the subject had said, "They are equal because I got the one from the other by mere crooking", the answer would have been as good as, or even more to the point than, the argument of reversal. The subject, however, would not have been counted among the true conservers, because he lacked reversibility.

This "reversibility" as a proof for "conservation" is the original meaning, but subsequently it has been used in many other and mutually unrelated senses. There are, however, also researchers who reject the reversibility argument. They postulate standard answers that have to be given in order to establish conservation. Then the question "why is this as long as that?" must not be answered by a material argument but by a formal one, if the subject is to be classified as a conserver; he should answer something like "because they have the same length". To the question "why do they have the same content?", it must be "because they include equal parts of space". It goes without saying that such investigators are even farther away from meaningful mathematics.

The lack of insight into the difficulties with the equivalence between "long" and "far" has already been mentioned. Often they are increased by a stress on intentionally misleading connecting paths – a pattern in the plane that suggests a system of paths or two points on the rim of a round table that invite marching along the edge – where the experimenter had, of course, meant straight paths.

These details may suffice. I would certainly not judge that all the investigations I have in view are worthless, but many of them suffer from wrongly placed stresses. The method of snapshots need not be rejected but in order to be applied it requires a background theory – or at least ideas – about the intermediate development. Such theories do exist, but they are so vague and general that anything can be fitted to them and they do not provide criteria for attributing relevance to certain questions or complexes of questions.

What is lacking here can be made clear parabolically. Let us assume somebody is investigating the development of flora during the year by snapshots. On trees and brushes he notices various kinds of buds at various stages. In the next snapshots he identifies leaves and petals at the same places. Later on the former have remained whereas the latter have been replaced by fruit. Then the fruit, and finally the leaves too, have disappeared. He has not paid attention to stamens, pistils and insects and does not know where the fruit and leaves went. Perhaps he does not even know that the leaves and flowers were locked up in the buds. His phenomenology was utterly fragmentary, he did not know what he had to look for, and there is a good chance that he will wrongly interpret what he has seen. Perhaps terms like *growing, blooming, bearing fruit* are lacking in his vocabulary, or they mean states rather than processes. Ideas about development would have given him a greater chance of noticing essentials.

1.26–29. MEASURING LENGTHS

1.26. *Yardsticks*

Measuring length requires instruments – measuring sticks or rules. At first the measuring instrument will be smaller than the thing to be measured. Remarks to the contrary in the psychological literature rest on misapprehensions about measuring, or on artificial experiments.

The first yardstick I see used by children is the step. For a long time they do not care whether all steps are equally long. Almost always they count one step too many (the zero step as one). From the beginning it is clear that fewer steps mean a shorter interval, though it is not as clear that composition of intervals goes along with addition of numbers of steps. At about the same time as measuring distances by steps, or somewhat earlier, one notices the activity of jumping over a certain number of pieces in patterns of tiles in order to see how far one can jump. I do not claim that this is really a measuring activity, though this kind of jumping may influence measuring by steps.

Bastiaan (4; 10) spontaneously measured the width of a path by steps. "This is six further". I show him I can do it in one step. He does the same with two steps. He continues measuring by pacing.

Bastiaan (6; 5) has made a large construction of roads, bridges, walls and tunnels in a sandpit. In order to make a drawing of the construction he measures distances with his two forefingers parallel at a fixed distance (about a decimeter), proceeding with the left forefinger in the hole made by the right one.

Bastiaan (almost 7; 6) measures distances with a span between thumb and little finger which he knows is one decimeter.

Measuring with a measuring instrument means laying down the instrument congruently a number of times until the length to be measured is exhausted. If the object to be measured is a distance between two points, rectilinearity of continuation must be practised as the measuring instrument is repeatedly laid down. It is surprising that even 12 years olds may neglect this. If the straight line between the two points is blocked, the path is partially replaced by a parallel one. It is a remarkable fact that usually parallelism is better observed than is the rectilinearity of the continuation in the non-blocked case. Indeed, the latter is more difficult. To do this reasonably, one has to develop a certain technique, which requires more geometrical insight than – unfortunately – is being taught in the primary school.

There is a rich variety of yardsticks. Most of the traditional length units are taken from the human body: inch (which means thumb), finger, palm, foot, short and long ell, yard, step, double step, fathom; for larger distances the stadium (= 100 fathoms = 600 feet), the Roman mile (= 1000 double steps), an hour's walk. The so-called metric measures are related by powers of 10: metre, kilometre, centimetre, millimetre, micrometre, picometre. At variance with them: light year, parsec.

