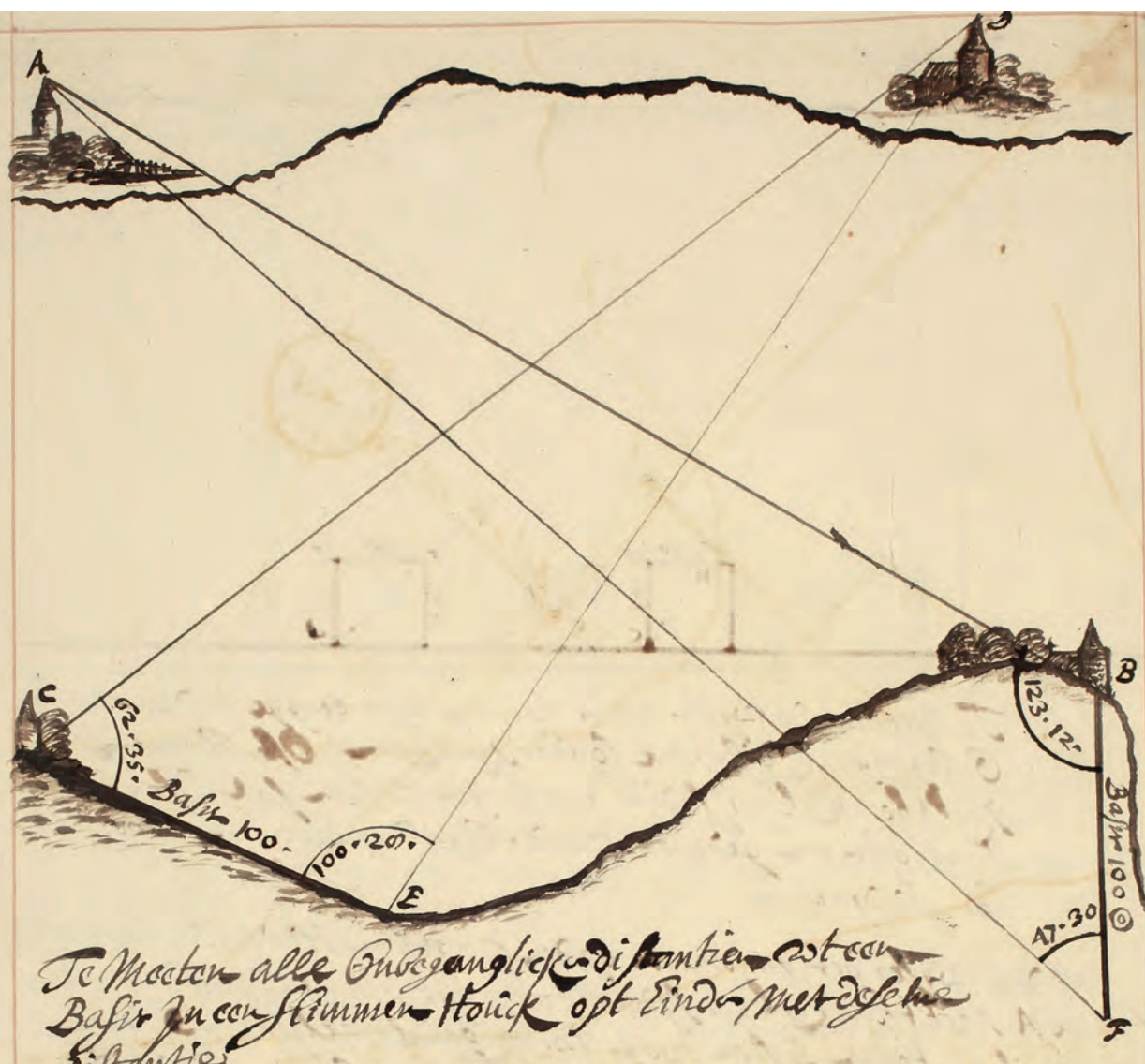


Kristín Bjarnadóttir, Fulvia Furinghetti, Jenneke Krüger,
Johan Prytz, Gert Schubring, Harm Jan Smid (Eds.)

“DIG WHERE YOU STAND” 5

PROCEEDINGS OF THE FIFTH INTERNATIONAL CONFERENCE
ON THE HISTORY OF MATHEMATICS EDUCATION
SEPTEMBER 19-22, 2017, AT UTRECHT UNIVERSITY, THE NETHERLANDS



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**Proceedings of the Fifth International Conference
on the History of Mathematics Education
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Editors:
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Utrecht University

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Introduction

The fifth International Conference on the History of Mathematics Education (ICHME-5) took place in the University Hall of the University of Utrecht, the Netherlands, from 19-22 September, 2017. The organisation of the conference and the production of the Proceedings were sponsored by the Descartes Centre and the Freudenthal Institute, both part of Utrecht University, and by the Dutch Association of Mathematics Teachers (NVvW). The local organising group consisted of Jan van Guichelaar (NVvW), Jenneke Krüger (NVvW, Freudenthal Institute), Heleen van der Ree (NVvW) and Harm Jan Smid (NVvW). The members of the International Program Committee were Kristín Bjarnadóttir (University of Iceland), Jan Hogendijk (Utrecht University), Jenneke Krüger (Utrecht University), Johan Prytz (Uppsala University), Gert Schubring (Universität Bielefeld, Germany/ Universidade Federal do Rio de Janeiro, Brazil) and Bert Theunissen (Utrecht University). Fulvia Furinghetti (University of Genoa, Italy) served as advisor to the International Program Committee. Nathalie Kuijpers (Freudenthal Institute) took care of the layout of these Proceedings.

Altogether 65 participants from 21 countries registered for this conference. In the opening lecture Danny Beckers (Free University, Amsterdam) presented a light-hearted overview of mathematics education in the Netherlands from 1275-2000. During the conference 40 contributions were presented. Of those, 24 peer-reviewed papers are published in these Proceedings. They may be categorized according to the following themes (bearing in mind that other categorizations would be possible):

Transmission of ideas

Evelyn Barbin; Frédéric Metin.

Arithmetic

Patricia Baggett and Andrej Ehrenfeucht; Luciane de Fatimai Bertini; Kristín Bjarnadóttir; Elisabete Zardo Búrigo.

Mathematics education in specific countries

Ezzaim Laabid; Andreas Christiansen; Man Keung Siu.

Teacher education

Luis Puig; Jose Matos and Ana Santiago.

The 1960s

Dirk de Bock, Michel Roelens and Geert Vanpaemel; Bert Zwaneveld and Dirk de Bock.

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Dutch mathematics education and educators, from the 18th to the 20th century

Henk Hietbrink; Martinus van Hoorn; Jenneke Krüger; Harm Jan Smid.

Influential people and their ideas

Viktor Freiman and Alexei Volkov; Agnes Tuska and András Benedek; Regina de Cassia Manso de Almeida.

Concepts and language in manuals and textbooks

Viktor Blasjö; Leo Rogers.

Ideas on the teaching of mathematics in journals and textbooks

Antonio Oller; Alexander Karp.

To emphasize the continuity of the project behind the conference held in Utrecht, the volume containing the proceedings keeps the original title of the first conference, i.e. “*Dig where you stand*” (followed by 5, which is the number of the conference). This sentence, which is the English title of the book *Gräv där du står* (1978) by the Swedish author Sven Lindqvist, stresses the importance of knowing the historical path of workers, craftsmen and professionals. In particular Lindqvist considers it important for these groups to know their own history in order to have a voice in society. However, these histories are often overlooked by traditional historians, which mean that you have to do the research on your own, where you stand. As was explained in the Introduction of the Proceedings of ICHME-3, we deem that “Dig where you stand” may be a suitable motto for those (historians, educators, teachers, educationalists) who wish to understand sensitively and deeply the teaching and learning of mathematics.

The editors:

Kristín Bjarnadóttir, Fulvia Furinghetti, Jenneke Krüger,

Johan Prytz, Gert Schubring, Harm Jan Smid.

On French heritage of Cartesian geometry in Elements from Arnould, Lamy and Lacroix

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Abstract

When Descartes wrote La géométrie in 1637, his purpose was not to write “Elements” with theorems and proofs, but to give a method to solve “all the problems of geometry”. However, in his Nouveaux Eléments de Géométrie in 1667, Antoine Arnould included two important Cartesian conceptions. The first one is the systematic introduction of arithmetical operations for geometric magnitudes and the second one is what he called “natural order”, that means Cartesian order which goes from the simplest geometric objects (straight lines) to others. This last conception led Arnould to numerous novelties, mainly, a chapter on “perpendicular and oblique lines”, and new proofs for Thales and Pythagoras theorems. In 1685, Bernard Lamy followed Arnould’s textbook in his Eléments de géométrie, in which he also introduced Cartesian method to solve problems. Our first aim is to analyze incorporations of Cartesian conceptions and Cartesian method into Arnould and Lamy’s Eléments. Our second aim is to analyze their impact for the heritage of Cartesian geometry into mathematical teaching, especially the “natural order” coming from Arnould and the “application of algebra to geometry” coming from Lamy. In this framework, we show that the geometric teaching of Sylvestre-François Lacroix played an important role in the 19th century and beyond.

Keywords: René Descartes, Antoine Arnould, Sylvestre-François Lacroix, Cartesian order, arithmetization of geometry

Introduction: Cartesian order and arithmetization of geometry

Towards the end of the 1620s, René Descartes wrote *Règles pour la direction de l’esprit* [Rules for the Direction of the Mind]. This text had never been achieved and published in his lifetime, but it is interesting to know that it had been read by Antoine Arnould. In his *Rules*, Descartes criticized Aristotle’s science based on syllogisms, because they can conclude with certainty but they banish obviousness (Rule X), and he gave his proper conception of science. Indeed, he wrote in Rule XII: “We can never understand anything beyond these simple natures and a certain mixture or composition of them with one another” (Descartes, 1998, p. 155). Hence,

all human knowledge consists in this one thing, to wit that we distinctly see how these simple natures together contribute to the composition of the other things (Descartes, 1998, p. 161).

In that way, he proposed to substitute an order of simplicity of things instead of a logical order of propositions. Descartes continued to call deduction the manner by which a composite nature can be obtained from simple ones. Thus, Aristotelian and

Cartesian deductions are different because, in the first one, propositions are deduced from others by logical rules and, in the second one, composed things are deduced from simple ones by simple operations.

Simple things and simple operations of geometry are introduced as soon as the first sentence of *La géométrie* (1637), where Descartes wrote:

Any problem in geometry can easily be reduced to such terms that a knowledge of the length of certain straight lines is sufficient for its construction. Just as arithmetic consists of only four or five operations, namely addition, subtraction, multiplication and the extraction of roots [...] (Descartes, 1954, p. 2).

So, simple things are straight lines and simple operations are arithmetic operations. This ‘arithmetization’ of geometry, leans on the introduction of one line called “unit” by Descartes, by analogy with arithmetic. Indeed, this unit permits us to obtain a product of two lines BD and BC , not as a rectangle, like in Greek geometry, but as a simple line. If AB is the unit, then BE is the product of BD and BC (figure 1 left). It also permits us to divide two segments and to obtain a segment. To consider a square root of a line, has no meaning in Greek geometry, but in Cartesian geometry, if FG is the unit then GI is the square root of GH (figure 1 right).

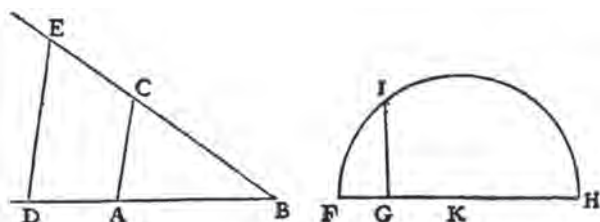


Fig. 1. Product of two segments and square root of a segment (Descartes, 1954, p. 4)

Descartes pointed out that often, it is not necessary to draw lines and it is sufficient to designate them by single letters, to which symbols of arithmetic will be applied. Moreover, thanks to the unit, it is possible to consider for instance a^3 or b^2 as simple lines, and, for instance, to consider the cube root of $a^2b^2 - b$ without taking into account the geometric meaning of this formula.

Descartes' purpose was to provide a systematic method to solve problems of geometry by deducing unknown lines from known lines. This method consists of translating problems by equations on lines and to solve these ones. In the First Book of *La géométrie*, Descartes used his method to solve, not elementary problems, but a difficult problem left to us by Pappus. In the Second Book, in accordance with his general conception and thanks to the unit line, he considered curves as composed by simple lines by means of arithmetic operations, when for a given line AG and for

each point C of the curve, there exists a single equation linking CM and MA . These lines are called “geometric” and the others “mechanical”. So, he did not introduce a “Cartesian coordinate system”. He used his method to find normal lines CP to a “geometric curve” (figure 2).

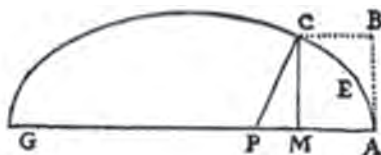


Fig. 2. Normal line to a “geometric curve” in *La géométrie* (Descartes, 1954, p. 97)

Cartesian order in Arnauld’s *Nouveaux Éléments* (1667)

In 1662, the Jansenists Antoine Arnauld and Pierre Nicole wrote *La logique ou l’art de penser* [Logic or the art of thinking], in which they gave the list of the defects of Geometers¹. The first one is “Paying more attention to certainty than to obviousness, and to the conviction of the mind than to its enlightenment” (Arnauld & Nicole, 1850, p. 331) while the fifth defect is “Paying no attention to the true order of nature” (Arnauld & Nicole, 1850, p. 335). They added about this defect:

This is the greatest defect of the geometers. They have fancied that there is scarcely any order for them to observe, except that the first propositions may be employed to demonstrate the succeeding ones. And thus, disregarding the true rule of method, which is, always to begin with things the most simple and general, in order to pass from them to those which are more complex and particular, they confuse everything, and treat pell-mell of lines and surfaces, and triangles and squares, proving by figures the properties of simple lines, and introducing a mass of other distortions which disfigure that beautiful science (p. 335).

That means that they considered it as a defect to not follow the Cartesian order in geometry: it is the order of nature. They wrote that Euclid’s *Elements* are quite full of this defect:

He measures the dimension of surfaces with that of lines. [...] It would be necessary to transcribe the whole of Euclid, in order to give all the examples which might be found of this confusion. (Arnauld & Nicole, 1965, p. 335).

They opposed the ‘method of doctrine’, found in Euclid, to the ‘method of invention’, that is Descartes’ one.

¹ Translations of Arnauld, Lamy, Lacroix’s texts by Évelyne Barbin.

'Natural order' accordingly to Arnauld

In 1667, Arnauld edited *Nouveaux Éléments de géométrie contenant un ordre tout nouveau* [New Elements of geometry containing a very new order], intended for the schools of Port-Royal. The “very new order” of the title is the Cartesian order, called “natural order” by him. He wrote in his preface:

It was a very advantageous thing to get accustomed to reduce our thoughts to a natural order, this order being as a light that clears up ones by the others [...] Euclid's Elements are so confused and muddled, that far from bringing to the mind an idea and a taste for a true order, on the contrary, they only make the mind used to disorder and confusion (Arnauld, 1667, np).

After four Books setting out the arithmetization of magnitudes, there are thirteen Books in natural order: straight and circular lines, perpendicular and oblique lines (Books V); parallel lines (Book VI); lines ended by a Circumference (Book VII); angles (Books VIII and IX); proportional lines and reciprocal lines (Books X and XII); plane figures according their angles and sides (Books XII and XIII); plane figures according their surfaces (Books XIV and XV).

To follow a natural order requires new proofs, for example for the theorems of Pythagoras and Thales; two propositions on simple lines but proven in Euclid by using triangles, more composite figures than lines. Arnauld wrote that natural order

gives rise to find more fertile principles, and clearer proofs. And indeed, in these *New Elements*, there are nearly very new proofs, which arise from principles by themselves, and which contain a great number of new proposals (Arnauld, 1667, np).

The first important principles concern perpendicular and oblique lines, that are defined and studied without using angles.

“Perpendicular and oblique lines”: new principles

Euclid defined perpendicular lines by using angles:

when a straight line set up on a straight line makes the adjacent angles equal to one another, [...] the straight line standing on the other is called a perpendicular to that on which it stands (Euclid, 1956, p. 153).

Arnauld explained that to form a more distinct definition of two perpendicular lines we can conceive that when two points of the cut line are equally distant from the cutting one, every point of the cutting line is equally distant from these two points of the cut line. For that, he introduced the notion of distance and $CA = CB$, $DA = DB$, $EA = EB$ (figure 3 left). Arnauld claimed that this statement is true because it is obvious:

I say that the consideration of the nature of straight lines only makes us see the truth of this proposition, and that it is impossible to keep the natural order of things in Geometry without this consideration [...]. So, we have to reject the scruple we could have, to receive this proposition as obvious by itself; we cannot do anything else without muddling the natural order of things and using triangles to show properties on straight lines, that means without using more compound to explain more simple, which is contrary to the true method (Arnould, 1667, pp. 87-88).

Here he followed Descartes' general rule of the *Discours de la méthode*: "things that we very clearly and distinctly conceived are all true" (Descartes, 1637, p. 33). The role of obviousness in Cartesian science permitted him to admit as axioms, sentences that are necessary for following the natural order.

Arnould continued with an explanation of the manner to consider oblique lines for understanding them better. Three lines have to be conceived together with three distances: kb for oblique line, kc for perpendicular line, bc for the distance away of the perpendicular line (figure 3 middle). He pointed out that bc and kc can be considered as oblique lines also with perpendicular line gc . He insisted on this explanation because

the consideration of these three lines [...] will help us to understand several things on oblique lines which cannot be explained by triangles, as it is a reversed order (Arnould, 1667, p. 95).

"Fundamental proposition on oblique lines" states that oblique lines Kf and Kg from the same point K to a same line z are longer when they are more distant of the perpendicular line KB . Arnould drew $KB = BC$, Cf and Cg (figure 3 right). By the more exact definition of a perpendicular line (Arnould), $Kf = Cf$, $Kg = Cg$, and by the means of Archimedes, KfC is shorter than KgC . Therefore, Kf (half of KfC) is shorter than Kg (half of KgC).

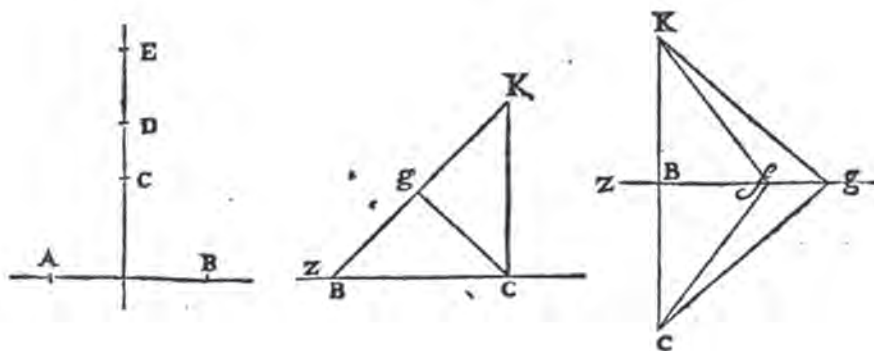


Fig. 3. Perpendicular and oblique lines in Arnould (Arnould, 1667, pp. 87, 95, 96)

Arnauld's new proof of Thales' theorem

We take this proof as an example because of its long heritage in teaching. Contrary to Euclid (Prop. 2, Book VI), Arnauld did not use triangles and gave “the very natural proof, that nobody ever gave I think” (Arnauld, 1667, p. 191). He began Book X on proportional lines with the “Fundamental proposition of proportional lines”: if two lines C and c are “equally inclined” in two “parallel spaces” A and E , then we have $P : p = C : c = B : b$ (figure 4 left). To prove it, he divided p in 10, 20, 500, 6000, 10000, &c. equal lines x (Arnauld, 1667, p. 191). He then considered parallels through the points of division; then c is divided in equal spaces because they have the same perpendicular. Then he wrote that the same can be done for P and C and he concluded, with taking in account the incommensurable case. First corollary of the “fundamental proposition” is Thales' theorem in a more general situation than in Euclid (figure 4 right): “several diversely inclined lines in a same parallel space are proportionally cut by parallels to this space” (Arnauld, 1667, p. 193).

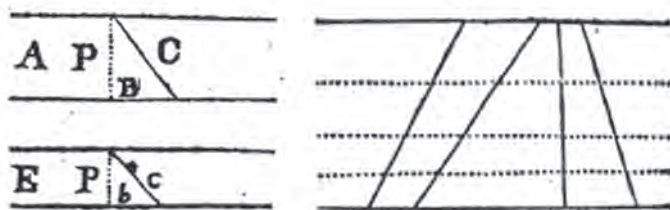


Fig. 4. Thales' theorem in Arnauld's *Nouveaux Éléments* (Arnauld, 1667, p. 193)

Arithmetization of geometry and method in Lamy's *Éléments*

Bernard Lamy was an Oratorian and a teacher of mathematics (Barbin, 1991). In 1676, he was banished from the University of Angers because of his Cartesian convictions. He published several textbooks and modified them all along their successive editions, like *Traité de la grandeur en général ou les éléments de mathématiques* [Treatise on magnitude in general or Elements of mathematics] with 23 editions from 1680 to 1765, *Les éléments de géométrie ou de la mesure de l'étendue* [Elements of geometry or measurement of extension] with 14 editions from 1685 to 1758, *Entretiens sur les sciences* [Conversations on sciences] with 12 editions from 1683 until 1768. In this last book, he showed himself an unfailing disciple of Descartes, Arnauld and Nicole. For instance, he wrote:

For perceiving well, we have to wait the clarity before to consent. We have not to do it before being forced by the obviousness of the truth (Lamy, 1966, p. 86).

In his *Éléments de géométrie*, Lamy followed the natural order of his predecessor Arnould, because as he wrote in his Preface:

[Geometry] has to be treated with method, which is not done by Euclid. He only thought to range his propositions, in such a way that they can serve for proving ones from the others; in this he succeeded. The truth is contained in his Elements; but besides there is so much confusion [...] as Monsieur Nicole complains in the Preface of *Elements of geometry* by Monsieur Arnould, which were printed in 1667 for the first time. It is in these *Elements* of Monsieur Arnould that we find this natural order, which is not in those of Euclid (Lamy, 1734, p. vi).