1.27. *Change of Yardstick*

If the object to be measured is not exhausted by applying the yardstick congruently a number of – say n – times, the problem arises of what to do with the remainder. In many cases one will resign oneself to the fact that a bit is left or is lacking, which means that the object is a bit longer or a bit shorter than n times the unit. Likewise the case where the remainder looks to be about half, one-third, or two-thirds the unit is not problematic. For greater precision a more systematic procedure is required. Two systems are familiar: common and decimal fractions. A less usual variation is binary fractions (or fractions with another base). A most natural system, now obsolete because of its complexity, is continued fractions, as I have explained elsewhere*. If a_1 is the measuring unit and a_0 the object to be measured,

$$a_0 = p_1 a_1 + a_2,$$

then the remainder a_2 ($< a_1$) is used as a new unit,

$$a_1 = p_2 a_2 + a_3,$$

and so one goes on, expecting that eventually the division will terminate, that is

$$a_{n-1} = p_n a_n.$$

Then a_n is a common measure of a_1 and a_0, and by reckoning backwards, one will find, say

$$a_0 = r a_n,$$
$$a_1 = s a_n,$$

which implies

$$a_0 = \frac{r}{s} a_1.$$

It is an advantage of this procedure that it involves a systematic search for a a denominator, provided a_0 is truly a rational multiple of a_1; that is, if the procedure indeed stops. But this need not happen. Then the procedure has to be stopped at a certain stage, the remainder is neglected, and the length of a_0 is expressed approximately in terms of a_1.

With the methods of decimal fractions one is saved the trouble of finding

* *Mathematics as an Educational Task*, p. 203.

a suitable denominator. The measuring unit is again and again divided into ten equal parts (even if such a partitioning is not yet marked on the measuring instrument), and one has only to see how often the subdivided unit goes into the remainder. It is a disadvantage of the decimal method that even simple fractional lengths such as $\frac{1}{3}$ of the unit can only be indicated approximately.

Length is one of the concepts by which common and decimal fractions can operationally be introduced. This subject will be resumed in the chapter on fractions.

1.28–29. *Measuring Length at an Early Stage*

1.28. Terms that should occur early in measuring length are "double", "three times", "half", and "a third". It struck me that 5–6 year olds who reasonably understood length did not know these terms, or at least, not as related to length; the dominance of the adjective "big" seems to block applying "double" and "half" to the linear dimension.

Bastiaan (5; 3), at a certain moment during a straight walk at the other side of our canal between two bridges at a large distance from each other, does not understand the question "Are we half-way?", but later spontaneously indicates the point where the "middle starts" (that is, the second half).

Terms like "half full" and so on (of a glass) function earlier and better.

Additivity of length is still a problem at this age. A long object is paced off anew after it has been lengthened by a second object. It is not noticed that the second pacing gives another length for the first piece.

One should realise that these are not trivial things – knowing

how lengths are composed,
that the results are again lengths,
that pieces of lengths are again lengths,
that the length measure of a part is smaller than that of the whole, and
that length measures behave additively under composing.

1.29. The length of flexible objects is measured after straightening. The circumference of curved figures is measured by means of a flexible object – a string – laid along side. It can also be done by rolling the curve upon a straight line. It is not at all trivial that this yields the same result. The length arising from rolling a circle is grossly underestimated by children, and even by adults.

Conversely, rolling a wheel can be used to measure linear distances (expressed by the number of revolutions of a bicycle wheel or a measuring wheel).

Geometrical knowledge can lead to more sophisticated methods of measuring distances. Some of them are possible at an early age. We will reconsider this question later.

Reading and designing maps with distance data does not necessarily presuppose acquaintance with ratio.

The relation between distances and the times needed to cover them does not necessarily presuppose an acquaintance with velocity.

Climbing stairs can be put into relation with distance.

The distances in a network of streets are accessible early.

THE METHOD

2.1. *Aspects of Phenomenology*

I started with an example to be used as a subject matter which I can appeal to when I explain my method. I chose "length" because it is both a rich and relatively easy subject.

First of all, what of the terms "phenomenology" and "didactical phenomenology"? Of course I do not mean "phenomenology" in the sense that might be extracted from the works of Hegel, Husserl, and Heidegger*. Though the clearest interpretation I can imagine is that by means of the example of chapter I, which is to be continued in the following chapters, nevertheless it is worthwhile trying something like a definition.

I start with the antithesis – if it really is an antithesis – between *nooumenon* (thought object) and *phainomenon*. The mathematical *objects* are *nooumena*, but a piece of mathematics can be experienced as a *phainomenon*; numbers are *nooumena*, but working with numbers can be a *phainomenon*.

Mathematical concepts, structures, and ideas serve to organise phenomena – phenomena from the concrete world as well as from mathematics – and in the past I have illustrated this by many examples**. By means of geometrical figures like triangle, parallelogram, rhombus, or square, one succeeds in organising the world of contour phenomena; numbers organise the phenomenon of quantity. On a higher level the phenomenon of geometrical figure is organised by means of geometrical constructions and proofs, the phenomenon "number" is organised by means of the decimal system. So it goes in mathematics up to the highest levels: continuing abstraction brings similar looking mathematical phenomena under one concept – group, field, topological space, deduction, induction, and so on.

Phenomenology of a mathematical concept, a mathematical structure, or a mathematical idea means, in my terminology, describing this *nooumenon* in its relation to the *phainomena* of which it is the means of organising, indicating which phenomena it is created to organise, and to which it can be extended, how it acts upon these phenomena as a means of organising, and with what power over these phenomena it endows us. If in this relation of *nooumenon* and *phainomenon* I stress the didactical element, that is, if I pay attention to how the relation is acquired in a learning–teaching process, I speak of *didactical*

* Is it by accident that – with Habermas included – the names of the most pretentious producers of unintelligible talk in the German philosophy start with an H?

** *Mathematics as an Educational Task*, in particular, Chapters II and XVII.

phenomenology of this *nooumenon*. If I would replace "learning–teaching process" by "cognitive growth", it would be *genetic* phenomenology and if "is . . . in a learning–teaching process" is replaced by "was . . . in history", it is *historical* phenomenology. I am always concerned with phenomenology of mathematical nooumena, although the terminology could be extended to other kinds of nooumena.

2.2. *The Part Played by Examples*

The piece of phenomenology with which Chapter I began was clearly an *a posteriori* constructed relation between the mathematical concept of length and the world of long objects structured by an operation of composing, ⊕. Length was interpreted as a function on this world. I did not analyse how I arrived at this function. Although this was indispensable, I omitted it because I had to tackle this question in the didactical phenomenological section and I wanted to avoid duplication. But as a consequence the didactical phenomenological section contains pieces of pure phenomenology, such as Section 1.15 about the congruence mappings and Sections 1.18–19 about the flexions. Likewise in the sequel I will not clearly separate phenomenology and didactical phenomenology from each other. As promised in the preface I would not sacrifice readability to systematics.

Where did I look for the material required for my didactical phenomenology of mathematical structures? I could hardly lean on the work of others. I have profited from my knowledge of mathematics, its applications, and its history. I know how mathematical ideas have come or could have come into being. From an analysis of textbooks I know how didacticians judge that they can support the development of such ideas in the minds of learners. Finally, by observing learning processes I have succeeded in understanding a bit about the actual processes of the constitution of mathematical structures and the attainment of mathematical concepts. A bit – this does not promise much, and with regard to quantity it is not much, indeed, that I can offer. I have already reported a few examples of such observations, and I will continue in the same way. I do not pretend that at this or that age this or that idea is acquired in this or that way. The examples are rather to show that learning processes are required for things which we would not expect would need such processes. In the first chapter I showed a child suddenly confronted with the necessity to differentiate "big" according to various dimensions, a child placing "far" into the context of "long" and learning about the connection between "half" and "middle". I am going to add another story, which happened a few hours after the event where "half" and "middle" were tied to each other:

Bastiaan's (5; 3) sister (3; 3) breaks foam plastic plates into little pieces, which she calls food. He joins her, takes a rectangular piece, breaks it in about two halves, lays the two halves on each other, breaks them together and repeats the same with a three-layered combination – the fourth piece was already small enough.