Book I concerns straight lines (perpendicular, oblique, parallel lines) then circular lines, while Book II concerns plane areas. Lamy added a Book V on Bodies to Arnould's *Éléments*.

Lamy's arithmetization of geometry for proving propositions

Book III gives “properties which suited every magnitude, applied to lines, planes, solids, and proven”. It begins with the four operations of Arithmetic on lines, planes and bodies. Like Descartes, Lamy introduced the unit line to operate arithmetically on all kind of magnitudes and to obtain lines as results, independently of their geometric names. In this manner, geometric properties on figures can be translated by arithmetic formulas on simple things that are lines. He wrote:

Also remark that, as it is advantageous to accustom one's spirit quickly to these kinds of calculations [...] we will delete all the figures used by Euclid and his interpreters for their proofs ordinarily [...], and it is now appropriate to make calculations with the pen in the hand (Lamy, 1731, p. 143).

Cartesian arithmetization of geometry is used by Lamy for giving new interpretations and new proofs of Euclid's Book II. He wrote in his Preface that

it is important to get used to see without images, and to be convinced that there are truths which are conceived otherwise than with bodies (Lamy, 1731, p. iii).

We take as an example Proposition VI of Euclid's Book II, that states that

if a straight line is bisected and a straight line is added to it in a straight line, then the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half equals the square on the straight line made up of the half and the added straight line” (Euclid, 1956, p. 385).

Euclid proved that (the area of) rectangle $ADMK$ juxtaposed to square $LHGE$ is equal to (the area of) square $CDFE$ (figure 5 left). While Lamy considered a simple

line and explained that we have to prove an arithmetic formula on lines, that is $AD \times BD + BC^2 = CD^2$ (figure 5 right). He named $AC = CB = b$, then the proposition results of the two equalities: $AD \times BD + BC^2 = 2bd + dd + bb$ and $CD^2 = (b + d)^2 = bb + 2bd + dd$

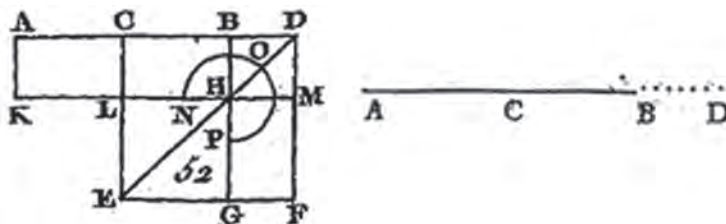


Fig. 5. Lamy's proof compared to Euclid's proof (Lamy, 1731, p. 146)

Cartesian method in Lamy's Book VI

In the last Book of his *Éléments*, Lamy wrote that there exists another method than the “method of doctrine”, that is the “method of invention”. Then he exposed Descartes' method to solve problems (without giving his name) and he applied it to solve many elementary problems.

In Problem I, he opposed the two methods by giving two manners to find a parallel DE to a basis BC of an isosceles triangle in such a way that $DB = DE$ (figure 6). For the first manner, he applied geometric propositions to prove that BE has to be the bisector of angle ABC . While, for the second manner, he named known lines $AB = a$, $BC = d$, and unknown line $AE = x$, he translated the problem by equations and obtained $a : d = x : a - x$ and $aa = cx$ as solution. He added: “this second analytic manner is general, and not particular to this problem” (Lamy, 1731, pp. 409-410). It is clear that the same method could be applied to find DE when knowing any kind of relation between DB and DE .

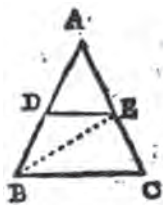


Fig. 6. Cartesian method in Lamy's Problem 1 (Lamy, 1731, p. 409)

Lamy's textbooks had been known in the 18th century by their numerous editions over a long period and by commentaries of the French philosopher Jean-Jacques Rousseau. Lamy's *Entretiens sur les sciences* and *Éléments de géométrie* were in his library

and they are discussed in Rousseau's books, like in his *Confessions* (1770) (Barbin, 2003). In this last book, Rousseau opposed Euclid to Lamy:

[Books] which combined devotion and science were most suitable for me, particularly those of the Oratory and Port-Royal, which I began to read, or rather, to devour. I came across one written by Father Lamy, entitled *Discussions on Sciences* a kind of introduction to the knowledge of those books which treated of them. I read and re-read it a hundred times, and resolved to make it my guide. I did not like Euclid, whose object is rather a chain of proofs than the connection of ideas. I preferred Father Lamy's *Geometry*, which from that time became one of my favorite works, and which I am still able to read with pleasure. Next came algebra, in which I still took Father Lamy for my guide (Rousseau, 1896, p. 238).

But, when he wrote on "application of algebra to geometry", it is clear that he did not follow Lamy on his proposal to "see without image":

I have never got so far as to understand properly the application of algebra to geometry. I did not like this method of working without knowing what I was doing; and it appeared to me that solving a geometric problem by means of equations was like playing a tune by simply turning the handle of a barrel-organ. The first time that I found by calculation, that the square of a binomial was composed of the square of each of its parts added to twice the product of those parts, in spite of the correctness of my multiplication, I would not believe it until I had drawn the figure (Rousseau, 1896, p. 245).

"Application of algebra to geometry" as a subject in textbooks

"Application of algebra to geometry" exists as a subject in textbooks from the beginning of 18th century. But it is possible that Rousseau took the expression into the *Encyclopédie* of Denis Diderot and Jean d'Alembert, where it appeared as an item. This item had been enriched in the *Encyclopédie méthodique* (1784), where d'Alembert wrote:

Application of algebra or analysis to geometry

It is in the Geometry of M. Descartes that we find the application of Algebra to Geometry, as well as excellent methods to improve Algebra itself: with that this great genius render an immortal service to Mathematics [...]. He was the first to learn how to express the nature of curves by equations, to solve problems of geometry with these curves; then to prove theorems of geometry with the help of algebra calculation (D'Alembert, 1784, p. 92)

Guisnée and Malezieu's textbooks in 18th century

Nicolas Guisnée was 'Royal teacher of mathematics'. In 1705, he published his *Application de l'algèbre à la géométrie* [Application of algebra to geometry], where he followed Descartes and Lamy by giving

methods to prove all theorems of geometry by algebra, and to solve and to construct all geometric and mechanical problems (Guisnée, 1705, p. ii).

After an introduction of 65 pages on algebraic calculus, Section I gives general principles to apply algebra to geometry and Section II concerns problems where solutions are given by determined equations of first or second degree, for instance a problem to inscribe a square in a given triangle (Guisnée, 1705, p. 38).

Other sections are ranged according to the types of equations obtained for solving problems, so curves are essentially considered as solutions of problems. Section VIII gives method to solve indeterminate problems of straight lines, circles and conics. Like in Descartes, coordinates of a point of a curve are straight lines not defined in a Cartesian system of axis. Guisnée treated conics and their propositions by algebra, but also mechanical curves, like spiral and cycloid, where coordinates are not straight lines but curved lines (Guisnée, 1705, p. 327).

Nicolas de Malezieu edited a new edition of *Éléments de géométrie de Monsieur le Duc de Bourgogne* [Elements of geometry of Sir the Duke of Burgundy] in 1722, where he used Arnould's order and devoted a part on "perpendiculars and oblique lines". This edition contains a part on introduction of application of algebra to geometry, where Malezieu explained the

great advantage of algebraic calculus [...] that permits to prove all theorems and to solve all problems with as much great facility as there are difficulties with the manner of the Ancients (Malezieu, 1722, p. 2).

His introduction to algebra covers around 80 pages. Then, he only gave some applications, for instance, to determine the area of a triangle from its sides, to obtain the equation of an ellipse from its Apollonius' definition, or to prove an Archimedean theorem on sphere and cylinder.

Lacroix's Elementary treatise on application of algebra to geometry

In 1798, Sylvestre-François Lacroix edited a *Traité élémentaire de trigonométrie et d'application de l'algèbre à la géométrie* [Elementary treatise of trigonometry and of application of algebra to geometry] for the Collège des Quatre-Nations, that had an 11th edition in 1863. After a historical part, where he referred to works of Descartes, Euler and Cramer, Lacroix explained that "there is no reason to imitate them and, on the contrary, one has to take an opposite way to the one they followed, because

one has to tend towards a very different purpose” (Lacroix, 1803, p. x). For him, the question is:

What a treatise on application of algebra to geometry has to contain, when it is intended for students who have to practice physico-mathematics, for young people who study to enter in Polytechnic school for instance? It is clear that we have to insert all that is necessary to understand the most recent and complete books that treat physico-mathematics, or lessons that are given in Polytechnic school. (Lacroix, 1803, p. xi).

Like Monge for descriptive geometry, Lacroix researched an “*élémentation*” for the application of algebra to geometry that put forward links between ideas (Barbin, 2015). Indeed, he wrote:

all that does not increase the power of methods or does not shorten the chain that links results between them has no to enter in elements (Lacroix, 1803, p. xii).

He explained that he wants to show the double point of view of the application of algebra to geometry, as a means to combine theorems of geometry and as a general means to deduce properties of extent from a little number of principles. He pointed out that:

This branch of mathematics, considered in general, contains, not only the research of properties of the extent by the means of the algebraic process, but it has also to show how we can represent all what means any algebraic expression by these properties, to reduce construction of figures to operations continually; and to come back from these last ones to the first ones (Lacroix, 1803, p. 73).

Lacroix began to solve elementary problems, for example to inscribe a square into a triangle, like Guisnée. He generalized Lamy’s Problem I (figure 6) by finding DE equal to a given line MN (Lacroix, 1803, p. 91). Then he exposed “Descartes’ fundamental idea” to represent curves by equations between two undetermined lines, by pointing out the role of the unit line:

Descartes, the first one, by remarking that figures and forms determine relations of magnitudes between straight lines, reached to apply algebra to theory of lines in general, and by this discovery, mathematics changed their face. If we conceive, for instance, that from all points of any line DE , we lead perpendiculars $PM, P'M', P''M''$, etc; to a straight line AB , given by its position, and that from A , we measure distances AP, AP', AP'' , etc., each of these lines and their corresponding perpendiculars will be linked in such a manner that we can deduce one from the other (figure 7). [...] Nothing prevents us to imagine that lines AP, PM , are related to a common line, taken as a unit, and from that, they can be represented by numbers or letters. If this relation, between AP and PM , between AP' and $P'M'$, etc. can be expressed by an

algebraic equation, this equation will characterize the line DE (Lacroix, 1803, pp. 105-106).

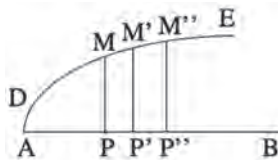


Fig. 7. Lacroix's figure to represent a curve by an equation (Lacroix, np)

Lacroix began with equations of a straight line and a circle. He constructed each conic corresponding to equations of second degree, then obtained properties of them by algebra. Later, he solved other problems, like duplication of a cube and trisection of an angle. He introduced the “Cartesian system of axis” only in an Appendix of 30 pages, devoted to curved surfaces and double curved lines. So, we can understand his teaching on application of algebra to geometry as an elementary way to combine objects and to solve problems, in the spirit of the “*élémentation*” of the new schools after the French Revolution (Barbin, 2015)

Application of algebra to geometry in the 19th century

From the beginning of the 19th century, textbooks on analytic geometry applied to curves and surfaces of 2nd order (those of Biot, Le Français, Boucharlat) had been edited for candidates for the École polytechnique. In those books curves and surfaces are determined by the means of a Cartesian system of axes. But other textbooks maintain Lacroix's branch application of algebra, like the one of Jean-Guillaume Garnier in the second edition of *Géométrie analytique ou application de l'algèbre à la géométrie* [Analytic geometry or application of algebra to geometry] (1813). He devoted a first Chapter on geometric constructions before defining points and lines in a Cartesian system of axis. Pierre Louis Marie Bourdon's *Application de l'algèbre à la géométrie* [Application of algebra to geometry] (1825) has three sections: Section 1 gives “a first method to solve questions of geometry by calculation”, while Sections 2 and 3 are devoted to “analytical geometry with two (three) dimensions”. Georges Ritt's *Problèmes d'application de l'algèbre à la géométrie* [Problems of application of algebra to geometry] (1857), intended for students of collèges, contains 122 elementary problems of constructions (see Moussard, 2015). These kinds of problems appear in French textbooks for students of collèges (14 years aged) until the years 1960. We find, for instance, Lamy's problem I (figure 6), where it is asked that $DE = BD + CE$ (figure 8) in a textbook of 1962 (Lebossé & Hémary, 1962b, p. 91).

133. Problème de géométrie. — Soit un triangle ABC dans lequel $BC = 13$ cm, $AC = 15$ cm, $AB = 11$ cm; par un point D du côté AB, on mène la parallèle à BC qui coupe AC en E. Déterminer D de façon que : $DE = BD + CE$.

Pour placer le point D il suffit de connaître la longueur BD. Posons $BD = x$ (en centimètres), et calculons EC. D'après le théorème de Thalès (fig. 4) :

$$\frac{BD}{CE} = \frac{AB}{AC} \quad \text{ou} \quad \frac{x}{CE} = \frac{11}{15}$$

d'où $CE = \frac{15x}{11}$

Calculons DE, en utilisant la similitude des triangles ADE et ABC. Nous avons :

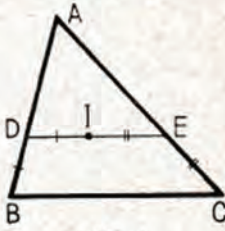
$$\frac{DE}{BC} = \frac{AD}{AB} \quad \text{ou} \quad \frac{DE}{13} = \frac{11 - x}{11}$$


Fig. 4.

Fig. 8. Lamy's Problem I in textbook of 1962 (Lebossé & Hémerly, 1962b, p. 142)

'Natural order' from Lacroix to 20th century

Lacroix wrote in his *Essais sur l'enseignement* [Essays on teaching] (1805) that Euclid introduced a kind of disorder and he continued by writing on Arnould's textbook:

Arnould (of Port-Royal) [...] undertook to correct this defect in his *New Elements of Geometry*, edited for the first time in Paris in 1667. This book is, I think, the first one where the order of propositions of Geometry corresponds to the one of abstractions, by firstly considering properties of lines, then those of surfaces and at last those of bodies. [...]. We could almost remark his idea to prove directly on lines, that parallels led by points taken at equal distances on sides of an angle, also cut the other side at equal distances [Proof of Thales's Theorem], proposition that those who followed Arnould's order took as basis of the theory of proportional lines (Lacroix, 1805, pp. 289-291).

As we saw already, Arnould's perpendicular and oblique lines is necessary to follow a 'natural order', in particular to avoid using of triangles in proofs of propositions concerning simple lines. So, it seems an obligatory way for all authors who adopted Arnould's order. That is why we will examine the presence or not of this theory in textbooks in the 19th and 20th centuries. In the same manner, we can research the presence or not of a new proof of Thales' theorem.

Perpendicular and oblique lines in Lacroix's Éléments de Géométrie

Lacroix's *Éléments de géométrie for the École centrale des quatre-nations* (1799) contains two parts. Section I of first part treats lines only: straight and circular lines, perpendicular

and oblique lines, theory of parallel lines, polygonal lines, especially inscribed in circular lines. Section II treats areas of polygons and circles. The second part concerns planes and bodies. But Lacroix introduced angles and triangles before the part on perpendicular and oblique lines, and so he reversed Arnould's 'natural order'. We can understand Lacroix's order as a way to break with the Euclidean order, adopted by Legendre in his *Éléments de géométrie* of 1794, and to put forward an order of simplicity of ideas that have to be combined (Barbin, 2007). This order corresponds to the "élémentation" promoted in *École normale de l'an III* (Barbin, 2015). Indeed, Lacroix wrote in the "Preliminary discourse" to his textbook:

The method of geometers is not the only cause of the certainty of their results, this certainty mainly comes from the nature of ideas they have to combine. [...] It is less in the method than in the simplicity of the first ideas and their obviousness in which the certainty of the reasoning consists (Lacroix, 1811, pp. xiv-xvvi).

Thales's Theorem is treated as in Arnould. Lacroix proved that if parallels AG , BH , etc. cut two straight lines AF and GM in such a way that AB , BC , etc. are equal then GH , HI , etc. will be equal. Then he proved that "three parallels AG , DK , FM , always cut two straight lines, AF and GM , in proportional parts, such that $AD : DF :: GK : KM$ " (Lacroix, 1811, p. 83) (figure 9). He discussed the case where the lines are incommensurable in a footnote.

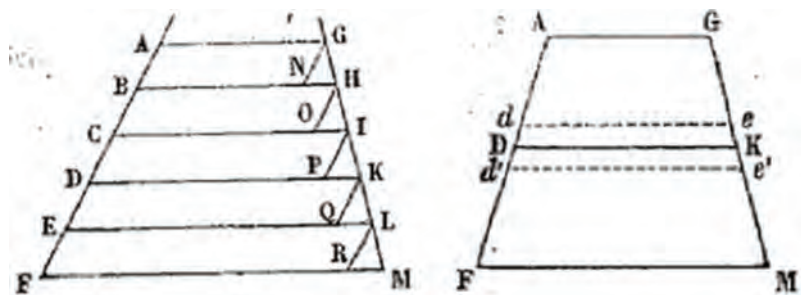


Fig. 9. Thales' theorem in Lacroix's *Éléments de géométrie* (Lacroix, 1811, np)

Perpendiculars and oblique lines as a part of textbooks

Perpendicular and oblique lines appeared in many kinds of textbooks, where sometimes it seems as a kind of obligatory passage. For instance, in the First lesson of his *Géométrie et mécanique des arts et métiers* [Geometry and mechanics of arts and crafts] (1826), intended for the Conservatoire Royal des arts et métiers, Charles Dupin wrote that Arnould's "fundamental proposition on oblique lines" has many applications for artists and mechanical workers (Dupin, 1826, pp. 28-29). Vincent and Bourdon, who taught in upper grades of Lycées, introduced many geometric

novelties in their *Cours de géométrie élémentaire* [Elementary Course on geometry] (1844, 5th ed.), like radical axes, theory of poles and polars, and elements of descriptive geometry. So, here figures are considered in space. But their Chapter I begins with a so-called “Theory of perpendicular and oblique lines” that uses the notion of distance (Vincent & Bourdon, 1844, pp. 33-37). Theorem I states that a perpendicular led from an exterior point O to a straight line AB is the shorter distance of the point to the line and is proven by rotation around OC (figure 10). Theorem II is Arnould’s “fundamental theorem”, proven by a motion of folding around OC . The notion of distance is later used to define bisector of an angle and parallels. Triangles only appear afterwards.

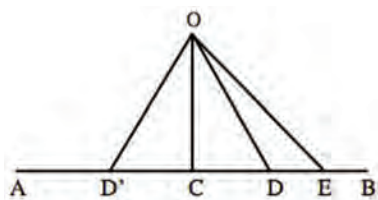


Fig. 10. Arnould’s theorems in textbook of 1844 (Vincent & Bourdon, 1844, np)

Perpendicular and oblique lines” in curricula of Collèges

In 1838, ‘Properties of perpendicular and oblique lines’ appeared in curricula for the 3th grade of Collèges (14 years old students), just after angles and before parallels and triangles (Rendu, 1846, p. 645). But in 1859, it appeared just after angles and triangles and before parallel lines. It is used to prove Thales’s theorem like in Lacroix. We find “perpendicular and oblique lines” in *Leçons nouvelles de géométrie élémentaire* [New Lessons on elementary geometry] of Amiot (Amiot, 1865) for Lycées, in textbooks of Combette (Combette, 1895) and Hadamard (Hadamard, 1898) and, after the Reform of 1902-1905, in Vacquant and Lépinay’s *Cours de géométrie élémentaire* [Course on elementary geometry] (Vacquant & Lépinay, 1909). It was a part of teaching in Collèges until the Reform of modern maths. During all this period, perpendicular and oblique lines had been treated after triangles, so the spirit of ‘natural order’ seemed to be lost. But, proof of Thales’s theorem by Arnould and Lacroix remained for a long time, until proofs were banished from the curricula.

More recently, perpendicular and oblique lines again appeared as a chapter of *Formes et mouvements, perspectives pour l’enseignement de la géométrie* [Forms and motions, prospects for the teaching of geometry] edited in 1999 by the team CREM in Belgium. It is the first theme before angles and triangles, of what is called “natural geometry” (CREM, 1999).

Conclusion: The Heritage of Cartesian Science in Elementary Teaching of Geometry

Cartesian science can be characterized as a new manner for conceiving deduction, which is to deduce composite things from simple things. As we have shown, two consequences are taken up in *Éléments* written by Arnauld and then Lamy in the 17th century. Firstly, Arnauld introduced a “natural order”, corresponding to the order of simplicity of geometric objects, and a theory of “perpendicular and oblique lines”, that is necessary to conform to this order. Secondly, Lamy developed an ‘arithmetization of geometry’, using the most essential feature of Cartesian geometry, that is the systematic introduction of a unit line, to translate theorems in formulas and problems in equations on simple lines.

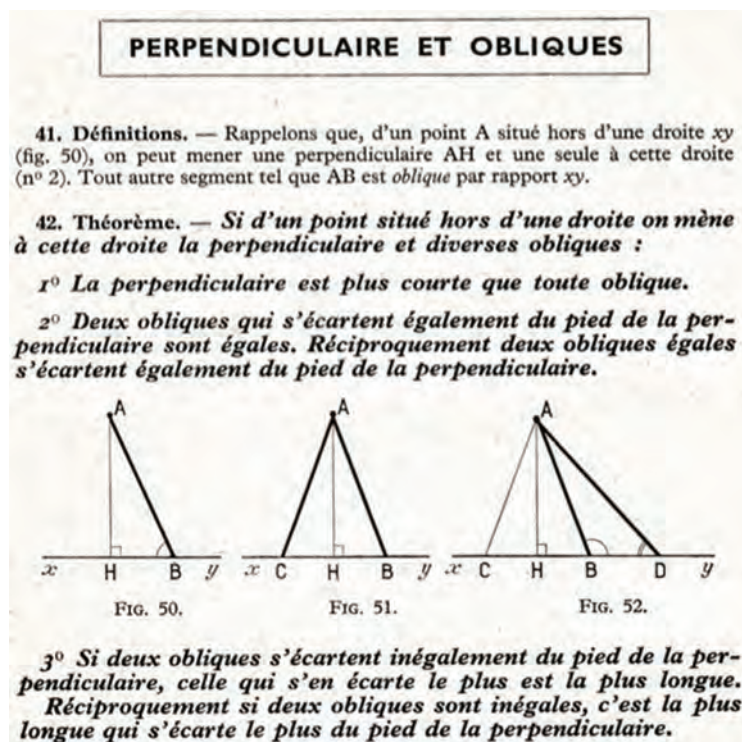


Fig. 11. “Perpendicular and oblique lines” in 1962 (Lebossé & Hémerly, 1962a, p. 142)

We can find traces of Arnauld and Lamy’s conceptions in elementary textbooks until 20th century (figure 11). But, we have to remark that Arnauld’s order had been followed despite that, from 19th century, new methods shown that it was fruitful to consider lines and figures in space at once. In the same way, Lamy’s arithmetization had been followed despite that, from 18th century, the introduction of the Cartesian system of axes provided a unified theory to conceive geometric objects.