I do not know where I should place this observation, whether I should classify it as mathematics, say geometry, or whether it belongs to general cognitive behaviour. I report this observation because I think it is one of the most important I ever made because it taught me a lesson on observing. I do not know whether the age of 5; 3 is an early or a late date for this kind of economic breaking; I do not know either whether Bastiaan imitated or adapted something he had observed before. I know only one thing for sure: that what he did is important and worth being learned. For myself it is fresh material to witness that in no way do we realise all the things children must learn. If I look at what people contrive to teach children, I feel inclined to call out to them: do not exert yourself, simply look, it is at your hand.

Why do people not look for such simple things, which are so worth being learned? Because one half of them do not bother about what they think are silly things, whereas those who do bother are afraid to look silly themselves if they show it. *Weeding and Sowing* is full of such simple stories. I told them in lectures. I do not care whether a large part of the audience interprets my reporting as senility, provided that by my example a small part of the audience is encouraged to follow suit – this, indeed, requires courage.

2.3. *Enactive, Ikonic, Symbolic*

Above I used Bruner's triad "enactive, ikonic, symbolic". Bruner* suggested three ways of transforming experiences into a model of the world: the enactive, the ikonic, and the symbolic representation. Corresponding to the dominance of one of these, he distinguishes phases of cognitive growth.

Bruner's schema can be useful. It has been taken over by others, and its domain of application has been extended, in particular towards the attainment of concepts in learning processes, where similar phases are distinguished. Later I will explain my objections to the idea of concept attainment as such, although I would not oppose the extension of Bruner's triad to concept attainment. As a matter of fact, in Bruner's work there is an example that shows how the three ways of representation can be extended to concept attainment: *enactively* the clover leaf knot is a thing that is knotted, *ikonically* it is a picture to be looked at, and *symbolically* it is something represented by the word "knot", whether or not it is accompanied by a more or less stringent definition.

There is a well-known pleasantry: ask people what "spiral" stairs are. All react the same way: they make their forefinger mount imaginary spiral stairs. Of course, if need be, they would be able to draw them. Does this mean that they are in the enactive or in the ikonic phase? Of course not. For the concept in question they possess a symbol, the words "spiral stairs", though if a *definition* is to be produced, one would have more or less difficulty in passing from the enactive or ikonic to the symbolic representation.

* *Studies in Cognitive Growth* (Edited by J. S. Bruner), Toward a Theory of Instruction, 1966, pp. 10–11.

Consider the number concept "three" and the geometrical concept "straight". Before the child masters these words, he can be familiar with what they mean: clapping his hands thrice and running straight to a goal if it is suggested to him (the enactive phase); sorting out cards with three objects or straight lines pictured on them (the ikonic phase). Mastering the word *three* (or *straight*) means he is in the symbolic phase, since "three" as a word is a symbol for the concept three (or "straight" is for *straight*). But likewise the three dots on dice can be a symbol; for instance, in playing the game of goose. A child that counts intelligently is in the symbolic phase even if this counting is accompanied by moving counters on the abacus. Adding on the abacus is enactive only for a moment. After the first experience it has become symbolic, though the symbolism differs from that of the written digits. The Roman numerals are as symbolic as the Arabic ones. Notches and tallies to indicate numbers belonged to the symbolic phase, even before people invented numerals – they are as symbolic as Roman and Arabic numerals. The cashier in the supermarket who prints amounts of money is neither enactively nor ikonically busy. A little child who claps his hands in joyfulness expresses his feelings symbolically even if he cannot yet pronounce the word joy. As early as kindergarten, children accept a drawing of a dance position where dancers are represented by strokes rather than manikins. If the doors of the men's and ladies' rooms are distinguished by plates of figures in trousers and skirts it does not mean that the decorator imagined the users to be in the ikonic phase; he did so because this difference is differently symbolised in the hundreds of languages that mankind speaks and writes – moreover the plates themselves are already symbols.

With these examples I intend to say that in learning–teaching situations, which are our main interest, Bruner's triad does not yield much. Bruner's domain of application is the psychology of the very young child, and in this period the phases can meaningfully be filled out.

2.4–5. *Concept Attainment and the Constitution of Mental Objects*

2.4. I would like to stress another idea, already stressed in my earlier publications. Let me start with a semantic analysis of the term "concept". If I discuss, say, the number concept of Euclid, Frege, or Bourbaki, I set out to understand what these authors had in mind when they used the word "number". If I investigate the number concept of a tribe of Papuans, I try to find out what the members of this tribe know about and do with numbers; for instance, how far they can count.