To understand these two contradictions, we have to take into account the role of Lacroix, who adopted Arnould and Lamy's conceptions at the turn of 18th and 19th centuries, as an answer to a demand of "élémentation" at work in new schools created after the French revolution. They constituted two Cartesian ingredients that remained particularities of French elementary textbooks until the 20th century.

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Pólya and Dienes: Two men of one mind or one culture?

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Abstract

George Pólya and Zoltán Paul Dienes played decisive roles in changing how mathematics was taught and learned in the twentieth century. Following the crucial ideas of their work and the turning points of their lives, we trace the origin and the goals of the two men, considering each 'paidagogos' in the light of the other. Contrasting their approaches and resources, both as mathematicians and teaching wizards, gives us the opportunity to compare their pedagogical backgrounds when faced with the same dominant teaching practice. The emergent questions concerning age dependent preconceptions of educational psychology not only help to resolve the differences of their attitudes, but also to recognize the complementary nature of their methods; a combination that can be utilized more widely for present-day didactic practice.

Keywords: heuristics, maverick philosophy of mathematics, embodiment, duration

Introduction

The names of George Pólya (1887-1985) and Zoltán Paul Dienes (1916-2014) are intertwined with mathematical heuristics and didactics, and the psychology and practice of problem solving. The closer we get to their work the more similarities appear. Still, even after the first look, we can detect characteristic differences in their approaches which seem more to complement one another rather than to suggest one and the same conception. Is it enough to find the common origin of their philosophy of mathematics, in contrast to Platonism, Formalism or Intuitionism, in considering mathematics as a human activity that is conceived in problem solving? Can we attribute their common views about the human side of this activity to the same point of departure, to the same educational objectives, or some other factors? Was it the milieu of their Hungarian background or, leaving it, a bi-, or multicultural identity that resulted in the apparent single mindedness of their pursuit of changing math education? Answering these questions, we compare the crucial ideas from their work and place the standard interpretations of their views into the historical context of the origin of modern heuristics.

Developers of new directions in mathematics education

Both Pólya and Dienes intended to renew the way mathematics was taught and learned throughout their life, and became world renown for their innovative efforts. They carried their Hungarian-born experiences with them in their mind-sets as seeds of their multicultural identity. Just to name a few highlights of their legacy:

- Pólya and Gábor Szegő published a two-volume *Problems and Theorems in Analysis* (Pólya and Szegő, 1925 in German, 1972 in English). In the preface of the book, they claimed:

This book is no mere collection of problems. Its most important feature is the systematic arrangement of the material which aims to stimulate the reader to independent work and to suggest to him useful lines of thought. We have devoted more time, care and detailed effort to devising the most effective presentation of the material than might be apparent to the uninitiated at first glance.

This book is a classic masterpiece of guided discovery, that both Pólya and Szegő were the proudest of, among their many publications. The two authors became life-long friends, and determining figures in the shaping of “Stanford Mathematics” (Royden, 1989, p. 250).

- Pólya was one of the founders of the California Mathematics Council (CMC) in 1942. The workshops and summer institutes he conducted for teachers (particularly in California and in Switzerland) throughout the years have influenced the teaching and learning of mathematics for generations of people (Taylor & Taylor, 1993, p. 92).
- Pólya’s *How to Solve It* (first published in the USA in 1945 and translated into 17 languages since then) and Rózsa Péter’s *Playing with Infinity* (first published in Hungary in 1957, the first translation into English by Dienes was published in 1961 in England) were written at about the same time (early 1940s). They both reflected “best practices” in Hungary (*How to Solve It*: on heuristic thinking and problem solving, and *Playing with Infinity*: on inquiry-based learning and student engagement) (Péter, 1976). Imre Lakatos translated *How to Solve It* into Hungarian (published in 1957) because he recognized its importance for Hungarian education, and had already begun work on his *Proofs and Refutations* during the 1950s. The work was first published in 1976 (after his death) as a book, based on the first three chapters of his four-chapter doctoral thesis, *Essays in the logic of mathematical discovery*, presented in 1961. It has many common roots with *Playing with Infinity* in terms of its approach and investigation of mathematical thinking.
- Dienes played a key role in the development of experimental research-based mathematics education. He had a unique position in the research community

as a well-trained mathematician and psychologist who also had extended experience in working with children and with school teachers worldwide. He compiled the report of the *International Study Group for Learning Mathematics* on mathematics in primary education in 1966. This initiative of UNESCO played a crucial role in changing mathematics education around the world (International Study Group for Mathematics Learning, 1966). The report provided the most comprehensive list of educational research from the 1960's by Dienes with contributions from Bruner, Jeeves, and Piaget, and many other major researchers of the time, such as, for example, Bartlett, Gattegno, Papy, Skemp, and Suppes. The report also included specific recommendations for teacher training that remain relevant in 2018, decades later. In addition, Dienes worked closely with Tamás Varga on “New Math” in Hungary¹ during the 1960s. They collaboratively examined, evaluated, and perfected the uses of many manipulatives in instruction, such as, for example, Vygotsky's logic blocks (Servais and Varga, 1971, pp. 38-46), Cuisenaire rods, and Dienes's Multibase Arithmetic Blocks (Servais and Varga, 1971, p. 107, Kántor, 2006).

Achievements

From our current standpoint, we attribute to Pólya the interdisciplinary ‘science’ of heuristics which, by now, has close links with didactics, cognitive psychology, and the contexts and methods of discovery in various fields from sociology or the history of science to computational models. After a period of more ‘practical’ views on ‘teaching’ giving rise to new interpretations he wrote:

Modern heuristic endeavors to understand the process of solving problems, especially the *mental operations typically useful* in this process. [...] A serious study of heuristic should take into account both the logical and the psychological background, it should not neglect what such older writers as Pappus, Descartes, Leibnitz, and Bolzano have to say about the subject, but it should least neglect unbiased experience. *Experience in solving problems and experience in watching other people solving problems must be the basis on which heuristic is built.* [...] The study of heuristic has »practical« aims; a better understanding of the mental operations typically useful in solving problems could exert some good influence on teaching, especially on the teaching of mathematics (Pólya, 1945, pp. 129-130, italics ours).

Dienes placed games and the issues of multiple embodiment at the intersection of mathematics education, theories of concept formation in epistemology, and developmental psychology. He described his philosophies about the ‘art and craft’ of active learning in the following way:

¹ A renewal of the content and method of math education but in radically different sense than “New Math” was conducted in the USA.

The mathematics I was bringing into the schools was really a Trojan horse. It was not just mathematics, it was a way to look at what learning is all about, or, even more fundamentally, what knowledge is all about. *To 'know' something surely is to know how to handle it. Handling means action: present action or at least past action, remembered accurately, burnt into our person as internalized action.* So if knowledge is internalized action, then learning must be the process of internalizing such action. If there is no action, then there is nothing to internalize, so no learning of any permanent nature can happen. It is philosophies such as these that climb out of the Trojan horse once it is smuggled into the educational system under the guise of essential learning, such as the learning of mathematics (Dienes, 2003 p. 317, italics ours).

Common points of their didactics

Their names not only intertwined with mathematical heuristics and didactics, the psychology (Dienes), and practice of problem solving (Pólya), but also there are deep commonalities in their approach to mathematics, and to learning and teaching in general.

Both of them freely and frequently turned to perceptual experiences, tools that augmented the learner's mind. They both trusted and built on the intelligence of all learners. They wanted to 'meet' the students at their current level, and lead them to a deeper understanding. They never complained about any lack of 'pre-requisite' knowledge of the learners. They could communicate about math with any age, gender, ethnic, and socio-economic group of learners. The learners were attracted to the tasks and challenges proposed by them.

They had lively and remarkable presence. Even one-time exposure to either of them built in many people the memory of a teaching wizard having nothing up his sleeves who taught the learners to make their own miracles. They collaborated intensively with the learners, thinking, playing, and exploring together. They viewed their role as teachers as a "play master participant", who has solid content knowledge and a clear vision of educational goals, but works with the learners, in the moment, with great presence and with great respect and admiration towards the ideas, attempts, and needs of the others (see, for example, the *Let Us Teach Guessing* video described in footnote 6 below). This sensitivity to others might have been a big part of their magic, which meant more than the kind of coaching that is so fashionable today.

The most important common conviction of the two 'paidagogoi' was, however, an attitude to mathematics that was novel compared to the contemporary western logical positivist, analytic philosophies of mathematics. They both considered mathematics (first and foremost) a 'human activity'. The legacy of this approach that comes through every aspect of their work is prominent in Imre Lakatos's philosophy

of mathematics and in the recent revival of the so called ‘maverick philosophies of mathematics’.²

Tibor Frank found the preface to a course that Pólya gave at Stanford University in the archives of the university’s library:

Start from something that is familiar or useful or challenging: From some connection with the world around us, from the prospect of some application, from an intuitive idea. Don’t be afraid of using colloquial language when it is more suggestive than the conventional, precise terminology. In fact, do not introduce technical terms before the student can see the need for them. Do not enter too early or too far into the heavy details of a proof. Give first a general idea or just the intuitive germ of the proof. More generally, realize that the natural way to learn is to learn by stages: First, we want to see an outline of the subject, to perceive some concrete source or some possible use. Then, gradually, as soon as we can see more use and connections and interest, we take more willingly the trouble to fill in the details. (Frank, 2004, p. 32)

Pólya’s “familiar or useful or challenging [situation, a] connection with the world around us” for Dienes meant “embodiment”.³ In Dienes’s view, structure, formalization, and proof must build on experiences gained *in situ*, which, for children, meant multiple embodiment through plays (Dienes, 1963, pp. 21-32).

Common cultural background behind the commonalities

When Pólya was taking courses in philosophy he fell under the influence of Professor Bernát Alexander, the grandfather of the eminent Hungarian mathematician, Alfréd Rényi. Alexander encouraged Pólya to take additional courses in physics and mathematics (Alexanderson 2000, p. 17).

Pólya was a successful student, in general, but mathematics was neither his strength, nor his interest (Alexanderson 2000, p. 16) until he realized the necessity of studying mathematics for advancement in philosophy and simultaneously, Lipót Fejér sparked his interest in mathematics. Fejér, himself, did not like or care

2 The reception of Lakatos was gradual and oft debated among mathematicians. He played an undeniably important role in twentieth century philosophy of mathematics in spite of his unexpected death in the midst of writing a book jointly with Paul Feyerabend, the first part of which came out as Feyerabend’s *Against Method* (1975/1979), without the second part in which Lakatos was to “restate and defend” the rationalist position. Concerning the recent flow of papers addressing the “maverick philosophy of mathematics” cf. Cellucci (2017) and related citations.

3 “Situation” here not only means the way in which something is placed in relation to its surroundings, but also the sum total of internal and external stimuli that acts upon an organism and the combination of circumstances within a given time interval, a “position in life”. In new or unusual state of affairs the familiar situation turns into a problem: a critical, trying challenge that calls for “awakening”, implies reaction, interpretation, and reply. (Cf. ~situation in Webster’s New Coll. Dictionary, 1974)

about mathematics until Zsigmond Maksay became his high school teacher. Maksay encouraged Fejér to work on challenging problems and submit his solutions to the high school math journal, *KöMaL*.⁴ Fejér became one of the best problem solvers of the journal. Pólya may have been deeply inspired by Fejér's personal 'journey', but also by Fejér's style of giving a personal background and flavor to problems. According to Pólya, the essential aspect of Fejér's mathematical talent was his love for the intuitively clear details:

It was not given to him to solve very difficult problems or to build vast conceptual structures. Yet he could perceive the significance, the beauty, and the promise of a rather concrete, not too large problem, foresee the possibility of a solution, and work at it with intensity. And, when he had found the solution, he kept on working at it with loving care, till each detail became fully intuitive and the connection of the details in a well-ordered whole fully transparent. (Pólya, 1961, p. 505)

Pólya and Károly Polányi were the founders of a student society called Galilei Kör [Galilei Circle]. Pólya gave a lecture there on the philosophy of Ernst Mach. The Galilei Circle (1908-1918) was the meeting place of radical intellectuals, mostly Jewish college students from the up and coming Budapest families of a new bourgeoisie (Frank, 2004, p. 28).

Zoltán Paul Dienes's parents were part of the same social and intellectual environment during the first decade of the 20th century as Pólya. Paul Dienes (the father of Zoltán Paul Dienes) and Pólya knew each other from the Galilei Circle and from the radical Társadalomtudományi Társaság (Social Sciences Association) of left-liberal artists and scientists gravitating around the social science journal *Huszadik Század* (Twentieth Century, ed. in chief Oszkár Jászi). They also worked closely with G. H. Hardy during the 1920's. Paul Dienes even invited Pólya to give a series of lectures at the University College of Swansea in Wales, where he worked in 1925 (Alexanderson 2000, p. 81; Cooke, 1960, p. 251).

Valéria Dienes (née V. Geiger, 1879-1978), mother of Zoltán Paul Dienes, was a philosopher, choreographer, dance teacher, and the creator of the dance theory *orkestrika* [orchestics]. She was among the first females to graduate from a university in Hungary. Her university work focused on studying mathematics and physics.

4 KöMaL (the Hungarian acronym for High School Mathematics and Physics Journal) is a monthly periodical for high school students. It was established in 1894. It has been the intellectual cradle for mathematicians like von Neumann, Paul Erdős, or László Lovász. Generations of future scientists, mathematicians, and also interested students have had their first mathematical experience by trying to solve problems posed every month in the journal's yearly problem solving contest (source: www.komal.hu/home.e.shtml). The attitude of the group to mathematics and the social network of teachers (central figures: Dániel Arany, Manó Beke, Sándor Mikola, László Rátz) behind the journal was a key factor in the genesis and survival of "martian" mathematics. Cf. Marx (2001).

Lipót Fejér was in love with Valéria, and he introduced Paul Dienes to her, claiming that Paul Dienes was a mathematical genius (Borus, 1978, p. 14).

Valéria began her doctoral studies of philosophy by listening to Bernát Alexander's lectures, just like Pólya did. She received her degree in the same ceremony as Paul Dienes in 1905 at Pázmány Péter University, Budapest. They exchanged engagement rings during that ceremony. Paul received his doctorate in mathematics. Valéria's doctorate was in philosophy as a major subject with a first minor in mathematics and a second minor in aesthetics. Her interest in aesthetics was based on her love of composing and playing music (Borus, 1978, pp. 15-18; Boreczky, 2013, p. 56).

Reforming ways of learning and teaching mathematics

Pólya concentrated on finding beautiful solutions to well-understood problems, and on the undeniable structure that arises from previous attempts and guesses. One of his favorite problems was the Riemann Hypothesis. He believed that the proof could be found after serious attempts, and he made several attempts. Doing mathematics was meant to give the pleasure of understanding and criticizing the reasoning of others. For him, mathematics seems to be analogous to an ever-perfected building under construction, that has some remarkable "free masons" who have worked on it (like Euclid, Archimedes, Descartes, Leibniz, Euler, etc.), and learning mathematics was equivalent to studying the methods and the reasoning of these masters so that the building could be renovated, copied, and expanded in the same, or in a new, ambitious artistic style. The problems novice learners were dealing with had been masterfully solved in the past, but had access points to these learners through which they could reconstruct some minor details of the construction. As in a guided tour, the leader of the group, who in this simile is the math teacher in the classroom, was to ask good questions to make the participants aware of the why's and how's of the constructions of the masters. Otherwise, the learning of mathematics could be disturbing, as it was to Pólya:

The author remembers [...] a question that disturbed him again and again: 'Yes, the solution seems to work, it appears to be correct; but how is it possible to invent such a solution? Yes, this experiment seems to work, this appears to be a fact; but how can people discover such facts? And how could I invent or discover such things by myself?' (Pólya, 1945, p. vi)

Dienes, in many senses, also continually reflected on his own 'journey' of personal concept formation, and on building a personal view of the mathematical world from interactions with objects and environments (multiple embodiment) and with people (social interactions during games). He concentrated more on the exploration of structures than on solving pre-formulated, 'posed' problems. There was no necessarily correct (or incorrect) answer in his mind, since there were no pre-formulated

theorems or problems, either. The immersion into intuitions based on a personal history of efforts was meant to provide the learner with the enjoyment of mathematics.

Dienes's childhood, just as Polya's in an earlier stage, was situated in a period of school reforms in Europe, and in Hungary. A series of teachers' societies emerged after the 'Ausgleich' (Compromise of 1867) that followed the 1848 Hungarian Revolution, and multicultural education catalyzed cross-cultural interactions between different layers, social and ethnic groups of the society, as well as transfer processes between arts and the sciences, between different disciplines, intellectual, political, and life-reform movements. The generative and fertile effects of these interactions which contributed to, and grew out from what is sometimes called the 'Hungarian Avant-garde' constituted a much more complex social development than it may look from the perspective of any single field. Its history is recently explored from the point of view of new arts, modernization, free schools, and reform pedagogy. (Beke, Németh and Vincze, 2013) Its effects in mathematics, the sciences, and on education were noted by several authors (e.g. by Frank 2012), but should be further explored as a process which relies and feeds on a considerable social network of personal and international relations as pointed out by Boreczky (2013).⁵ It was a period when the rapid development of psychology and Durkheimian sociology brought new ideas for social understanding and catalyzed educational experiments facing the dominantly Herbartian pedagogy of the end of 19th century. This special intellectual era which in the end of the war concluded in the attempts of the 1918 and 1919 revolutions forced radical changes. Foreseeing, or in consequence of their failure, many thinkers who believed in a 'New Hungary' left, or had to leave the country. The following Christian-conservative turn brought about much more étatist educational conceptions and transformations.

For Dienes's parents this turn meant difficult years. After his parents left Hungary for Vienna and divorced, Zoltán went to live with his mother and brother in Nice, France, at a commune set up by Raymond Duncan, brother of dancer Isadora Duncan, who brought 'free dance' into Europe from the USA. The commune was a social experiment, where all the children were 'owned' in common (Dienes, 2003, p. 23). His mother eventually fled the commune with her children and made her way to Bavaria where Zoltán met again with his father before he returned to Hungary with his mother. His childhood and teenage travels in Europe provided him colorful social, cultural and linguistic experiences contributing to his open minded, observant attitudes. In his autobiography, Dienes reflects on his childhood experiences related to learning and discovering a new language as a way to understand learning itself (Dienes,

5 The complex, interdisciplinary, social interaction, and network theoretic approach was initiated and taken by Ágnes Boreczky and her research group for the reconstruction and understanding of the "social and symbolic space [that] considerably went beyond the avant-garde" (Boreczky, 2013, p. 55), catalyzing modern civil attitudes. It existed until the 1930s in a politically irreconcilable situation that Zoltán left behind moving to England to his father at the age of 15.

2003, pp. 59-61). His views are similar to Michael Polányi's (1962, pp. 92-95) conception of "tacit knowledge" that can not only be contributed to direct exchanges but to common school, and life experiences in their youth, to resources coming from reform theories of "natural learning" through excursions, body culture or dance-schools:

We must consider for a moment the difference between *natural* and *artificial* learning, for this provides important clues to the understanding of children's difficulties when they are confronted with artificial scholastic learning situations. If a child is taken to a foreign country where his mother-tongue is not spoken, within a few months he can speak the new language as well as his newly acquired friends, whereas his parents may still be struggling with grammar years later, trying to learn the language 'properly'. It is, of course, the child who has learnt it 'properly', and the reason is that he learnt it naturally. Fortunately, it is impossible to learn skating or riding a bicycle from a book, or many people would have a try. The 'fiddling around' with the data is the only way of making sure that you do not eventually take a tumble on the ice or fall off your bicycle. Natural learning is not invariably preferable to artificial learning; it is however *a priori* probable that it will be more effective. (Dienes, 1964, p. 24)

Pólya's views on learning and teaching mathematics resemble Dienes's assertions above. They suggest that the teaching of mathematics need to involve not only demonstrative reasoning or 'artificial scholastic learning', but also plausible reasoning and fiddling around with mathematics by making guesses. This is the 'natural way' of learning mathematics:

Mathematics has two faces. Presented in a finished form, mathematics appears as a purely demonstrative science, but mathematics in the making is a sort of experimental science. A correctly written mathematical paper is supposed to contain strict demonstrations only, but the creative work of the mathematician resembles the creative work of the naturalist: observation, analogy, and conjectural generalizations, or mere guesses, if you prefer to say so, play an essential role in both. A mathematical theorem must be guessed before it is proven. [...] Older writers, as Euler and Laplace, did not fail to notice that the role of inductive evidence in mathematical investigation is similar to its role in physical research, but more modern writers seem to have forgotten this remark almost completely [...] an ambitious teacher of mathematics [...] should, more than any particular facts, teach his students two things: First, to distinguish a valid demonstration from an invalid attempt, a proof from a guess. Second, to distinguish a more reasonable guess from a less reasonable guess. [...] At any rate, we should not forget an important opportunity of our profession: *Let us teach guessing!* (Pólya, 1984, p. 512-520).⁶

6 Pólya's lecture, demonstrating his ideas 'in action' by instructing a mathematics class for university students with the title *Let Us Teach Guessing* is the first video recording made by the Mathematical Association of America in 1966.