It seems to me that this double meaning of "concept" is of German origin. The German word for concept is *Begriff*, which etymologically is a translation of Latin "conceptus" as well as "comprehensio" and which for this reason can mean both "concept" and "(sympathetic) understanding". "Zahlbegriff" can thus mean two things, number concept and understanding of number; "Raumbegriff," concept of space and geometrical insight; "Kunstbegriff," concept of art and artistic competence.

Actually, in other languages too "concept" is derived from a word that means understanding (English, *to conceive*; French, *concevoir*) which, however, does not have the misleading force that the German *begreifen* has. I cannot say whether it has been the influence of German philosophy – in particular, philosophy of mathematics – that created the double meaning of number concept, of space concept, and in their train as it were, of group concept, field concept, set concept, and so on. At any rate the confusion has been operational for a long time and has been greatly reinforced by the New Math and by a rationalistic* philosophy of teaching mathematics (and other subjects) which in no way is justified by any phenomenology. It is the philosophy and didactics of concept attainment, which, of old standing and renown, has gained new weight and authority in our century thanks to new formulations. In the socratic method as exercised by Socrates himself, the sharp edges of concept attainment had been polished, because in his view attainment was a re-attainment, recalling lost concepts. But in general practice the double meaning of concept has been operational for a long time. Various systems of structural learning have only added a theoretical basis and sharp formulations. In order to have some X conceived, one teaches, or tries to teach, the concept of X. In order to have numbers, groups, linear spaces, relations conceived, one instills the concepts of number, group, linear space, relation, or rather one tries to. It is quite obvious, indeed, that at the target ages where this is tried, it is not feasible. For this reason, then, one tries to materialise the bare concepts (in an "embodiment"). These concretisations, however, are usually false; they are much too rough to reflect the essentials of the concepts that are to be embodied, even if by a variety of embodiments one wishes to account for more than one facet. Their level is too low, far below that of the target concept. Didactically, it means the cart before the horse: teaching abstractions by concretising them.

What a didactical phenomenology can do is to prepare the converse approach: starting from those phenomena that beg to be organised and from that starting point teaching the learner to manipulate these means of organising. Didactical phenomenology is to be called in to develop plans to realise such an approach. In the didactical phenomenology of length, number, and so on, the phenomena organised by length, number, and so on, are displayed as broadly as possible. In order to teach groups, rather than starting from the group concept and looking around for material that concretises this concept, one shall look first for phenomena that might compel the learner to constitute the mental object that is being mathematised *by* the group concept. If at a given age such phenomena are not available, one gives up the – useless – attempts to instill the group concept.

For this converse approach I have avoided the term *concept attainment*

* In the 18th century sense of *a priori* concepts epistemology.

intentionally. Instead I speak of the constitution of mental objects,* which in my view precedes concept attainment and which can be highly effective even if it is not followed by concept attainment. With respect to geometrically realisable mental objects (square, sphere, parallels) it is obvious that the constitution of the mental object does not depend at all on that of the corresponding concept, but this is equally true for those that are not (or less easily) geometrically realisable (number, induction, deduction). The reader of this didactical phenomenology should keep in mind that we view the nooumena primarily as mental objects and only secondarily as concepts, and that it is the material for the constitution of mental objects that will be displayed. The fact that manipulating mental objects precedes making concepts explicit seems to me more important than the division of representations into enactive, ikonic, and symbolic. In each particular case one should try to establish criteria that ought to be fulfilled if an object is to be considered as mentally constituted. As to "length" such conditions might be

> integrating and mutually differentiating the adjectives that indicate length, with "long, short",
> comparing lengths by congruence mappings and flexions,
> measuring lengths by multiples and simple fractions of a measuring unit,
> applying order and additivity of measuring results, and
> applying the transitivity of comparing lengths.

2.5. In opposition to concept attainment by concrete embodiments I have placed the constitution of mental objects based on phenomenology. In the first approach the concretisations have a transitory significance. Cake dividing may be forgotten as soon as the learner masters the fractions algorithmically. In contradistinction to this approach, the material that serves to mentally constitute fractions has a lasting and definitive value. "First concepts and applications afterwards" as it happens in the approach of concept attainment is a strategy that is virtually inverted in the approach by constitution of mental objects.

* Fischbein calls them *intuitions*, a word I try to avoid because it can mean inner vision as well as illuminations.