Dienes enjoyed listening to adults' mathematical conversations when he was a child, and bonded with his father through discussing mathematical ideas. These positive childhood experiences may have inspired him to give similarly positive experiences to other children, 'replicating the magic' that happened with him. He was capable of grasping some sophisticated mathematical concepts at an early age. Many of his educational experiments were to see whether any(!) mathematical concept could be taught to children, through providing adequate multiple embodiments and contrasts (Dienes, 1964, p. 40). These methodological points which were influenced by Piaget's structuralism and the developments of experimental psychology in the 1920's and 30's that contrast theories of formal operations after World War 2 (WW2) can be traced back to artistic theories of abstraction of modern Constructivism and set in parallel with Polányi's conception.

Contrasts and differences

Proofs, operations, structures, and embodied procedures

Pólya placed higher importance on proofs because of the higher age-group of his students. He considered exploring ways of finding different proofs for the same theorem essential and a *bona fide* result of the heuristic study of a problem. He encouraged his students to find special cases first, and then to generalize by testing various conditions and models. He often studied proofs to explore the problem-space and to widen applicability or generalizability of solutions. He extended the understanding of mathematical concepts and theorems to applications in other fields and *vice versa*. His mathematical works showed an extremely wide interest in applied mathematics, but always addressed problems that were mathematically fundamental. There are more than forty concepts, methods, and theorems named after him, and many of them are considered deep insights that opened up new fields of application (Royden, 1989, p. 251).

Operations, not only abstract ones, but perceptual ones, either as embodied actions or as manipulation of physical objects, and structures of the environment that can be experimentally experienced were of key importance for Dienes. From his childhood, he considered indirect proofs as genuine ones, while being second rate ways for understanding mathematical reality. In teaching mathematics, especially for young learners, he preferred activities that turned the features of numbers or topological properties into a human experience and used representations that could be physically perceived. He used steps, jumps, and hops as moves, for experiencing number theoretic operations, and made the learners build toy harbours and waterways at the banks of a river to provide experiences about various concepts of topological

connectedness (Dienes, 2003, p. 328).⁷ He used musical tones, not just to represent mathematical relations, but to transpose and transform regularities into musical experiences of abstract operations (e.g. multiplication) or relations, making their essence understandable, and amused children by teaching them to sing and express with arm positions the similarities of multi-tonal, transposed musical structures in chorus, in the form of “mirror” or “augmentation canon” in the spirit of the Kodály method (Dienes, 2007, pp. 11-12).

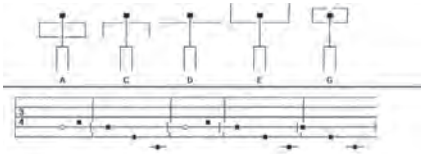


Fig. 1. Musical tones expressed by arm positions (Dienes, 2007, pp. 11-12)

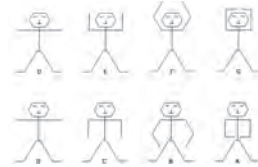


Fig. 2. Symmetricity of arm positions relative to note D (Dienes, 2007, p. 55)

Interdisciplinarity from the point of view of problem solving (Pólya) and concept formation (Dienes) belonged to different contexts with different meaning for the two heuristic thinking instructors. However, both of them were against the division of the intertwined aspects of human activities into different disciplines. For Pólya, it meant the universal presence and applicability of heuristics in all disciplines throughout historically interconnected problems. For Dienes, it meant “threads”, such as time, form, contrast, meaning, or the use of symbols that “run through” different fields of human experience (e.g. mathematics, language, music, bodily movements). These threads, as embodied procedures, that establish “connexions between spatial and temporal sequences”, can be organized (e.g. in games) so that they form patterns (Dienes, 2007, pp. 43-52, Kántor, 2006). The ‘interdisciplinary threads’ also provide the basis for jumping from one domain of abstraction to another discipline, understanding the same structure in different settings. It helps to operationalize, also in a computational sense, the construction of functions according to his functions principle (Dienes, 2007 p. 58, p. 68, Benedek, 2018). His slogan was: “Give me a mathematical structure and I’ll turn it into a game”. He could lead children from bodily movement experiences and the description of positions of chairs in the room through the playful explorations of physical transformation paths of chairs to an integrated abstract experience of operations of a 3D (or higher) transformation group, and wrote a whole book of “algebra stories” (Dienes, 2002).

⁷ For a representative collection of potential influences originating from 20th century art movements that could also have an effect on Dienes’s methods cf. Weibel, P. (Ed.) (2005).



Fig. 3. The 24 position of chairs in the “Chairs-game” (Dienes, 1989, p. 153)



Fig. 4. Dienes playing the *Chair-game* with children who compete in finding the shortest inverted path between two positions reducing cycle length



Fig. 5. The discovered transformation groups and their relations (Dienes, 1989, p. 154)

Time: Duration versus Historicity

According to the standard interpretations, Dienes merged his math didactics with Piaget’s psychology of concept formation, while Pólya did so with the history of mathematical thinking and personal progression of thought.

It is Dienes’s system of thoughts, however, that can be traced back to Henri Bergson or the 20th-century schools of Process Philosophy, pointing towards the role that games, activities and stories play in children’s thinking. One of Dienes’s initial insights was that numbers, in the early concept formation of a child, within the Piagetian pre-operational developmental period, rely on the experience of succession (ordinal, rather than on cardinal number (Dienes 2007, pp. 3-4). This insight, that goes against contemporary formalisms of the 50’s and set-theoretic conceptions of the ‘New Math’ of the 1960’s, had computational consequences and originated from many sources for and against Piaget’s work. Dienes’s first resource included childhood experiences with time and rhythm in dance, music, and bodily movements, and later, transfers from, or rather “resonances” to, his mother’s system of thought. His mother, Valéria Dienes played a major role in teaching and conceptualizing modern dance. Between 1908 and 1912, she studied in Paris with Bergson and translated Bergson’s works into Hungarian. Inspired by Bergson and the dancer Isadora Duncan, she founded the school and theory of orchestics, which is the first modern dance conception that builds up a communication theory. Duration, the central concept of Bergson’s philosophy of time, plays a crucial role in Valéria Dienes’s dance-theory. In her theory, sequences of bodily movements and positions, are perceived and understood as quasi-grammatical semantic components in the space-time structures of communicative human expressions (Dienes, 2016). Similarly, the perceptual experience of kinaesthetic patterns is just as good for creating, as for understanding, the regularities of basic mathematical operations associated with sounds or bodily movements in Zoltán’s game-based didactic methods (Holt & Dienes, 1973, p. 110). In this way, he creates personal knowledge, in Polányi’s sense,

that involves first perceptual experience, and afterwards, via kinaesthetic variation and sequential repetition, constitutes the abstractive structural affordances of this knowledge. Lastly he extends, in an analogical manner, the expressive time dependence of acts of meaningful motion (as those of modern dance) to experience with symbolic operations, and to mathematical thought expression as an abstractive movement of thought.

The processual aspects of action and understanding (“awakening”) as fundamental features of the subjective, but communicative, and intentionally meaningful human reality on behalf of Valéria Dienes on one hand, and the perception and abstraction of the regularities of experienced phenomena via embodied operations as key components in the understanding of mathematical structures, on behalf of Zoltán Dienes, have common roots. They can be linked to the ‘Systems Movement’ of the ‘Hungarian Avant-garde’, Bergson’s time conception, phenomenological music theory, and artistic Constructivism. Dienes raised certain questions that both Piaget’s theory and Brunner’s school struggled to answer which can be summed up in the following question: “What makes the jump between different levels of abstraction and what constitutes the transformation from pre-operational to operational stages: how can the “move” be taught and reproduced?” (Dienes 2007, pp. 15-16). He spelled out his position on the developmental stages of abstraction in math psychological terms (Dienes, 2007, 62-86, Benedek, 2018) but preferred demonstrating instead of arguing that the jump is ‘teachable’.

Pólya’s historical approach is linked to questions that address the role of time, and temporal order in mathematical discovery. He considers the feedback of the history of thought and temporal emergence of solutions in individual problem-solving a “hermeneutic circle” (Kiss, 2009). He connects history with the mental development of understanding: with the ways of (re)producing the processes of awakening in the arts and the sciences.⁸ This was a major theme of inter-disciplinary discourse with the involvement of many fields before WW2, and an issue of the ‘Genetic Method’ that received continued attention afterwards, partially due to Hadamard (1945). It is most concisely expressed in the 1962 Memorandum of 75 mathematicians (Pólya played an active role in its formulation) as

... a general principle: The best way to guide the mental development of the individual is to let him retrace the mental development of the race - retrace its great lines, of course, and not the thousand errors of detail. This *genetic principle* may safeguard us from a common confusion: If A is logically prior to

8 In contrast, Polányi was interested in the relationship of ontogenetic and the phylogenetic development and the logic of emergence. (Polányi 1962, pp. 393-94, 337-39, and fn. 3, 4). Dienes, studying transfer processes between structures, looked for principles that explain the moves between stages of personal cognitive development (Dienes and Jeeves, 1970).

B in a certain system, B may still justifiably precede A in teaching, especially if B has preceded A in history.⁹

The historical aspects of discovery and Pólya's heuristics became a basic line of thought in Lakatos's philosophy of mathematics and rational reconstructions. Their relationship, as well as Lakatos's acquaintance with Polányi's conception, is documented by now (Kiss, 2009). Their common roots are, however, less known. Similarly, the origins of Dienes's methods were the same artistic and scientific backgrounds from whence Polányi's interest emerged in the temporal process of the development of personal knowledge. They originate in the progressive counter-culture and in the continuing professionalization into which most representatives of this short period of post-feudal modernization and social-political progress took refuge from the upcoming 'Christian Course' nationalism after the war.

The 'late' Hungarian fin-de-siècle

The closing of the 19th century and the onset of the 20th up to the 1920s was a special period in the intellectual history of Hungary. It was similar to, but in many respects different, late blooming of spiritual powers from the avant-garde movements of the European fin-de-siècle to which it was strongly tied. This 'late' development, however, did not mean the same decadency, degeneration, cynicism, or pessimism as elsewhere in Europe. It was instead a rapid cultural and economic modernization that was due already in 1848 but came rather late, absorbing every intellectual movement of European thought with fresh eye and critical reception. After the feudal and agricultural backwardness of the country and in the light of the administrative and political reforms 'civilization' had a different, less pessimistic meaning, and insights about its 'degeneration' allied with critical reconsideration of the developments of industrial societies. By the arrival of millenarianism and the World Fair of 1900, Budapest showed the face of a modern capital strongly linked to Vienna, Berlin, München, Paris, London, and aspired to belonging to an integrated global whole with the cosmopolitan German, French, Anglo-Saxon new world of the twentieth century.

This blooming was not the result of a revolution exchanging the ruling political establishment but the result of the 'Ausgleich' of the Austrian and Hungarian elite in 1867, which brought about a second reform period that changed its social and cultural composition. "The rise of a new urban middle class affected the school system." (Frank, 2012, p. 358) With the failed 1848 Freedom Fight, the educational and emancipatory

9 Before the above lines it reads: "**Genetic method.** »It is of great advantage to the student of any subject to read the original memoirs on that subject, for science is always most completely assimilated when it is in the nascent state« wrote James Clerk Maxwell. There were some inspired teachers, such as Ernst Mach, who in order to explain an idea referred to its genesis and retraced the historical formation of the idea." (Memorandum, 1962)

initiatives of the civil (but noble-led) revolution were extinguished as result of the consequent political retorsion, but by the 1849 *Entwurf der Organisation der Gymnasien und Realschulen in Oesterreich* and after 1867, Franz Joseph I. established top-down educational reforms similar to the German transformation of the elementary, secondary, and university system and the modernization of the state. It was a change that coincided with the non-state controlled modernization efforts of an emerging civil society and bourgeoisie industrialization that also kept an eye on Italian, French and Anglo-Saxon educational reforms including new approaches of Felix Klein in Germany, John Perry in England and E.H. Moore in the US. Baron Joseph Eötvös (Minister of Religion and Education during the 1848 Revolution) could return to his position and was able to put through his crucial liberal reforms concerning educational rights, obligations and emancipation. Many Hungarian teachers, such as, for example, Mór Kármán, Manó Beke, László Rátz, and Gyula Kőnig (Szénássy, 1992, pp. 217-218) and intellectuals at all levels (Béla Bartók, László Moholy-Nagy, Ferenc Molnár, just to name a few key artists) experimented with new methods, invented new approaches, and took an active part in the formation of the cultural life of Europe. Members of the professional educational circles were integrated into the European scientific and artistic elite in various ways: through Masonic lodges, scientific associations, by family relations, or as private tutors.

[T]his modernizing group came partly from the decaying landed gentry of feudal origins and partly from intellectually aspiring members of the assimilating (predominantly German and Jewish) middle class. While creating metropolitan Budapest in the intellectual sense, they constituted themselves as a group through what proved to be a completely new and unique social and psychological experience (Frank, 2012, p. 358).

This experience reached Pólya during his university studies of philosophy, when the renewal of art and scientific life peaked. Therefore, his constructive and reconstructive heuristic movements of thought rely on the problem solving practices of a reflective age, both historically and in a developmental sense underlying a belief in reflective human attitudes. Since he left Hungary by the 1910's starting his career as a mathematician, he missed several direct consequences of Hungarian politics and of the pedagogical and life-reform movements that played decisive roles in the lifeway of Dienes's parents. This is the period of women emancipation, new cults of childhood, the growing psychological attention to children's upbringing, and of alternative school, and life experiments. The fall of the Monarchy, the 1918-19 revolutions, during which Zoltán's father, Paul Dienes, was responsible for reorganizing the university, was a period, followed by the White Terror, the *Numerus Clausus* act, and revisionist territorial nationalism, that changed the political climate. It brought for Zoltán a series of diverse experiences, since both his father and his mother, who later divorced, were forced to flee the country. While Pólya started his 'random walks' at the age of 23, (in Vienna, 1911; continued in Göttingen 1912-13 after getting his PhD in 1912 at Budapest; then

in Paris 1914; and in Zurich, 1914-1940, spending one year during 1924-25 in Oxford and Cambridge with G.H. Hardy), Zoltán started his 'petite promenades' at the age of 3 (in Vienna, Nice, Paris, Bavaria, returning to Hungary, Pápa, then to Budapest) and experienced a latent family counter-culture in opposition as a child. The mentality of alternative schooling, the educated but creative upbringing, was instilled in Dienes by his parents and their social circles. These circles included Mihály Babits, Anna Lesznai, Alice Jászi (né Madzsar), Ervin Szabó, Béla Balázs, Zoltán Kodály, Lajos Bárdos, and their Austrian, French, German and English friends who helped them after World War 1 (WWI). The diversity of cultural circles may partially explain his highly developed skills of communicating with children as a result of being introduced into new schools and children groups, from Duncan's commune through Hungary to England where he joined one of the most avant-garde schools of the country (Dienes, 2003, p 97-98, 123-124). His early interest in the foundation and philosophy of mathematics and his turn to the psychological and experimental aspects of mathematical concept formation was motivated both from his father's and mother's side, though his devotion to lower age groups came probably from the latter side. Dienes's father, in a younger age also his mother and his math teacher, Zoltán Ferenczy, imprinted the legacy of Hungarian mathematics in his mind in the post-WWI period. Pólya's and Dienes' father's circles were essentially the same both in Hungary and in England including G.H. Hardy, but Zoltán Dienes had different experiences at his mother's side at different times. Pólya, just as Dienes's father, was influenced directly by Lipót Fejér and the social, scientific and artistic trends in which Austria and Hungary brought forth its pre-war intellectual achievements. In light of Zoltán Dienes's temporally consecutive, culturally manifold inheritance, the two heuristic thinkers, with their own interpretations and reflections, seem to represent different periods in the organic development of theoretical and experimental approaches to mathematical discovery and creative thinking in Hungary and Europe, torn by two wars and political cataclysms.

A most needed outcome: Connecting the two heuristics

Dienes's innovations were often confused with the New Math of the 1960s that he substantially criticized. Even in Hungary where Tamás Varga succeeded in implementing Dienes's methods at a time when Pólya's heuristics were relatively well known, several factors hindered joining their heuristics both theoretically and in praxis. Procuring this defalcation by a synthesis is more and more exigent.

Dienes's and Pólya's approaches complement one another in many ways. (1) Their (1a) embodied and (1b) reflexive heuristics can be built on one another as the age and personal development of the students grow. (2) The rich perceptual experience obtained in games, movements and embodied manipulation of the situation can be combined with the more conscious planning and exploration of the problem space (2a) turning tacit knowledge into abstract concept formation, and (2b) using

invented intersubjective symbolic expressions. The joint affordances of 2a and 2b pave the way for improving guesses and the discovery of proof ideas. (3) The levels of abstraction and operational stages of early developmental and former experimental periods can be gradually turned into methods that use symbolic expressions at every level. It (3a) preserves a sense for subsuming syntactic and formal tools to case-based meaningful understanding and (3b) the learner can use operations on domain dependent structures for the development of higher semantic architectures. (4) Linking events and occurrences along the students' (4a) temporal route to personal knowledge with (4b) the reconstruction of histories that (4c) retrace the mental development of the solutions of problems within a historical subject matter integrates the selection of domain dependent heuristic methods and heuristic selection methodologies in a shared common pool.

The combination of Dienes's and Pólya's methodologies makes reflection on our heuristic generation processes possible at all levels of conceptual and methodological development, applying symbolic representation for raising awareness. Such a synthesis could embed mathematical discoveries into the whole spectrum of temporal and historical development of conscious human activity, opening new ways for its symbolic and computational representations.

We indicated that their heuristics built on the cultural and artistic reform movements of the pre-, and post-World War I. period of Hungary and Europe. These movements incorporated visions and reform pedagogies that we can extend today to social learning, computational media and technologies, and to our social life. Adopting them, we were able to implement the essence of these reforms realizing the goals of the progenitors. Adapting their principles and methods we could go beyond what their contemporary practitioners could only partially achieve because WW2 interrupted their attempt to reform industrial civil society, and its education. We believe that the reconstruction of the methodological insights and the humanistic visions behind Dienes's and Pólya's methods of teaching creative thinking proves to be as appropriate now, as ever!

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Problems in the teaching of arithmetic: records in French school notebooks (1870-1914)

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Abstract

This study analyzes how problems were present in French primary school based on what was recorded in school notebooks between 1870 and 1914. The corpus of French documents used in this investigation is composed of 13 notebooks from the Fonds Histoire de L'Education collection of the University of Limoges. The analysis shows that both daily-lesson and monthly-lesson notebooks from that period prioritized "type-problems" that involved the idea of proportionality regarding situations, to be solved through cross-multiplication. The existence of "type-problems" also involved a model for spatial organization and solution procedures composed of a sentence in order to explain what each operation referred to and was followed by the referred operation, which was represented horizontally. These models can be observed in all notebooks.

Keywords: history of mathematics education; problems; primary school; school notebooks

Introduction

Problems gained prominence in French legislation at the end of the 19th century, with proposals that involved knowledge useful for social life and development of moral values (Sarrazay, 2003). The French studies that identified the space taken up by problems, like Sarrazay's (2003) and D'Enfert's (2003), approached legislation and problems, documents in which Teaching Regulations and Guidelines were exposed in the form of prescriptions provided by the government. However, how were the guidelines and regulations consumed? How were proposals for the use of problems in the teaching of arithmetic present in primary school classrooms? School notebooks constitute a possibility to search for answers to these questions.

For this reason, this study aims to analyze how problems were present in French primary schools, based on what was recorded in school notebooks between 1870 and 1914.

1 Consumption is understood as presented by Certeau (2014).

School notebooks as source material for research in a historical perspective

In general, notebooks have different uses and purposes. A notebook can be used for personal notes, accounting or as a diary, among many other possible applications. In some cases, the determination of how it will be used depends solely on its owner, whereas in others it will depend on the person who will read the records contained in it. As a kind of material to be used for students who follow their teacher's instructions, school notebooks may also have had different purposes and ways of use over time, but they differ from those used outside school. For children to be able to use school notebooks, they have to learn specific sets of rules (Viñao, 2008; Santos, 2008; Lopes, 2008), which concern the school universe. Hence, notebooks constitute a product of school culture.

This perspective has consequences for the process of doing research in which notebooks are used as documents, because it is important that the rules which generate the activities contained in the notebooks be "known so that researchers understand and interpret the resulting school product" (Viñao, 2008, p. 26). Besides, when a notebook is used as a historical source, it is also important to combine its analysis with analyses of other historical sources (Viñao, 2008, p. 27).

Gvirtz and Larrondo (2008, p. 45) broaden the previously presented discussion by defending that the notebook is not a neutral source and

must be understood both as a product and a producer of school culture, as a generator of specific speeches and specific effects.

The authors state that it is possible to build your own chronology through questioning the notebooks used as sources. These chronologies may or not coincide with cycles of educational policies or with the evolution of scientific subjects.

The same authors (2008, p. 35) also highlight that notebooks are a privileged source for educational research because

the school notebook – a space of interaction between teacher and student
– allows the effects of this interaction to be seen, that is, schoolwork itself.

This privileged access to schoolwork might offer the researcher clues concerning the activities that were set as priorities to be recorded in the notebook, as well as clues to school time organization. It also allows the researcher to observe rhythms, sequences and moments through the identification of daily, weekly, monthly and yearly uses (Viñao, 2008) as far as schoolwork is concerned.

The time frame and the corpus for the research

We decided to study the time period 1870-1914 due to D'Enfert's studies (2003) on official documents related to the teaching of mathematics in French schools, in which the author identified this period as a time of educational renewal.

According to D'Enfert (2003), the 1870s represented a movement of ideological renewal. Such movement led to the nomination of Jules Ferry² for the Ministry of Public Instruction and to the passing of important education laws in the 1880's that recommended the application of the intuitive method. In general, the proposal was to respect children's 'natural' predisposition to start learning from what was known, and therefore easy, then proceed to the unknown and difficult with great emphasis on observation and use of concrete objects. Such guidelines resulted, according to D'Enfert (2003), in some recommendations for the teaching of mathematics, like: the use of the abacus; the teaching of the metric system through observation or manipulation of conventional measurements; mental calculation not to exercise memory, but as an initiation to arithmetic and to develop intuition skills in children; the teaching of geometry through observation and concrete objects and thinking. Besides, in that period, there was the proposal for "concentric" education: the same program should be studied in each of the three levels of primary school (elementary, intermediate and superior³) and reviewed in more depth as levels progressed (D'Enfert, 2003).

The 'corpus' of French documents used in this investigation is composed of notebooks from the *Fonds Histoire de L'Éducation* collection of the University of Limoges. At present, this library is developing a project⁴ that involves the cataloging and digitalization of school notebooks used between 1878 and 1987. The collection holds 408 cataloged notebooks in the digitalization stage; online availability of this collection has been started.⁵

Thirteen notebooks from this collection were selected as sources for this study, because they were originally used in the time period we chose for this study and they contain mathematics classes' records. Out of the 13 notebooks, 7 are monthly-lesson

2 Jules Ferry (1832-1893), a rich and cultured lawyer, was an enthusiastic advocate for the Republic. He was nominated Minister of Public Instruction in February 4th 1879. The 1882 law, which established secular and compulsory primary school, is one of the laws that were passed during his term as a Minister (Albertini, 2006).

3 According to the first article of the decree on educational organization and study plans for public primary schools of July 27th, 1882, primary school in the French public educational system was divided in three courses: elementary course (*cours élémentaire*) from 7 to 9 years old; intermediate course (*cours moyen*) from 9 to 11 years old; superior course (*cours supérieur*) from 11 to 13 years old.

4 Further information on the Project can be accessed on the library's website: <http://www.unilim.fr/scd/2016/06/06/fonds-de-leducation/>

5 Available online: <http://www.unilim.fr/histoire-education/>

notebooks and 6 are daily-lesson notebooks. The daily-lesson notebooks contained the records of different subjects studied on each day, whereas the cover of each monthly-lesson notebook contained instructions on how they should be used, as shown below:

Every student, when entering school, will receive a special notebook that they have to keep throughout their school life. The first lesson of every month, of every subject, will be written in the notebook by the student, in the classroom and without any help by others, in such a way that the group of lessons allows to follow the sequence of exercises and appreciate the student's progress year by year. This notebook will be deposited in school. (in free translation)

We understand that 13 notebooks can be a relevant number in a study that has a historical perspective because the analysis promotes discussions between the records found in the notebooks and data from other historical sources, as well as from other previous studies. This perspective is supported by results of educational research conducted on notebooks, such as the results presented in Mignot (2008). Such results show that notebooks can contribute to educational research, even if they provide only small samples, for the tasks performed in the classroom enable us to take a closer look at the teacher-student interaction.

In general, despite being of different types, the notebooks present a type of organization that is similar in some aspects: all records kept by children were written with the same ink color - black or blue - except for drawing activities, which were recorded in pencil; there were no personal remarks made by the children, only the lessons; they offered indicators of date with day, month and year and also titles for each activity like arithmetic, grammar, moral, etc.; teacher's marks were in different colors, mostly red, and were made in the left margin of the notebook or over the student's writing. The marks over the student's writing were corrections, whereas the marks in the left margin were related to assessing the child's work, sometimes presenting a grade from zero to five, other times presenting indicators like "good", "bad" or "exact".

Problems in French school notebooks

When analyzing a notebook involves an observation of how problems were present in them, it is necessary to start with identifying the problems. In a historical analysis, it is important that this identification be based on the understanding that people had at that time of what a problem would be.

So, by analyzing texts by Leyssenne (1888) and Serrazy (2003), which approach the use of problems for the teaching of mathematics during the historical period studied in this research, it is possible to observe that the activities identified as

problems, in the notebooks, were the ones related to the teaching of arithmetic that explicitly had the nomenclature “problem” or the ones that involved real life situations or moral values.

Monthly-lesson notebooks

In the monthly-lesson notebooks, students recorded activities throughout their school life, so they did not refer to a specific learning level because they were used throughout the years. Six out of the seven notebooks analyzed in this study belonged to students who shared a common teacher (Table 1).

Table 1. Monthly-lesson notebooks⁶

Student	Period of use	Teacher
René	1897-1900	Alain
Pierre	1909-1910	Alain
Armand	1910-1912	Alain
Louis	1910-1912	Alain
Jean	1912-1913	Alain
Sophie	1913-1917	Alain
Lorran	1902-1903	Nicollas

It was possible to identify that Pierre, Armand, Louis and Jean were in the same classroom throughout the years since the notebooks used in the same school years had the same activities recorded on the same dates. Thus, those four notebooks allow us to observe the activities presented by teacher Alain from 1909 to 1913.

Table 2. Period in which each notebook was used.

	1909	1910	1911	1912	1913
Pierre					
Armand					
Louis					
Jean					

Using this kind of notebook as determined by law, according to Hébrard (2001), allowed a process of evaluation, that is, of checking students’ progress because all tests taken throughout a student’s school life are gathered in the same document. In that sense, it was possible to notice that problems were a favorite activity to assess

⁶ All names used to identify students and teachers are fictitious.

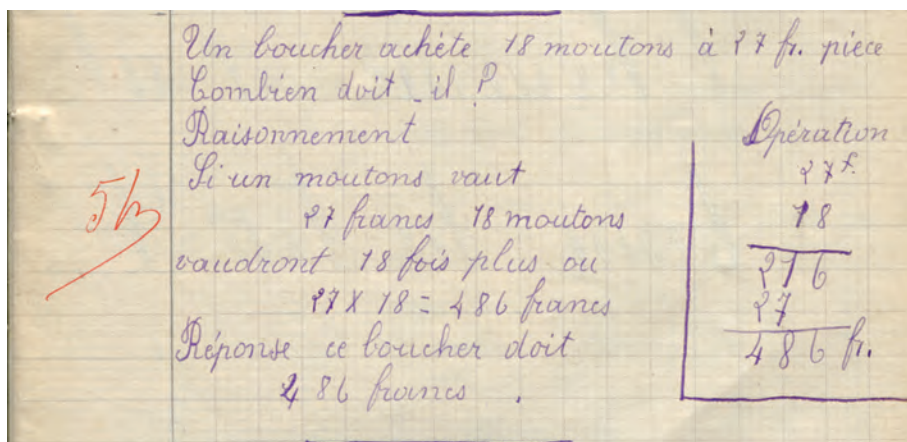
children's performances in arithmetic. Activities like writing, dictation, drawing and others were performed on different days every month. The activities related to the teaching of arithmetic were always performed around the 20th of each month and involved the solving of two problems. Activities like number writing and calculations with no relation to problems were identified two times, at most, in the same notebook, which implies a very low representativeness rate considering the fact that the notebooks were used for more than one year.

In most cases, problems were presented under the title "Calculation". The option of using two problems in each monthly test and the use of the term "Calculation" seem to indicate a connection with the study certificate exam, which contained, at that time, two arithmetic questions referring to the use of calculations and the metric system with a justified solution.

The way solutions were recorded also presented a close relation to the instruction given for the study certificate because it highlighted the record of the justified solution. All children applied the same method to record the solution. In his analyses of French school notebooks from the 19th and 20th centuries, Hébrard (2001) drew our attention to the existence of a spatialization of language in the notebooks, which is revealed through a set of graphical demands for the recording of each activity in models of space management of the notebook. According to the author, this was also observed regarding arithmetic problems

First, and all over the line, the wording of the problem; then there is a division of the page in two unequal columns (one-third, two-thirds). In the narrow column, identified by the title "Operations", additions, subtractions or multiplications were copied in the form of "calculations"; in the wide column, dedicated to the "Solution" (sometimes called "Development" or, to please more demanding teachers, "Developed Solution"), stereotyped formulas in written language explain the operations contained in the other column. [...] in the last line of the "Solution" column, or sometimes all along the page extension, the "answer" is explicitly formulated according to the terms presented in the wording. (Hébrard, 2001, pp. 131-132)

The same spatial organization was observed in the notebooks analyzed in this study. There was also, in the records, the teacher's correction mark in a score that ranged from zero to five. The act of grading the student's work is justified because this notebook was destined to contain the children's assessments.

Fig. 1. Problem solved by student Louis on January 19th 1912.

A butcher has bought 18 sheep for 27 fr. (francs) each.
How much does he owe?

Reasoning

If one sheep costs

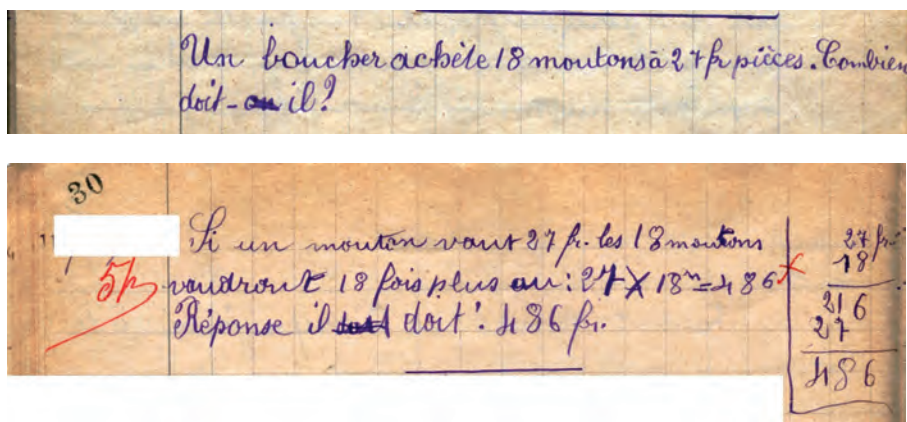
27 francs, 18 sheep

cost 18 times more, that is

$$27 \times 18 = 486 \text{ francs}$$

Answer: the butcher owes

486 francs.

Fig. 2. Problem solved by student Armand on January 19th 1912.

A butcher has bought 18 sheep for 27 fr. (francs) each. How much does he owe?

If one sheep costs 27 fr., 18 sheep

cost 18 times more, that is: $27 \times 18 = 486$ f

Answer: he owes 486 fr.

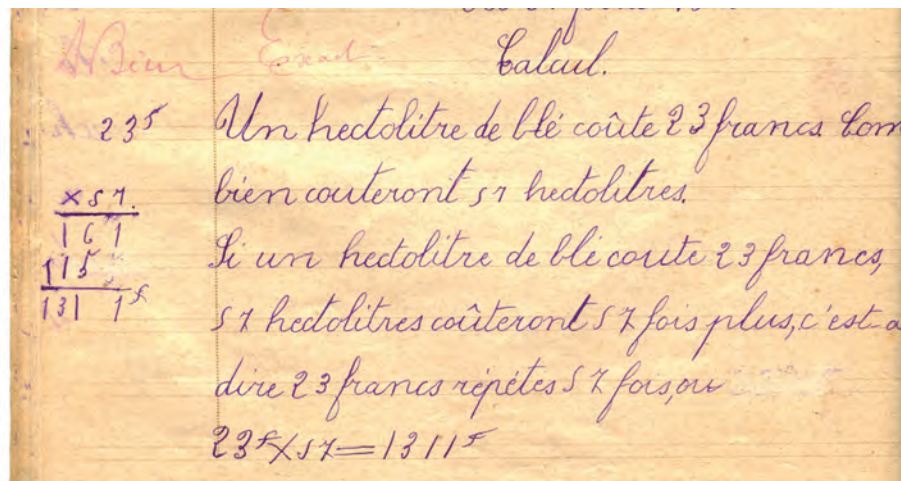


Fig. 3. Problem solved by student Lorrain on May 28th 1902.

A hectoliter of wheat costs 23 francs. How much will 57 hectoliters cost?

If one hectoliter of wheat costs 23 francs, 57 hectoliters will cost 57 times more, that is 23 francs repeated 57 times, or

$$23f \times 57 = 1311f$$

Answer: 1311 francs.

*

In addition to a model of graphic organization for the recording of problems, there was a model regarding the way solution procedures were written down in teacher Alain's class. The sentences used by students to express the "Reasoning" and the "Answers" were pretty similar, sometimes even identical.

The existence of a solution model was obvious when we analyzed the same problems solved by different children (Figures 1 and 2) and also when we observed the same student solving problems over time or students' notebooks of different teachers (Figures 2 and 3).

As years went by at school, problems sometimes needed more complex solutions (solutions that involved a larger number of operations). However, spatial organization and solution-writing models remained. The "solution" was always composed of a sentence that aimed to explain what each operation referred to and was followed by the referred operation, which was represented horizontally.

The existence of a model for problem-solving that involved not only spatial organization, but also solution procedures allowed us to recognize that what was

presented in these notebooks is related to what Sarrazy (2003) identified in school documents: the teaching of rules for the solution of “type-problems”, which indicates that the teaching of problems was based on memorization and repetition, according to the author.

Another aspect that is noticeable in the solution to most problems in the analyzed notebooks is that the indication of the unit species that was being calculated was written out in full in the sentences presented in the “Solution” and represented with one letter (first letter) or word next to the number in the indication of the operations, both in horizontal organization and also in a vertical calculation organization. For example, the letter “f” to express an amount in francs (Figure 3), the letter “m” to express quantity in meters or “d” for decimeter.

In that historical period, teaching guidelines involved specific emphasis on observation and utilization of concrete objects (D’Enfert, 2003) and problems related to social life – commerce, industry, agriculture, domestic management (Sarrazy, 2003). The need to somehow indicate what was being calculated seems to be a way of guaranteeing – in all solution procedures – the connection between the calculated quantities and social-life situations. In solutions, concrete numbers gained prominence, as defined by Minet and Patin (1913, pp. 5) in their manual for the intermediate-course: “a number is considered **concrete** if the species unit is mentioned. Example: forty soldiers.”⁷. The value of this kind of indication is confirmed by the teachers’ corrections, who complemented the information when it was not provided by students.

Single daily-lesson notebooks

Out of the six single daily-lesson notebooks, three are from intermediate-course students (IC) and three are from the superior course (SC) – the latter are from the same student (Table 3). None of the notebooks identified the teacher.

Table 3: Daily-lesson notebooks (1870-1914)

Student	Period of use	Class
Anne	1895 (March, April)	IC
Simon	1895 (April, May)	IC
Yves	1903 (October, November)	IC
Guy	1914 (May, June)	SC
Guy	1914 (June, July)	SC
Guy	1914 (July)	SC

⁷ Original French text: “Un nombre est concret, s’il désigne l’espèce d’unité. Exemple: quarante soldats.”

In the single daily-lesson notebook students recorded activities done over the same day. In all of them there was first the date and then the activities separated by “titles”, which categorized them according to content, subject or kind of activity. For example: moral, grammar, writing, dictation, science, history, calculations, problems, demonstrations, geometry, drawing, algebra and others.

According to Hébrard (2001, pp. 135), this kind of notebook “is irrefutable evidence of all the work done”, which would not have been possible if a different notebook had been used for each subject. The author argues that the single notebook allowed the family to supervise students’ work and inspectors to supervise teachers’ work, so it became a way of controlling the “work done by the teacher over each student’s piece of work”.

As in monthly-lesson notebooks, problems held a prominent place in single daily-lesson notebooks likewise. There were around two problems on virtually every day on which activities were registered, which cannot be observed in any other activity related to mathematics. As far as spatial organization is concerned, it is the same as Hébrard’s observations (2001) and as our observations in this study.

Still regarding problem-solving structures, student Guy’s notebook was the only one that contained illustrated records. In this case, the drawings were always related to problems that involved calculating surfaces (rectangle and trapezium) and volume (parallelepiped). The pictorial record in these cases was accompanied by notes of the measurements of the figures’ sides; these records did not seem to be mere illustrations, but a way of organizing the problem data that would be used in the explanation of the procedures adopted for the solution. In two of the notebooks there were problems with area and volume calculations. However, pictorial records were not used in any of them.

Although problems had a prominent place in activities involving contents related to mathematics, the notebooks presented other activities that allowed us to search for an establishment of relationships between the contents approached in problems and in other activities.

In student Anne’s notebook, which was used in March and April 1895, in addition to problems, for every day on which the student kept records, there were also “Calculations”. The calculations referred to operations performed in vertical form and, in this notebook, they involved mainly multiplication and division with decimal numbers.

It is interesting to observe that all problems proposed in this notebook involved the calculation of a surface of a rectangle or calculations involving proportionality in situations to be solved using cross-multiplication. Therefore, all problems required the use of multiplication or division with decimal numbers.

Repeated occurrences of a similar kind of problem all along the notebook and the order in which activities were recorded are indicators of a desire to teach students to solve specific “type-problems”. Moreover, there was the notion that “calculations” offered students opportunities to “practice” the operations, prioritizing those which were used to solve problems – multiplication and division were the most frequent ones in the activity “Calculations” and also the ones that were harnessed in problems involving proportionality and calculation of surfaces.

In Simon’s notebook, from the intermediate course, which was used in April and May 1895, the idea of proportionality was present in all problems. It was not possible to identify a relation to other activities because they were not recorded in the notebook; there was only the indication of titles about “principles related to multiplication and division” and “general properties of fractions” without the recording of the activities.

Despite approaching the notion of proportionality, Simon’s notebook contains problems with more data and more relations among the different pieces of information contained in the data, which demands a bigger number of operations (up to nine) for the solution than what is recorded in Anne’s notebook.

Yet, Matthew, an intermediate-course student, used a notebook in the months of October and November 1906, in which, besides problems, there were activities (under the title “Exercise”) related to the metric system: square meters, measurement conversion, surface measurements. In the notebook there were some problems that harnessed the use of measurement conversion, but it did not happen in most examples.

In this notebook, we have also identified problems related to moral values: one of them questions how much bread one could buy for his family if he had saved the money he had spent at the cabaret; the other involved money one would save if he stopped buying cigarettes. Sarrazy (2003) and D’Enfert (2006) highlighted, in their studies based on official documents, that problems involved necessary knowledge for adult life in that historical period, as well as moral values. In the notebooks analyzed in this study, we observed that domestic economy was a very popular theme and it was only related to moral values – like spending on cigarettes and cabarets – in one notebook: Matthew’s.

In all three Guy’s notebooks from the superior course, which were used between April and May 1914, it was possible to detect some relations between problems and other activities recorded in the notebooks. It happened regarding contents explored in activities under the title “Algebra”, like reduction to the same denominator, systems of equations and first-degree equations, which were harnessed in problems four days later.

It was also possible to notice a solving procedure which was only used in daily-lesson notebooks in the superior course: the use of algebraic solutions to problems, which appeared with arithmetic solutions. Both solutions were separated and individually named.

The algebraic solutions involved the representation of the data of the problems in equations or systems of equations, and the answer to the problems was found with the solving of those equations or systems of equations. One problem in Guy's notebook is used as an example:

A manufacturer sold 225 meters of canvas and 240 meters of cotton cloth in a first negotiation for 1098 francs. In a second negotiation, he sold 180 m of canvas and 375 meters of cotton cloth for the same price. Find the price of each material.

He started the algebraic solution by identifying the price of canvas as X and the price of cotton cloth as Y . He wrote the equalities in a system of equations, which he later solved:

$$225x + 240y = 1098$$

$$180x + 375y = 1098$$

Other contents explored under the title "Algebra" (special binomial products and second-degree equations), as well as the content recorded under the title "Geometry" (point and line relationships, geometric constructions of triangles), were not harnessed in any of the problems recorded in the notebook.

It is important to highlight that activities involving a specific content were recorded before that content was harnessed in problem-solving. This is an indicator that problems made use of what had been previously studied. However, in all notebooks, most problems did not have a relation to other activities and, in addition, it was possible to notice that many problems repeated the same format, the same proposal, the same "type" of problem with some variations regarding themes and data. Thus, it seems that problems were applied to teach students to solve "type-problems" (Sarrazy, 2003), which were of the same "type" as those observed in monthly-lesson notebooks.

Some Considerations

Both in daily-lesson and monthly-lesson notebooks, there was the prioritization of "type-problems" that involved the idea of proportionality regarding situations to be solved through cross-multiplication.

The themes used in the problems both in daily-lesson and monthly-lesson notebooks also presented a “type” that was mostly used in all notebooks: the relation to men’s adult work life. Situations involving money were the most common (buying, selling, profit, amount received for a job, amount saved). The ones who bought, sold, got paid, saved and profited were workers, shopkeepers, family men, butchers, farmers, gardeners, tailors, potters, etc. The few situations that mentioned women’s adult life approached their role as mothers or dressmakers. It is also important to highlight the great representativeness of problems that, in these themes, involved data referring to the use of the metric system (length measurements).

Although some problems involved only two pieces of information that needed to be used to perform an operation to solve the problem and others involved a higher complexity of relations with the presentation of different data that needed to be related in many operations so that the problem’s question could be answered, the themes involved in the narratives contained in the wordings were very similar.

Another common aspect in problems contained in monthly-lesson and daily-lesson notebooks was related to the magnitude of the numbers used. Mostly they involved numbers up to the first digit of thousands, that is, up to nine thousand. Numbers of greater magnitude were more common in problems that were solved with addition and subtraction, while numbers of lower magnitude appeared more in problems that involved multiplication and division.

Here we resume the relations between this study and research studies by D’Enfert (2003), Sarrazy (2003) and Hébrard (2001). D’Enfert (2003) and Sarrazy (2003) identified in legislation and in textbooks the indication of problems in the teaching of arithmetic as a way of bringing social life into primary school from 1880 on. In the notebooks, just like Hébrard (2001) had observed before, such problems gained centrality because many times they were the only recorded activity in relation to arithmetic.

In his analyses, Hébrard (2001) pointed out the graphical demands for the use of the notebooks, which resulted in a specific spatial organization for problem solving. In this study, we observed the same spatial organization, but in addition to a spatial organization model, we detected a model for the solution: a model for writing the solving procedures.

The occurrence of “type-problems” whose solution would be learned by students based on memorization and repetition (Sarrazy 2003) was also observed in the notebooks that we analyzed. Here the discussion is expanded through exploring the relation between problems and all other arithmetic activities. In the notebooks, the recording of activities related to some arithmetic contents happened before those contents were used in problems, which indicates that those other activities could be used as tools in problem-solving.

Another possible contribution from the analysis of school notebooks is the observation of ‘how’ some orientations were used, “consumed”. In the historical period approached in this study, educational guidelines demanded a connection between the teaching of arithmetic and concrete objects (D’Enfert, 2003) and to social life situations (D’Enfert, 2003; Sarrazy, 2003). In the notebooks, the use of “concrete” numbers with the identification of species units in operations seems to be one way of meeting those guidelines. Moreover, a link to social life was present in the problems recorded in the notebooks through themes related to adult life and to the monetary system.

The analysis enabled us to identify a specific type of problem which was mostly used in arithmetic classes between 1870 and 1914. This kind of problem involved: the idea of proportionality; themes related to social life and to men’s adult life; models for spatial organization and solution procedures. It is important to highlight that the analyzed corpus does not represent teaching in French in its entirety. However, in this paper we discuss important aspects of the use of problems in the teaching of arithmetic in French primary schools. The results that we present here will soon be linked with other results, as the research continues with records of arithmetic classes in later periods in order to observe how this proposal can be characterized over time.

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Two textbooks on unconventional arithmetic: Reactions of influential persons

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Abstract

The paper describes two Icelandic arithmetic textbooks, published respectively in 1780 and 1911–14, both written at the beginning of an era when enhancing general education was placed on the agenda for restoring the Icelandic society after natural disasters. Both textbooks emphasized unconventional and mental arithmetic. Both books were criticized by influential persons. The focus is on the grounds of their arguments. Both books survived the criticism, considering the circumstances, while few positive comments on them have been found in sources from their times.

Keywords: Enlightenment, arithmetic textbooks, mental arithmetic, number tricks, Rule of Three

Introduction

The influence from the Enlightenment was strongly felt among Icelandic students in Copenhagen from the 1770s onwards (Sigurðsson, 1990). Iceland had then been part of the Danish realm from the end of the 14th century. The Danish authorities felt responsibility when Iceland suffered various calamities during the 18th century, such as exceptionally cold climate with pack ice in the 1750s. In the period 1770–1780 a number of informative printed texts were distributed in Iceland, even for free, by proponents of the Enlightenment, supported by the Danish government or gentry. The two first substantial arithmetic textbooks, written in Icelandic by young Icelanders in Copenhagen, Olavius (1780 and Stefánsson (1785), were published with such financial support. We shall examine the reception of one of them, *Greinileg vegleiðsla til talnalistarinnar* [A clear guide to the number art] by Ó. Olavius of 1780.

The period of cold climate in the mid-18th century was followed by volcanic eruptions and earthquakes in the 1780s which caused death of livestock and consequent famine. It was called the Haze Famine due to poisonous gases from the volcanos. The population sank from about 50,000 inhabitants below 40,000. The two Latin schools were closed down temporarily. From 1802 until 1930 there was only one school at that level in the country. These conditions and the bankruptcy of the Danish Kingdom in 1813 following the Napoleonic wars, hindered results of the efforts of the Enlightenment proponents. For example, no other mathematics textbooks written in Icelandic were published until the 1840s.

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In the 1860s, another cold period with pack ice set in, continuing into the 1890s, resulting in large-scale emigration to North-America. However, small-scale industrialization and mechanization of fishing vessels grew in the same period, gradually improving living conditions. From the mid-19th century, a movement had risen towards independence, and enhancement of education was a factor in that effort. From 1869 until the Great War, a stream of arithmetic textbooks for beginners by at least eight authors were published. We shall examine a reaction to one of the later arithmetic textbooks of that period, *Reikningsbók* [Arithmetic] by S. Á. Gíslason, published during 1911–1914.

In both textbooks, those by Olavius and by Gíslason, the authors emphasized mental and unconventional arithmetic as an aid to written arithmetic but were received with some mistrust. The arguments for this mistrust will be examined.

This paper is a study of reactions of influential persons to these books. The question is on what grounds they founded their opinions. Both persons may be considered as authorities in educational matters for their time. The study is based on one single source each.

Background

Education frame

The two Latin-schools, which became only one in the year 1802, laid their main emphasis on Latin and religious studies as they educated the clergy until the mid-19th century, but most of the time they taught very little arithmetic. Students were given Stefánsson's (1785) arithmetic textbook but they studied arithmetic only if they so wished and were then guided by older students, not the teachers (Helgason, 1907–1915). Other schools did not exist until the 1870s, apart from scattered privately run primary schools. By the King's order of 1791, children were to be able to read, otherwise they could not be confirmed. Knowledge of arithmetic was not widespread, even among the clergy who had the duty to oversee education provided by the homes.

Law no. 2 of 1880 on education in writing and arithmetic required homes to teach children the four arithmetic operations in whole numbers and decimals. Primary school legislation was enacted by law no. 59 of 1907 when the communities were to overtake the responsibility of providing education; in schools in urban areas, and by itinerant teachers in the more rural areas. Both legislations gave sparks to arithmetic textbook publications.

Unconventional arithmetic and mental arithmetic

Mental arithmetic has been part of popular culture in many societies through the centuries. In a summary of researches made by Threlfall (2002), it emerges that most calculations in adult life are done mentally. Mental work develops insight into the number system, also termed “number sense”, and mental work develops problem solving skills.

Ancient Egyptian multiplication and the related Russian peasant multiplication are widely known. They primarily involve creating repeated doubles of the multiplicand and adding up those relevant to fill up the multiplier (Seppala-Holtzman, 2007). (An example of this method is multiplying $13 \cdot 21$ as $(8 + 4 + 1) \cdot 21$. The multiplicand 21 is doubled as many times as necessary, here three times, and the individual results are used until the powers of 2 add up to 13). The widespread use of doubling in societies where access to school education and/or writing material was limited, indicates that doubling was considered more accessible than general multiplication. In extant Latin translations of Al-Khwarizmi's arithmetic textbook, *Kitāb al-bisāb al-Hind* of the 9th century, such as *Dixit Algorizmi* and *Liber Alchorismi*, doubling and halving were counted among seven or even nine arithmetic operations (Allard, 1992, p. xxxi).

Arithmetic textbooks of the 19th and 20th century have generally taught written procedures to perform arithmetic operations which may have made mental arithmetic redundant in the minds of students as well as their teachers. The procedures have provided a sense of security which no less applies to the widespread use of pocket calculators in modern everyday life.

Mental arithmetic was, however, still considered important in the mid-20th century. The mathematician Gustave Choquet, a guest-speaker at the Royaumont seminar, emphasized that children must know how to do simple and rapid mental calculations, and they must be accustomed to finding very quickly an order of magnitude for a total of a product, that is to use estimation (OEEC, 1961, p. 66).

Researchers have attempted to assess methods to train strategic flexibility of students in mental arithmetic. Threlfall (2002) proposed that mental calculation is an interaction between noticing and knowledge, and ...

... if the aim of teaching mental calculation is flexibility, what children and adults actually do to calculate efficiently should not be distilled into general descriptions of methods or ‘strategies’ and promoted as holistic approaches to calculation, offered as models to emulate, or taught as procedures to learn. Rather it suggests that solutions to problems would be better approached as specific examples of how particular numbers can be dealt with, how numbers can be taken apart and put together, rounded and adjusted, and so on. (Threlfall, 2002, p. 45)

Threlfall's conclusion was that flexibility cannot be taught as 'process skill' but will rise consequentially through the emphasis on considering possibilities for numbers rather than by focusing on holistic 'strategies'.

Conventional arithmetic of proportional problems

For centuries, proportional problems were solved by a method called in Latin *Regula Trium*, the Rule of Three. The method consists of finding the fourth proportional to three known quantities. It is traced back to Italian merchants in late medieval times, described in arithmetic books, the Italian *libri d'abbaco* (Van Egmond, 1980). The way of thinking in the Rule of Three can be found in ancient Indian works, e.g. by Brahmagupta (597–668) and Bhaskara II (1114–1185) (Tropfke, 1980, pp. 359–361).

In *Arithmetica Danica*, the oldest arithmetic textbook known to have been available in Iceland, the solution method is described. Four numbers are to be arranged so that the first and the third are of the same kind and the second of the same kind as the fourth. When one wants to know the fourth number, the second and the third numbers are to be multiplied together. The product is then to be divided by the first number to give the sought after fourth number (From, 1649, pp. 77–78). This method echoes in the textbooks by Olavius (1780, pp. 176–177) and Stefánsson (1785, pp. 132–137).

E. Briem wrote an influential arithmetic textbook, first published in 1869. It was used in the emerging lower secondary schools from the 1870s into the 1910s. The search for the unknown in direct Rule of Three was to find a number that is as many times greater or less than the middle term, as the rear term is greater or less than the front term in the previously mentioned sequence of the three known proportional numbers. This was to be found by multiplying the middle term by the rear term and then divide by the front term (Briem, 1898, p. 63–65).

A Clear Guide to the Number Art of 1780 by Olavius

The author

The first comprehensive printed mathematics textbook in Icelandic was *Greinileg vegleiðsla til talnalistarinnar* [A clear guide to the number art] by Ólafur Olavius (1780). Olavius (1741–1788) studied philosophy in Copenhagen during 1765–1768. He was one of the founders of the Icelandic Enlightenment society, in 1779, called *Lærdómslistafélagið*, the Society of the Learned Arts, established by students and intellectuals in Copenhagen. From the beginning of printing in the 16th century, the sole printing press in Iceland had exclusively printed religious books. Olavius brought a new printing press to Hráppsey Island in 1773. For about a year he oversaw there

the first printing of secular books in Iceland. Then he left, presumably due to disagreements about finances. After that he travelled around the country during the summers of 1775–1777 under the auspice of the Danish King in order to write an economic description of Iceland, published in 1780. Olavius stayed in Denmark until his death; during 1779–1788 as customs officer in Skagen. Thus he stayed only for one year, 1773–74, and three summers in Iceland after his Latin school graduation in 1765 (Ólason, 1951, pp. 72–73).

During 1770–1786 Olavius's voluminous informative writing was published, including works about economics and the use of products of nature, all intended to improve and remedy Iceland's economy. The arithmetic textbook *Clear guide* was financed by an unknown sponsor. About 1300 copies were distributed for free in Iceland where the population was around 50000 and the estimated number of homes was 7000. Records indicate that several copies existed in private libraries in one county in northwest Iceland some 50 years after its publication so it must have been studied by some number of people (Jensdóttir, 1969).

The content

Clear Guide (Olavius, 1780) begins by an address in Danish to Count Schach Ratlau, Knight of the Elephant Order, a possible sponsor, and to whom the author dedicated the book. He described the miserable life of Icelanders who had no education in the art of reckoning and must do with carving dashes in a piece of wood for numbers. He wondered how people had been able to survive there through the centuries. The book was destined for use in the Latin Schools as well as for other children of the country that had desire for exercising arithmetic. It was, however, never used at the Latin Schools, which were in bad shape during the following decades due to the Haze Famine.

In his following address in Icelandic to the reader (Olavius, 1780, pp. ix–xx–viii), the author revealed that he had modelled his book after the German textbook *Demonstrative Rechenkunst* by Christlieb von Clausberg (1732), republished in 1748 and 1762. The author said that he had also taken examples from Danish textbooks, one of them *Arithmetica Tyronica* by Chr. Cramer (1735, 1755, 1762, 1766, ...), but to a lesser degree. As none of the well knowledgeable Icelanders had written such a book, Olavius dared to present this one. Without such knowledge, one could hardly avoid loss of money or other capital in trade, e.g. with foreign merchants. He had strived to use Icelandic words for the general public who might not understand or articulate the foreign terms.

Olavius said that he had struggled to provide clear explanations as he did not know of anyone out there [in Iceland] who taught the general public anything in the reckoning art, which he said was in line with most other situations in that 'un-country',

apart from small teaching activities in the schools. Indeed, each common person who wished to learn something had to be his/her own teacher. Therefore, he must produce a great number of solved examples. Much more was to be learned from one solved problem than from ten unsolved ones.

In continuation, Olavius explained what he meant by what he called ‘talnabrögð’, number tricks, a term that he used to translate the German term ‘Vortheilen’, in English advances or benefits. These were to be used either for a quicker or easier work or for less heavy thinking. In more detail:

1. addition and subtraction could be applied in place of multiplication and division
2. multiplication applied instead of division
3. smaller numbers used than mentioned, and
4. fractions be avoided, even if working with them might sometimes be quicker and easier (Olavius, 1780, p. xx).

The author expressed his feeling that these tricks or advances could be used on one third to one half of the problems. But certainly, time had to be devoted to studying them. It was also advantageous to know more than one method to solve a problem in order to confirm its correctness.

Before giving examples of his tricks, Olavius explained how multiplication by chosen numbers could be simplified. It seems that multiplying by 8 was assumed to be done by doubling three times mentally, and even multiplying by 32 by doubling five times (p. 75). In multiplying by 11, digits for unities and tens should be added, then digits for tens and hundreds and so on. For instance, in multiplying 26748219 for 11 he wrote 9 on the farthest place on the right, then added 1+9 mentally, writing 0, 2+1, adding 1, writing 4, 8+2, etc. to reach 294230409 (p. 66). Not quite mental arithmetic, but saving writing space. He discussed what to do when multiplying by 0, both at the right end of a number, as by 0 in 80, and also internally, as the 0 in 207 (p. 67). He remarked that multiplying by 25 could be done by multiplying by 100 and dividing by 4 (halving twice); 375 is $\frac{1}{8}$ of 3000; $1\frac{2}{3}$ is $\frac{1}{6}$ of 10; $12\frac{1}{2}$ is $\frac{1}{8}$ of 100, etc.

We shall look at some examples of each case of Olavius’s number tricks:

1. Dissolving multiplication partly into addition or subtraction:

$$\begin{array}{r}
 \text{1. D. } 4325 \text{ med } 276 \\
 \underline{25950} \quad (6) \quad \begin{array}{c} 6 \\ 30 \\ 240 \end{array} \\
 129750 \quad (5) \\
 1038000 \quad (8) \\
 \hline
 1193700.
 \end{array}$$

Fig. 1. Multiplication by dissolving 276 into $6 + 6 \cdot 5 + 6 \cdot 5 \cdot 8$ (Olavius, 1780, p. 71).

Olavius first multiplied by 6, then the product by 5, and subsequently multiplied that product by 8. Then he added up the products by 6, 30 and 240.

$$\begin{array}{r}
 \text{1. D. margfald } 96 \text{ med } 39. \quad \text{St. } 342 \text{ med } 199. \\
 \underline{3840} \text{ edr } 40 \div 1 \quad \underline{68400} \text{ edr } 200 \div 1 \\
 \text{vert. } 3744. \quad 68058.
 \end{array}$$

Fig. 2. Multiplying by powers of 2, then multiplying by powers of 10, followed by subtracting once (Olavius, 1780, p. 87).

$$\begin{array}{r}
 \text{1. D. } 2374 \text{ med } 65. \quad \text{2. D. } 3589. \text{ med } 577. \\
 \underline{18992} \quad (8) \quad \begin{array}{c} 8 \\ 56+1 \end{array} \quad \underline{25123} \quad (7) \quad \begin{array}{c} 7 \\ 560+10 \end{array} \\
 132944 \quad (7) \quad 200984 \quad (8) \\
 154310. \quad 2070853. \\
 \\
 \text{3. D. } 24783. \text{ med } 1277. \quad \text{4. D. } 3243. \text{ med } 12034. \\
 \underline{272613} \quad (11) \quad \begin{array}{c} 1100 \\ 77 \end{array} \quad \underline{3243} \quad (11) \quad \begin{array}{c} 11000 \\ 33+1 \end{array} \\
 1908291 \quad (7) \quad 107019 \quad (3) \\
 31647891. \quad 39026262.
 \end{array}$$

Fig. 3. Multiplying by 8, 7 and 11 (Olavius, 1780, p. 73).

We see from the examples taken in Figures 2 and 3 that the author used his easy methods to multiply by 10 and 11. He also liked to multiply by 8, while he had to multiply by 7 in three examples in Figure 3.

2. Multiplication applied in place of division, and
3. smaller numbers used than mentioned

$$\begin{array}{r}
 \text{2. D. deil } 38976 \text{ med } 87\frac{1}{2}, \\
 7) 5568 \\
 \hline
 \text{qvótr } 445 \frac{1}{2}, \text{ edr } 445\frac{1}{2}.
 \end{array}$$

Fig. 4. Dividing by $87\frac{1}{2}$ which is equal to $7/8$ of 100 (Olavius, 1780, p. 286).

Dividing by $87 \frac{1}{2}$ can be done by noticing that $87 \frac{1}{2}$ is $\frac{7}{8}$ of 100 so the division equals dividing by 7 and multiplying by 8, which is fortunate, and then dividing by 100. Easy!

According to his foreword, Olavius knew of criticism of that kind of tricks. The critics say, he said, that they cannot be used in all cases, they fit seldom, and the method is difficult to handle, and more incomprehensible and more confusing than the ordinary method. Olavius protested and explained that a suitable method had to be chosen for each case and there was no reason to deem methods impracticable even if they did not fit everywhere. Time and knowledge were needed. Those who read and cogitated would be rewarded with discovering much usefulness and many advantages (Olavius, 1780, pp. xx–xxii)

Who were the critics that Olavius referred to? It is unlikely that they were residents of Iceland. The book had not yet been published when the foreword was written, and Olavius did not stay in Iceland. The critics must have had its model in mind, that is Clausberg's book, or a similar one. Few Icelanders read German, and no sources are available about Clausberg's book in Iceland. The critics are more likely Olavius's pals in the Society of the Learned Arts in Copenhagen.

The critique

In the 18th century there was no platform in Iceland for educational discussion. The first Icelandic journal ever, the *Journal of the Icelandic Society of the Learned Arts* was established in 1780, the same year as the *Clear Guide* was published. The only sources available about such discussions are private letters and memoirs such as Helgason's (1907–1915).

Private letters by the reverend G. Pálsson, an ex-headmaster at one of the Latin schools, have been edited and published (Sveinsson, 1984). Pálsson was an authority in education in Iceland and a renowned teacher. He published a noteworthy primer (Pálsson, 1782), printed by the controversial printing press in Hrappsey. There he devoted three pages in an appendix to numbers and the multiplication table. For various reasons, both external and personal, he had financial difficulties that he was unable to handle even though he dedicated his primer to the most powerful persons in his neighbourhood in hope for support (Sveinsson, 1982).

Pálsson said in a letter, dated October 5 1780 – February 20 1781, to his successor, the headmaster of one of the two Latin schools, that he had read the foreword of the *Clear Guide* and was not impressed but had ordered the book. He seized the phrase 'non-country' and others similar, and deemed that as an opinion of mean people with foreign taste. He mentioned the term 'number art' [talnalist] in the title which he found project affectation, while it indeed refers to the German title of

von Clausberg's (1732) *Rechenkunst*. Pálsson would have preferred the term 'number wisdom' [tölvísi]. In a letter dated August 11 1781, Pálsson recounts that he had not yet seen the book, and in a letter dated January 29–March 25 1782 he is still patiently waiting for the book. Actually, he would have preferred that an arithmetic book by a colleague minister, available in a manuscript, had been printed, not this one. In his letter dated April 24–May 31 1782, Pálsson recounts that he saw the book externally but his guest, who had read it, had not liked its style. Pálsson expressed the opinion that it was neither prudence nor correct didactics that the author expressed himself so affective as Pálsson had seen in the foreword (Sveinsson, 1984, p. 372–373, 383, 387–388, 394–395).

Why should Pálsson, well knowledgeable in arithmetic, have had such negative opinion of the first substantial printed textbook in arithmetic of 374 pages, written in Icelandic, distributed for free; a book that he had not read when he wrote his letters? According to the letters, Pálsson was repelled by Olavius's descriptions of the misery in Iceland. Financial complications, due to Olavius's import of the printing press, where Pálsson was involved, may also have contributed to his negative feelings.

No other sources have been found available as yet about others' opinions of the content of the book, while the reverend Pálsson's unfavourable review has echoed in historical texts until today (Guttormsson, 1990). Another matter is that the book may have been ambitious for the Icelandic community, where, quoting the author: "each common person who wished to learn something had to be his/her own teacher".

Already in 1785, another arithmetic textbook in Icelandic was published by another proponent of the Society of the Learned Arts, the district governor, later governor of Iceland, Ólafur Stefánsson (1785). That book was immediately legalized for the Latin schools (*Lövsamling for Island*, 1855, p. 244) and was presented to the students for free. Olavius's *Clear Guide* was never used in the schools. However, due to the bad shape of the country and the schools and that the teachers did not teach mathematics (Helgason, 1907–1915), Stefánsson's book was also of little use in the schools. But both books were listed in private libraries half a century later so that they may have been used for self-instruction in the homes (Jensdóttir, 1969).

Arithmetic by S. Á. Gíslason

The author

Sigurbjörn Á. Gíslason (1876–1969) studied theology at the Icelandic School of Theology for a tertiary degree, to become eligible for teaching at secondary schools, as he could not afford to go abroad to study mathematics at a university. Gíslason

became a successful mathematics teacher in Reykjavík during 1897–1945, and wrote a six-volume series of an arithmetic textbook, *Reikningsbók* [Arithmetic], for students from the age of seven into secondary colleges. Gíslason managed to take a year-long study trip to Denmark in the early 1900s. He may have become acquainted there with Pestalozzi's pedagogical theories which were well known and favoured in Denmark at that time (Hansen, 2009).

Gíslason lived in Reykjavík all his adult years. In early 1900s, Reykjavík was a fast growing town with a large primary school, the sole Icelandic grammar school (the previous Latin-School), a commercial college, a teacher college, a secondary school for girls, and an engineering college. Gíslason taught at all these schools for some periods of time except at the grammar school; the longest period at the engineering school. His books covered the then current syllabi of these schools except the grammar school where Danish textbooks were used.

The content

In his foreword to the first volume, Gíslason emphasized that mental arithmetic was the main issue in all general and simple exercises. The students were to gradually get used to writing down the exercises with correct symbols and explain orally why they do the particular operation as they do. Rote learning was worth nothing but regrettably was commonly practiced. Gíslason followed up his vision by starting most sections through volume six by exercises in mental arithmetic.

The content of the first four volumes matched the curriculum prescribed by law no. 59 of 1907. Every child, who reached the age of 14 years, should have learned:

§4. in arithmetic the four operations in whole numbers and fractions, and be able to use these in order to solve simple problems coming up in daily life, e.g. to calculate the area and volume of simple bodies; he/she should also be skilled in mental arithmetic with small numbers.

Volumes 5 and 6 were more advanced. Vol. 5 contained equations, proportions, percentages, and interests. The Rule of Three was treated in equations as proportions. Vol. 6 continued with powers, exponents, square root and cubic root, stocks, compound interests, and logarithms. Most sections in Gíslason's Arithmetic began by exercises in mental arithmetic. No special technique was presented, only simple exercises, chosen to throw light on the concepts being worked on.

The exercises in themselves were quite trivial. As a sample of exercises for 10–12 year-old children, questions were posed about how much a person earned a day if the salary for a 6-day work week was given. The amounts to be divided were 18, 9 and 42 crowns. More interesting for the modern reader is to see that a man-worker

had double the salary of a woman and a member of parliament had more than twice as much as an ordinary worker, both men.

Wondering about the ultimate goal of these exercises in mental arithmetic, a guess is that it was developing number sense, while written arithmetic was certainly also necessary in a country with growing trade, import and export. Gíslason taught at colleges where accounting and learning how to produce proper bills were important. Another aspect is that textbooks, paper and pencils were still expensive for families with small income so some training in mental arithmetic was useful in daily life.

In 1929, three arithmetic textbook series were legalized¹ for use at primary school level (Elíasson, 1944). One of them was Gíslason's Arithmetic, volumes 1–4, and another was a textbook series by Elías Bjarnason (1927–1929). The final topic for primary level was addition of fractions with different denominators. We shall compare Gíslason's and Bjarnason's presentations of finding a common denominator.

Bjarnason presented the following procedure, shown in fig. 5, for finding the Least Common Multiple, LCM, for the denominators of the fractions $\frac{1}{2}$, $\frac{8}{9}$, $\frac{5}{16}$ and $\frac{7}{24}$.

Gerum ráð fyrir, að finna eigi samnefnara brotanna $\frac{1}{2}$, $\frac{8}{9}$, $\frac{5}{16}$ og $\frac{7}{24}$. Allir nefnararnir eru skrifaðir í röð áfram og strik dregið undir röðina, þannig:

$$\begin{array}{r}
 2) \ 2 - 9 - 16 - 24 \\
 \hline
 2) \ 9 - 8 - 12 \\
 \hline
 2) \ 9 - 4 - 6 \\
 \hline
 3) \ 9 - 2 - 3 \\
 \hline
 3 - 2 - 1
 \end{array}$$

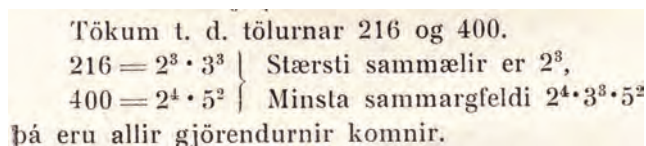
Fig. 5. Finding the Least Common Multiple of four fractions (Bjarnason, 1957, p. 32).

Bjarnason explained that the denominators were to be lined up and then be divided repeatedly by the lowest factor found to divide into any two or more. The other factors were to be pulled down. When no more could be done, the factors were multiplied to become $24 \cdot 32 = 144$. Prime numbers and prime factors were not mentioned.

Gíslason presented two methods, one similar to Bjarnason's method, and another one which included prime factoring and was easier to argue for. The example

¹ No information is available about legalizing textbooks and what it meant. Legalizing is presently only known to have been done twice: Stefánsson's arithmetic textbook was legalized in 1786, and a list of textbooks in all subjects in primary school was announced legalized in 1929.

he took was to find the Least Common Multiple, LCM, of 216 and 400, and simultaneously, Greatest Common Factor, GCF, of the two numbers, see fig. 6.



Tökum t. d. tölurnar 216 og 400.
 $216 = 2^3 \cdot 3^3$ } Stærsti sammælir er 2^3 ,
 $400 = 2^4 \cdot 5^2$ } Minsta sammargfeldi $2^4 \cdot 3^3 \cdot 5^2$
 Þá eru allir gjörendurnir komnir.

Fig. 6. The GCF and the LCM of 216 and 400 (Gíslason, 1913, p. 11).

Gíslason then remarked that all the prime factors were included in the multiple of the LCM, $2^4 \cdot 3^3 \cdot 5^2$, and the GCF, 23, in total $27 \cdot 33 \cdot 52$; and in general that

$$\text{LCM}(a, b) \cdot \text{GCF}(a, b) = a \cdot b.$$

Thus Gíslason's book offered flexibility in important arithmetic procedures.

Gíslason's *Arithmetic* became quite widespread. In foreword for the second edition of volume 1 in 1913, the author reported that it had been reprinted in 4000 copies. That is a large number in a country where the total population was about 87,000. This may have applied to the sum of copies of the five volumes then published.

Gíslason's *Arithmetic* survived into the 1930s as a legalized textbook series for primary level. In 1938, Bjarnason's series (1927–1929), which then was published in a revised edition, and was more in favour of training procedures, was chosen for free distribution in primary schools (Sigurgeirsson, 1987). This hindered publication and distribution of other textbooks.

The critique

In the early 1900s, teaching was becoming a growing profession. A private teacher training college operated during 1897–1908, and a state-run teacher training college was established in 1908. Teachers had their own journal, *Skólablaðið*, as a platform to express their views. During 1909–1916, its editor was Jón Thorarinsson (1854–1926). He had been the headmaster of the private teacher training college and became the first state secretary of education in 1908.

Gíslason's textbook series was mentioned several times in that journal. In 1912, a teacher praised that the exercises were expressed in words, not only by numbers. Another advantage was a great number of exercises. A special advantage was the great emphasis that the author laid on mental arithmetic in volume one. In the teacher's point of view, several exercises in volumes two and three were too difficult for the assumed age of pupils. Volume four was the best one. Still another advantage was that the books contained various pieces of knowledge: the map of Iceland, America in the eyes of Christopher Columbus, poetry by a beloved poet,

etc. However, the author used the old measuring systems too much in comparison to the new metric system (Jónsson, 1912).

In 1916, another review of Gíslason's series appeared, saying that it was good in many respects, but too large and far too expensive for small schools and itinerant schools. Another arithmetic textbook (Brynjólffsson and Arason, 1914) collected the whole material in one volume for a much lower price (Hjartarson, 1916)

In 1916, the Secretary of Education, Jón Thorarinsson, refused when asked to recommend Gíslason's series for a grant from the National Budget. In his reply, he mentioned that others had given volumes 1–4 good reviews. He remarked, however, that in volume 5 (which actually was not intended for compulsory school level but for the colleges) he found the mental arithmetic exercises far too difficult and randomly arranged. He remarked in particular an exercise in mental arithmetic where the students were asked to compare 4 ‰ of 9000 crowns and $4\frac{1}{2}$ ‰ of 8000 crowns (Gíslason, 1912, p. 53). Furthermore, the series was too expensive. He concluded by saying that the best arithmetic textbook among the many that were being published at that time would win in their competition and that it was not right to use national funds to support one over the other (Skjalasafn Fræðslumálaskrifstofunnar 1976-C/2).

Thorarinsson was undoubtedly the person most knowledgeable about educational matters in the early 1900s. He had devoted his early life to studying education in Denmark, Germany and England and had been headmaster in the first teacher training college in Iceland. Understandably, he had concerns about the price of school books and his remark on the win of the best book was well grounded. However, he failed to see the simple solution of the aforementioned problem: to find 1 ‰ of by thinking something like “of one thousand 4 crowns, of nine thousand $9 \cdot 4 = 36$ crowns”; and similarly $8 \cdot 4\frac{1}{2}$, also 36, as had been pointed out in the introduction to the chapter on percentages. It is likely that Thorarinsson learned Briem's method, to create the sequence of front term, middle term and rear term: $1000 - 4\frac{1}{2} - 8000$, multiply middle term and rear term, $4\frac{1}{2} \cdot 8000$ to gain 36000, and then divide by the front term, 1000, seemingly a complicated process, instead of using the primitive method of finding 1 ‰ of 8000 to begin with. Then the continuation should be easy.

Another aspect is that students studying the textbooks had seen similar tasks before and were presumably better prepared than the Secretary of Education who may just have taken a look at the book and tried to recall a method he learned long before.

One should not underestimate that money was important in the poor society that Iceland was in the beginning of the twentieth century when it had not yet gained sovereignty and was still far away from independence. Anyhow, no harm was

done, Gíslason sold his books quite well and they survived until the late 1930s when Bjarnason's textbook was chosen for free distribution and no competition existed any more.

Discussion

The two arithmetic textbooks of 1780 and 1911–1914 may be considered progressive for their time. Both emphasised unconventional and flexible methods, involving mental arithmetic. Olavius (1780) explained a great number of strategies built on thorough knowledge of arithmetic facts. For a reader who is well trained in arithmetic it arouses admiration and inspires flexibility. It may have repelled less experienced readers, but Olavius made it perfectly clear that it was not intended for beginners.

One would for example have expected the Rev. Pálsson to admire the many ideas presented in the book, but his standpoints about the book were expressed before he had had opportunity to read more than its foreword. One must conclude that his opinions were based on negative attitudes unrelated to the book's content. One reason could be that Pálsson felt that Olavius was in his foreword patronizing Icelanders in their misery. Another reason could be malice due to financial difficulties concerning the printing press that Olavius imported in Pálsson's county. Olavius was young and enthusiastic to bring the first printing press to print secular books, while Pálsson was an elderly resident in the area, trapped in his own financial difficulties. We do not know the exact reasons for their disagreements but Olavius left and some mistrust remained.

It is somewhat surprising that Thorarinsson did not appreciate the mental arithmetic exercise in Gíslason's (1911–1914) series that he took as an example nor the collection of exercises. Possibly, his thinking was fixed in methods laid down centuries ago on the Rule of Three which prescribed fixing the given numbers in a certain order and continue according to a fixed procedure.

Threlfall (2002) proposed that mental calculation be thought of as interaction between noticing and knowledge, and concluded that flexibility cannot be taught as 'process skill' but will rise consequentially through the emphasis on considering possibilities for numbers.

The prominent persons, Pálsson and Thorarinsson, who reviewed the arithmetic textbooks briefly with other aims than training their own personal skills, did not notice the advantages of the strategies presented in the books. This is in accordance with the opinion of Threlfall who did not recommend focusing on holistic 'strategies' in mental calculation. It is not very likely either that Icelanders were receptive to strategies as presented by Olavius in the 1780s, and hardly under the Haze Famine.

In the second case, Thorarinsson's mind did not operate in a flexible way which he cannot be blamed for. Problems that descend upon people, more or less unprepared, can create feelings of insecurity. It was, however, unfortunate, as he was a powerful person and an authority in educational matters, that he did not take into account the cumulative nature of mathematics. A whole series of books can hardly be judged by one exercise in its fifth volume, not even intended for the compulsory school level. He must have been familiar with Jónsson's more balanced review which was published in a journal of which he was editor.

Concluding remarks

On what grounds did Pálsson and Thorarinsson express their opinions? It is regrettable that they did not focus on the advances of the arithmetic textbooks that they expressed their views about. Instead of bringing out their mathematical merits, they discussed extra-mathematical matters, such as language use and prices, certainly valid viewpoints but very onesided. But did their opinions exert any influence on the distribution of the books?

Pálsson wrote personal letters to his successor headmaster in one of the two Latin schools. In his time, the only platform available for discussion was private letters. Pálsson's view may have had some influence in that school and on other colleagues in the vicinity, but ultimately the book by the district governor was legalized as the prescribed arithmetic textbook for the two Latin schools and there was no place for another one. Thorarinsson's letter is a reply to a query from the author about a grant for his publications. It is not likely that many others knew about his reaction. Thus, the reactions to the two books had hardly any widespread influence.

A documented report exists that Gíslason's book was considered far too expensive by the opinions of Thorarinsson and Hjartarson (1916). Iceland was taking its first steps into modernity, it was not yet independent, and frugality prevailed. It is not known if Gíslason was disappointed that his application for a grant was refused. At least his series continued to sell and was chosen for legalization in 1929. It was not chosen for free distribution in 1937 which was not unnatural considering that it had been written a quarter of a century earlier. Two other series were chosen, written about fifteen years later than that by Gíslason. Similar aspects concerning prices may have prevailed in the 1930s during the Great Depression as in the pre-Great-War times.

One can, however, state that both books served their purpose and were read as widely as may be expected in their milieu.

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Attitudes toward intuition in calculus textbooks

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Abstract

Intuition was long held in high regard by mathematicians, who considered it all but synonymous with clarity and illumination. But in the 20th century there was a strong tendency to vilify intuition and cast it as the opposite of rigorous reasoning. Calculus in particular became a battleground for these opposing views. By systematically surveying references to intuition in historical and modern calculus textbooks, I look at how its status has changed across the centuries. In particular, I argue against the veracity of the self-fashioned origin story of the modern anti-intuition movement, which relies heavily on a particular historical narrative to portray the demise of intuition as an inexorable triumph of logic and reason.

Keywords: intuition, calculus, rigour, pathological functions

The standing of intuition in mathematics has suffered fluctuating fortunes. One school of thought takes it to be the antithesis of proof and careful reasoning. Hans Hahn epitomises this view. In his famous 1933 lecture “The crisis in intuition,” Hahn argued that “the failure of intuition” is a historical fact, which had the inevitable consequence that “intuition gradually fell into disrepute and at last was completely banished” from mathematics (Hahn, 1933, pp. 84, 76). Mathematicians thus came to realise that we must

confess our faith ... in careful logical inference ... as opposed to bold flights of ideas, mystical intuition, and emotive comprehension (Hahn, 1930, p. 30).

This account has become canonised mathematical folklore (e.g. Bell, 1945, 278, 292, 294, 387; Boyer, 1959, 5, 13, 25, 59; Gray, 2008, 60, 62, 75, 118, 217, 275). It is nowadays a veritable party line in modern textbooks. Calculus students in particular are nowadays inculcated with a narrative that paints intuition as a corrupting temptation that must be resisted. Insofar as intuitive arguments are presented at all, textbooks make sure to undermine them at once by hastening to emphasise that they don’t count as real mathematics. “These intuitive arguments do not constitute proofs” (Thomas, 2004, p. 85), we are always warned — a message that can also be efficiently conveyed by denigrating scare quotes: “Intuitive ‘Proof’ of the Chain Rule” (Thomas, 2004), p. 192). The dichotomy between intuition and proof is an admonition almost all modern calculus textbooks feel obligated to emphasise at every opportunity (Stewart, 2012, pp. 63, 87, 142, 199, 304, 312, 722, 982, 1008; Lang, 1986, pp. 162-163, 176, 218, 306). Likewise, intuitive conceptions of the

fundamental notions of calculus must also be purged: “the intuitive definition of a limit” is “inadequate” and “vague” (Stewart, 2012, p. 72), and in the same way all other intuitive conceptualisations must be replaced by formal ones (Stewart, 2012, pp. 76, 142, 284, 352, 353). An emblematic formulation in one prominent calculus textbook even makes intuition the direct antonym of mathematics itself: “Intuitively, ... Mathematically, ...” (Strang, 1991, p. 62). In sum, although some modern calculus books occasionally pay lip service to intuition, their persistent phraseology pitting it against proof and rigour ensures that their most conspicuous message is that intuition is not real mathematics.

The anti-intuition movement has been very successful in portraying itself as the inevitable outcome of rational progress. I challenge this triumphalist narrative. The wave of anti-intuition sentiments that dominated much of the 20th century is not the end of history and the definite “right” view of mathematics; rather, it is an ideology that happened to fit the needs of a particular era. The circumstances that gave rise to it are complex and include internal mathematical developments as well as a broader philosophical context (see e.g. Volkert (1986), Jahnke (1993), and Gray (2008)). Rather than admit this, however, the movement fashioned for itself a more flattering origin story.

According to this standard account, the history of the calculus is a key battleground on which the anti-intuition attitude proved its superiority. An especially decisive proof of the folly of intuition, the story goes, is the existence of continuous, nowhere differentiable functions:

A curve does not have to have a tangent at every point. It used to be thought, however, that intuition forced us to acknowledge that such a deficiency could occur only at isolated and exceptional points of a curve, never at all points. It was believed that a curve must possess an exact slope, or tangent, if not at every point, at least at an overwhelming majority of them. ... Ampère ... attempted to prove this conclusion. ... It was therefore a great surprise when Weierstrass announced [in 1872] the existence of a curve that lacked a precise slope or tangent at any point. (Hahn, 1933, p. 82)

There are several major problems with this potted history. First of all, Hahn’s diagnosis of Ampère’s error is driven by his ideological commitment to discredit intuition rather than by a serious analysis of the case. For what grounds are there for taking intuition to be the culprit in his failed proof? Ampère himself doesn’t say a word about intuition. Rather he claims that his proof is based on “the most rigorous possible” methods (Ampère, 1806, p. 156). Why not conclude, therefore, that rigorous mathematics is fallible, as opposed to it being intuition’s fault? Of course, if one simply defines, as Hahn and others sometimes seem to do, rigorous mathematics to be exact and true mathematics, and “intuition” to be non-rigorous mathematics, then sure enough it follows that “intuition” is to blame for all errors in mathematics.

But this is a terminological sleight of hand, not a historical conclusion as Hahn would have it.

It is simplistic to claim that, according to intuition, any continuous function must be differentiable almost everywhere. The notion of function or curve involved in Weierstrass's proof is a highly formal one. A more balanced and reasonable reaction to Weierstrass's function would be to conclude that the kinds of curves we have geometric intuitions about does not correspond exactly to the particular formal definition of function assumed in this proof. And if the error lies in assuming these two classes to be equal, then this is hardly an error of intuition, but rather the error of making naive and unwarranted assumptions about formal objects.

This was in fact exactly the reaction of many people at the time. Köpcke, for instance, raised the question: To what kind of curve do our intuition apply? Curves generated by the motion of a point, or boundaries between two regions of the plane? Arguably, only the former are 'intuitable,' yet something like the latter is the notion of continuous function required for Weierstrass's proof (Köpcke, 1887, p. 136). Others made very similar points (Klein, 1894, p. 42; Perron, 1911, p. 204).

This more nuanced and less dogmatic view of the role of intuition helps explain why Hahn's ideology did not triumph until a full sixty years after Weierstrass had supposedly provided the clinching argument for it. To be sure, Hahn's view was not without precursors, even quite numerous ones. Nevertheless, given how supposedly compelling the historical evidence allegedly is, it is remarkable how many leading mathematicians of the late 19th century did not follow the script. This includes many of the major figures famous for their work in formalising mathematics and in particular the calculus.

Weierstrass himself, for one, does not seem to have drawn any anti-intuition conclusions from his own work. In his collected works, I count three mentions of the word intuition and its cognates: one in a general discussion of teaching, where he argues that the best teacher not only announces and justifies results but also makes them intuitive (Weierstrass, 1894, vol. III, p. 321), and two other occurrences where providing an intuitive interpretation of particular results is presented as a positive (Weierstrass, 1894, vol. II, p. 237, vol. IV, p. 346).

Set theory quickly became the language for formal, as opposed to intuitive, definitions and proofs in calculus and beyond. But Cantor, when he introduced the notion, had no intention of banning intuition. On the contrary, he explicitly bases the notion of set upon it: "By a set we understand any collection M of definite and separate objects m of our intuition" (Cantor, 1895, p. 481). Earlier in the century, Dirichlet (1837) had done much the same: when giving his celebrated formal definition of the integral, he explicitly takes area as an intuitively given notion.

Pro-intuition sentiments like these are also the norm in calculus textbooks from the time. Authors state with pride that their treatment is “based on geometrical intuition” (Serret, 1899, pp. iii), and invoke it repeatedly in guiding their exposition (Serret, 1899, pp. 5, 6, 7, 10, 278; Worpitzky, 1880, pp. 752, 754). Gaining “a clear intuition” of the material is explicitly stated as a goal (Bergbohm, 1892–93, pp. I.4, II.97). Giving intuitive meaning to results, beyond their formal content, is highly valued (Lipschitz, 1880, pp. 572, 633; Kiepert & Stegemann, 1897, p. 577; Kiepert & Stegemann, 1894, pp. 12, 320); one example is how the geometrical concept of curvature gives meaning to the quadratic term of a Taylor expansion (Worpitzky, 1880, p. 693). We also regularly find phrases like “as geometrical intuition allows us to recognise easily ...” (Harnack, 1881, p. 64), “by means of this intuition, one easily convinces oneself that ...” (Lipschitz, 1880, p. 664), and so on. In sum, Courant speaks for a long tradition when he says in his famous calculus book that “it is my aim ... to give due credit to intuition as the source of mathematical truth” (Courant, 1927, p. v).

The formal theory of the calculus is of course also acknowledged, but phrases like “this theorem follows already intuitively ..., but can also be proved as follows ...” (Lübsen, 1855, §158), “intuition teaches us the same thing directly” (Worpitzky, 1880, p. 714), or “this corresponds precisely to our intuition” (Kiepert & Stegemann, 1897, p. 578) suggest that intuition and formalism coexist and are both valid. Rather than one being real mathematics and the other only half-baked pseudo-understanding, they are both useful perspectives. For any given situation or purpose, one or both may be suitable for the task at hand. Nobody is saying that intuition can do everything, or that it must be banished from mathematics. It is notable that explicit support for intuition along such lines comes even from some of the pioneers of rigorous real analysis. Dedekind, for example, envisions such a balance:

Resort to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful, from the didactic standpoint, and indeed indispensable if one does not wish to lose too much time. But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny. (Dedekind, 1872, p. 1)

Lipschitz too is famous for his formal analysis work, but in his calculus textbook intuition is by no means shunned. He does temper the role of intuition with the warning that

the geometrical interpretation serves only to make the analytically defined concepts more graspable with the help of intuition, not as foundation for the proofs. (Lipschitz, 1880, p. 502)

But he does not carry this insistence as far as modern textbooks. For instance, he is perfectly happy to consider the notion of area to be intuitively given when defining

the integral (Lipschitz, 1880, p. 102), and in general he is keen to highlight that “the geometrical intuition and its analytic representation correspond to one another” (Lipschitz, 1880, p. 10).

Toeplitz was another leading analyst who saw a positive role for intuition:

The greater number of students do not yet possess the same ability for abstract thinking in their first hour of university lectures, but have, rather, a hunger for intuitive and productive notions. The intuitive path aims to satisfy that hunger. Kiepert-Stegemann, in its earliest editions, is a perfect example of this trend carried out in a pure way; this work must contain a spark of real didactic genius from which it derives its success. (Toeplitz, 1926/2015, p. 298)

In his own semi-historical calculus textbook, Toeplitz does not repeat Hahn’s story of the inevitable demise of intuition. Instead we find phrases like “this is the computational equivalent of the intuitively seen fact that ...” (Toeplitz, 1949, p. 51) which treat intuition as a viable viewpoint that is respectable and indeed often equivalent to formal methods.

Altogether, Hahn’s quasi-historical narrative about the cleansing of intuition is at odds with the historical record. Hahn uses a caricature of history to justify his ideological stance. Time and time again, key mathematicians who by Hahn’s logic should have despised intuition instead give it a respectable place in mathematical thought.

I have used calculus textbooks in particular as indicators of the mathematical community’s attitudes toward intuition. I have done this for two main reasons. Firstly, Hahn’s narrative is based primarily on examples drawn from calculus. Secondly, although there is a vast philosophical literature on intuition—including authors with much affinity to mathematics such as Descartes, Kant, and Brouwer—my concern is not with philosophy but with the working mathematician’s everyday attitude toward intuition. Calculus textbooks, I would argue, is where the rubber hits the road and we see how philosophical commitments play out in actual, hands-on mathematics.

I tried to extend my investigations also to earlier time periods, but in this endeavour I met with limited success. I went through many textbooks and did full-text searches for intuition and cognate terms in various languages, but I found that older texts contain very few explicit mentions of intuition. The vast majority of books never mention intuition at all. This includes the following: Wallis (1656), Leibniz (1678-1714), Newton (1687), Bernoulli (1692), l’Hôpital (1696), Ditton (1706), Berkeley (1734), Reyneau (1736), Reyneau (1738), Deidier (1740), Maclaurin (1742), Simpson (1750), Kästner (1770), Tempelhoff (1770), Lagrange (1797), Lacroix (1802), Neubig (1817), Cauchy (1821), Jephson (1826), Hall (1837), Raabe (1839), Snell (1846), Spencer (1847), Church (1850), Miller (1852), Woolhouse (1852),

Autenheimer (1856), Price (1857), Smyth (1859), Greene (1870), Williamson (1877), Dölz (1878), Todhunter (1881), Byerly (1882), Knox (1884), Bass (1887), Bayma (1889), Kleyer (1889), Jordan (1896), Perry (1897), Czuber (1898), Murray (1898), Lorentz (1900), Thompson (1910), Landau (1934).

Nevertheless one can say something about the earlier period as well. I believe it is safe to say that intuition was held in high regard in the early history of the calculus. Like many other leading figures, Leibniz never mentioned intuition in his mathematical publications. But in his philosophical works he was very positive toward it. For example:

The most perfect knowledge is that which is both adequate and intuitive. (Leibniz, 1969, p. 291) When my mind grasps all the primitive ingredients of a concept at once and distinctly, it possesses an intuitive knowledge. This is very rare, since for the most part human knowledge is ... confused. (Leibniz, 1969, p. 319)

Leibniz surely considered mathematics no exception to these general pro-intuition convictions. Interestingly, I did find an old calculus textbook that expresses the same idea—of intuition as the opposite of confusion—specifically in the context of the calculus: "... then our Knowledge will be ... more intuitive: ... Now we see through a Glass darkly, or in a Riddle; but then Face to Face." (Stewart, 1745, p. 478, echoing Corinthians 13:12) I believe this passage can be taken as quite indicative of attitudes toward intuition in the early calculus generally.

The 18th century, as is well known, saw the calculus turn away from geometry and become heavily focussed on analytic expressions. It is natural that this would be accompanied by a diminished estimation of intuition. Indeed, in Euler's calculus textbooks we do find two down-putting references to intuition: "... which we will be able to prove rigorously ... In the meantime it is not so difficult to see intuitively that this is true." (Euler, 1755, §170) "The truth of these formulas is intuitively clear, but a rigorous proof will be given ..." (Euler, 1748, §166)

But, as Schubring (2005) has observed,

soon after 1800, however, the pendulum moved the other way ... The defamed synthetic method was restored as dominant value, and the requirement that concepts be generalizable was replaced by that of their being of easy intuitive grasp. (p. 152)

A clear expression of this revival of intuition is found in Carnot (1813), who explicitly argues that intuition is favourable to algebraic analysis:

Far from using analysis to establish elementary truths, we must disengage them from all that prevents us from perceiving them as distinctly as possible. ... Those who succeed in making us see almost intuitively the results to which

we had only arrived before them by the aid of a complicated analysis, do they not always procure us as much pleasure as surprise ...? (§161)

Thus the low tide of intuition in the era of Eulerian analysis soon gave way to “a new dominance of geometry, in the name of intuitiveness” (Schubring, 2005, p. 295).

In conclusion, intuition was largely held in high regard by leading practitioners of the calculus for nearly 200 years. It suffered a temporary dip in fortunes in the 18th century, but this was due to no fault of its own but rather to a focus on the programmatic algebraisation of mathematics. Its stock again plummeted in the first half of the 20th century. Perhaps the reasons were much the same this time, namely a zeal for a tout-court programmatic reform of mathematics along symbolic and analytic lines. The leaders of this movement, however, succeeded in portraying the demise of intuition not as their ideological doing but as an inevitable historical conclusion objectively forced upon us all by factual mathematical developments. It has been my primary goal in this essay to challenge this ingrained narrative. If history is any indication, the time may well be ripe for intuition to bounce back once again like it did two hundred years ago.

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Daily life traits in arithmetic word problems: a glance at 1950s school notebooks

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Abstract

The paper focuses on the representations of everyday life featured in arithmetic word problems studied in elementary schools in the 1950s in the state of Rio Grande do Sul, in the south of Brazil. The official guidelines at that time recommended the study of daily life problems, elaborated by the teacher or the students, that evoked situations similar to those experienced or devised by the children. The analysis of the notebooks of two fourth graders in the 1950s shows partial compliance with the guidelines. Math word problems that were to connect students to everyday activities in local contexts reveal the teachers' intentions to reconcile preparation for practical life with the exercise of arithmetic operations studied in class, but these are interspersed with improbable scenarios and wordings referencing artificial contexts. Word problems featuring "daily life" situations that children might experience are few; in the text of most word problems, the teachers turn to a collection of issues common to the school tradition, introducing in them local traits and colors with the purpose of lending them credibility and familiarity. In both sets of notebooks, the arithmetic word problems mimic real life but are formulated and solved within the confines of the school itself; in that sense, they can be considered school creations.

Keywords: history of mathematics education, school notebooks, elementary school, word problems

Introduction

In the 1920s and 1930s in different regions of Brazil, elementary school programs were reformed, resonating ideas of the Progressive School, referred to in Brazil as Escola Nova¹. Mathematics and arithmetic word problems, in particular, occupied a prominent place in these programs, allegedly to prepare pupils for real life situations (Carvalho, Silva, Sant'Ana, Fernandes, & Santana, 2016; Almeida & Leme da Silva, 2014).

In the southern state of Rio Grande do Sul, new programs were enacted in 1939. According to their guidelines, the then usual practice of solving lists of problems of the same type, drawn from textbooks, should be abandoned. Instead, the teacher or the pupils should formulate arithmetic problems similar to or extracted from everyday life. The situations evoked were to be familiar to the children. Therefore it was necessary to consider the students' schools, their out-of-school daily lives,

¹ See Vidal (2013). Following this designation, we will use the adjective *escolanovista* to refer to the ideas propagated by this broad educational chain.

their ages, the urban or rural environment of the school location, traces of the local culture, and other aspects. Thus it was expected that teachers would have autonomy and take the initiative in the selection of themes and situations, in the choice of data and writing, or in the revision of arithmetic word problem statements.

How were these guidelines considered, interpreted, carried out, or even disregarded by teachers? How were the problems presented to the pupils or which problems did they themselves formulate? What were the contexts their statements dealt with, or how was everyday life represented in these problems?

As Viñao (2008) argues, access to students' schoolbooks can give us clues about the curriculum practiced in the classroom. For example, notebooks can be elucidative with regards to the frequency with which a certain type of activity is performed or the language or examples evoked by the teacher in addressing a particular topic of the program.

The use of school notebooks as a source, however, is not a widespread practice among researchers in the History of Mathematics Education. One reason for this is the difficulty of accessing collections of notebooks that can be considered representative of teaching in a given level, time and region. Another possible explanation is that few surveys have focused on mathematics teaching practices (Ackerberg-Hastings, 2014).

Reflecting on studies carried out in Brazil, Leme da Silva and Valente (2009) argue that students' notebooks are very rich documents. However, the effort to set up shareable collections of notebooks is incipient. Rios, Búrigo, Fischer, and Valente (2017) present an overview of ongoing research using as a source the digital collection² organized by GHEMAT (History of Mathematics Education Research Group in Brazil).

In this article, we take as sources two notebooks belonging to that collection, used by pupils in the fourth grade of elementary school in the 1950s. We present considerations about the approach of arithmetic word problems in the classes these pupils attended, confronting notebooks' excerpts with the official prescriptions about the subject.

The official prescriptions on arithmetical problems

In Brazil in the 1930s, the prevailing autonomy of the states gave way to the centralization of power by the Vargas federal government, aggravated by the establishment in 1937 of the authoritarian regime known as *Estado Novo*. In this timeframe, several policies for the modernization of culture and education were implemented,

2 The collection is available online at <https://repositorio.ufsc.br/handle/123456789/160300>.

including the creation of the National Institute of Pedagogical Studies (INEP) in 1938. One of the roles of INEP was “to provide technical assistance to state, municipal, and private education” (Decree n. 580 of 1938).

The new curriculum of Rio Grande do Sul was prepared in the midst of this centralizing trend. Its wording fell to a committee of teachers and graduates of the Normal School of Porto Alegre, the leading state institution dedicated to the training of primary school teachers; Lourenço Filho, then president of INEP, revised the text³.

Decree n. 8020 of 1939 established the programs which were valid for all, mandatory even for schools maintained by communities of descendants of German and Italian immigrants who, until then, had enjoyed curricular autonomy. In addition to the contents that should be studied in each grade of the primary course, the Decree stipulated the purposes and guidelines for the study of each discipline (Decreto n. 8020, 1939/1957).

In the mathematics section, the item “arithmetic problems” appeared at the end of the list of contents for each grade, as an activity that should involve the use of other knowledge (numbers, operations, measures).

The guidelines for the study of arithmetic word problems reiterated recommendations that these problems should be “formulated by the teacher or the student” (Decreto n. 8020, 1939, p. 85), taking advantage of “situations arising in the life of the student or the class” (Ibid., p. 96), and

[...] arranged by the teacher to present the mathematical facts seized in the forms capable of occurring more frequently in life; it is advisable to get the students to identify themselves with the characters presented in the problem⁴ (Decreto n. 8.020, p. 96).

The writings were to evoke or replicate real life, involving

data taken from the child’s experience in the environment that surrounds him/her: spending on meals, clothing, transportation, school supplies, etc., using price lists organized or collected by students, advertisements, etc. (Ibid., p. 85).

The problems could be inspired by another subject’s contents, games or occasional activities, such as excursions, visits, or projects. But with the situations evoked being quite varied, it was expected that the pupil would

3 The program was written by Graciema Pacheco, Maria Fialho Pereira, and Marieta da Cunha e Silva, and reviewed by Olga Acauan Geyer (Novo programa de ensino..., 1939).

4 All translations from Portuguese to English are the responsibility of the author. The originals can be consulted as they are available online.

[...] express his thoughts with clarity, order, and neatness, though not sacrificing these in favor of the most essential quality of correct and quick responses (requiring, for example, that the pupil should record all the calculations, even those done mentally or applying a systematic analysis of all the problems) (Ibid., p. 107).

Traces of ‘escolanovista’ or Progressive School thought are clear in these guidelines that value, on one hand, previous experiences and pupils’ involvement and, on the other, the acquisition of correct, fast, and effective reasoning. In particular, it is possible to identify resonances of Edward Lee Thorndike’s writings and, more precisely, those of his manual *The New Methodology of Arithmetic*, published in Portuguese in 1936 by Editora Globo in Porto Alegre, capital of the state of Rio Grande do Sul.

In this manual, Thorndike argues that, concerning arithmetic problems in primary school,

[...] every problem should preferably (1) address situations that are likely to occur in real life many times; (2) be treated in the way they would be in practical life; (3) be presented in a way neither much more difficult nor much easier to understand than they would be if presented to the pupil’s senses in reality; (4) awaken, to a certain extent, the same degree of interest that follows the resolution of the problems encountered in the actual course of their activities (Thorndike, 1936, p. 153).

Nevertheless, it is widely known that an emphasis on approaching problems similar to or drawn from reality is not exclusive to Thorndike’s work. For instance, concurrent appeals of the same type were frequent in the official French guidelines (D’Enfert, 2011).

The guidelines disseminated in the official pedagogical journal

Soon after the new programs were determined, an apparatus was set up in the state of Rio Grande do Sul to “monitor the progress of work in educational establishments” (Peres, 2000, p. 129). This apparatus was coordinated by the Educational Research and Guidance Center (CPOE), an agency of the Department of Education created in 1942, which brought together so-called education technicians, primary or normal teachers who had undergone special training in education (Quadros, 2006).

The CPOE took on the roles of production, dissemination, and evaluation of the implementation of the official guidelines for primary education (Quadros, 2006). One of the instruments for disseminating these guidelines was the pedagogical journal *Revista do Ensino*, which, after a nine-year interval, had its publication resumed in 1951. Its yearly eight journals were widely distributed annually to elementary schools of the state (Bastos, 1997).

The first issue of the journal, published in 1951, already contained an article dealing with arithmetic word problems written by a CPOE member, Suelly Aveline (1951). Consistent with the precept that arithmetic problems should take advantage of “situations arising in the life of the student or the class” as established by Decree n. 8020 (p. 96), the article presents suggestions of problems for each elementary grade related to seed and tree seedling planting, activities to be carried out by the pupils.

In 1954, the newspaper published an article by Sydia Sant’Ana Bopp, also a member of the CPOE team. Unlike the previous article, which included proposed activities, this one presented theoretical considerations and general orientations on the approach of arithmetic problems in school with questions such as “Why do our children find so much difficulty in the learning of mathematics? What should we do to help facilitate the child’s reasoning, ultimately leading him or her to solving problems?” (Bopp, 1954, p. 6).

In the same vein, the author considered that one of the main explanations for children’s difficulties in their resolution was the presentation of

[...] unreal problems, those external to childhood and that in no way can be experienced by the child. If the reasoning is based on observations, how are we going to offer the children problems that do not fit their experiences? (Ibid., p. 6).

From this evaluation, the author reiterates and details the Decree guidelines, identifying situations and contexts that could inspire the formulation of a problem’s wording:

All the activities at school and in the private life of the child offer opportunities for problems: attendance, lunch, school accounting, mail, bank, library, class notes, etc. Problems about school lunch, the price of food, with the possibility of making comparisons with older tables; problems with the price of paper, books, etc., problems with Brazilian import and export trade, historical dates, tram ticket expenses, health care, teeth treatment, etc. (Bopp, 1954, p. 7).

We see that Bopp’s (1954) article conveys a fairly elastic interpretation of what would be “child-experienced” problems, including references to the bank and typical school subjects such as historical dates. Furthermore, references to money, costs, and prices dominate the above-mentioned contexts. In this respect, the guidelines that were announced as innovative seem to replicate those traditionally present in textbooks.

Other articles dealing with the topic continued to be disseminated throughout the decade and would reiterate the diagnosis of the difficulty and the precepts of

the reliability of the word problem statements and the children's familiarity with the situations evoked (Carvalho *et al.*, 2016).

The CPOE's reiteration of these evaluations and precepts suggests that the team blamed the teachers and their practices for the pupils' difficulties with solving arithmetic word problems.

In a visit to Rio Grande do Sul in the 1950s for a study commissioned by INEP, Roberto Moreira (1955) found, based on interviews and classroom observations, that there was "no correspondence between the guidelines for the implementation of school programs and actual practice within the classroom" (Ibid., p. 129). The practice of active methods was, according to the author, infrequent, as the teachers emphasized memorization and the mechanisms that would be tested in the exams to which the pupils were subjected at the end of each year.

The school notebooks of Juvenal and Gladis

Since the 1990s, school notebooks have been treated by educational historians as a source and as a research object. To the topic addressed here, it should be noted that school notebooks are rich in evidence of past school practices.

As Viñao (2008) warns us, it is not a question of taking the notebooks as portraits of the classes they witness: they do not contain records of gestures, oral communication (with the exception of dictations), games, dramatizations; according to the use and purpose assigned to the notebook, it may contain a somewhat accurate record of what the teacher dictated or wrote on the chalkboard and of the students' written activities. In addition, it should be noted that notebooks are not only products but also producers of school culture: the use of notebooks as a support for durable records (different from, for example, scribbles on the blackboard) enable and favor certain practices; learning to use the notebook or producing certain writings becomes an additional purpose of the school.

Chartier (2007) also observes the risks of anachronism in the analysis of notebooks because when we find records of activities that seem familiar to us, we tend to infer that they were performed as they appear to us in the memories we keep from our own school life, just as we tend to perceive activities that were not common when we were students or teachers to be archaic.

It would be necessary to analyze a large number of notebooks to infer from them about the school culture at a given time and place. In this paper, we deal with only two students' notebooks that were used in the fourth grade of primary education by their authors in the mid-1950s. This approach is exploratory: from the

arithmetic problems found in these notebooks, we intend to raise some reflections to be considered in more extensive investigations.

The first notebook belongs to Juvenal Rosa Nunes and was used in 1954 when he studied in the Grupo Escolar Ramiz Galvão; the others comprise a set of four notebooks belonging to Gladis Renate Wiener, used in 1956, while she studied at the Colégio Farroupilha.

Grupo Escolar Ramiz Galvão was a public school of the state network located on the outskirts of the city of Rio Pardo, in the interior of Rio Grande do Sul. The municipality, located in a region dedicated to cattle raising and rice cultivation, had only 40,000 inhabitants in 1950 (Fundação de Economia e Estatística, 1981). The school was inaugurated in 1938 as part of a major commemoration of the 50th anniversary of the abolition of slavery in Brazil.

Colégio Farroupilha was a private school, founded in 1886 by a charity and maintained by entrepreneurs who were descendants of German immigrants. It was located in the city center of Porto Alegre, the main industrial and commercial center of the state with about 400,000 inhabitants in 1950 (Fundação de Economia e Estatística, 1981).

The difference between the literacy rates in the two populations reveals the disparity in schooling processes at a time when the school was strongly associated with urbanization in Brazil. In 1950 in Porto Alegre, 80% of people aged five years and over stated that they could read and write; in Rio Pardo, the index was only 48%.⁵ However, schooling rates were rising: Juvenal recalls that, in the Grupo Escolar, the classes were offered in three condensed periods to meet enrollment applications.

The teacher of Juvenal's class was a young woman named Marina Celeste da Cruz⁶. At that time, the city of Rio Pardo had a first cycle Normal School, which offered the equivalent of a middle school education, but with a teacher training component; but since the grupos escolares welcomed the most qualified teachers, it is probable that Marina held a higher diploma.

Gladis' teacher was Irene Marta Fischer Petrick, a graduate of Colégio São José, a Catholic institution run by nuns of German origin, which offered teacher training equivalent to first cycle Normal School. Irene began teaching at Colégio Farroupilha in 1938 and was an experienced teacher in the 1950s (Jacques, 2015). Her surname

5 *Grupos escolares* offered graded schooling provided by graduated teachers. The expression *grupos escolares* can be roughly translated as "school grouping". In fact, the first schools named as *grupos escolares* came out of the assembly of former so-called "isolated schools".

6 According to the testimony given to Nicolas Giovanni da Rosa in January, 2017, and data collected from the school's website: <<http://escolaramizgalvao.blogspot.com.br>>.

and background suggest that her entry and tenure in the school were in some way related to her belonging to and identifying with the Teuto-Brazilian community.

Juvenal reports that he entered primary school at age thirteen; before that, he lived in a rural location that had no schools. He was probably sixteen when he entered fourth grade. He attended the classes of the second period, which began at eleven o'clock in the morning, and he worked in the morning and in the afternoon, that is before and after school. Juvenal also reports that he left school "before finishing fourth grade, because of work; I had to work⁷."

Gladis was ten when she attended the fourth grade of Colégio Farroupilha. Her parents were Jewish German immigrants who arrived in Porto Alegre in 1937; they were trained podiatrists and owners of a company manufacturing orthopedic products. After primary school, Gladis attended secondary school at the same institution; she studied mathematics and later became a university professor.

For the working class, the class to which Juvenal belonged, the only accessible school was the primary school; in fact, most students in the state network never progressed beyond the first grade, and not even a third of them completed the fourth grade (Cunha, 1980). The children of the middle or upper classes, in turn, were encouraged to pursue studies, take competitive exams to enter public secondary schools, or attend private schools, as was the case with Gladis.

In both schools, teachers were required to comply with the state program of 1939, which was still in force. For the fourth grade, the program prescribed the study of these items: multiples, divisors, and divisibility rules; comparison and simplification of fractions, reduction of fractions to common denominator; decimal fractions, multiplication and division of decimals; metric system; notion of surface and area; perimeters of quadrilaterals and triangles; problems, including "more detailed oral analysis and writing" (Decreto n. 8020, p. 107).

But despite the curriculum being the same, the settings of the two schools were quite diverse, as was the composition of the student bodies. Considering these differences, it can be assumed that the objectives Juvenal's and Gladis' primary teachers assigned to the teaching of mathematics were also diverse.

The uniformity of contents and the dissimilarity of the contexts make the examination of the Juvenal and Gladis notebooks particularly interesting. We presume that the notebooks can give us clues about how the prescribed curriculum was interpreted or how it was fulfilled by the two teachers. Considering the practices in the primary school of Rio Grande do Sul at that time, we can assume that the same

7 According to the testimony given to Nicolas Giovani da Rosa in January, 2017.

tasks were assigned to all students in each class, with each student performing them individually using his or her notebook.

It is necessary to remark that we know little about the autonomy of teachers in each of the schools or the degree of control exercised by the administration, parents, or senior supervisory bodies. We also know little about the sources of inspiration or didactic materials to which the two teachers had access. We do not have access to their class diaries or autobiographical documents. So the focus is not on analyzing the planning processes of their classes, but rather on considering what the notebooks tell us about the tasks assigned to students.

The place of arithmetic problems in the notebooks

Viñao (2008) notes that schoolbooks can take many different forms according to the uses attributed to them.

The notebook Juvenal kept as a memento of school days was entitled “my diary.” The first records are from March 15, 1954, and continue for 6 months until September 29, possibly the date he left school. The notebook contains records of various subjects. He explains:

We wrote each subject in one notebook; in another notebook, then at night at home, we wrote it out to the diary. We transcribed it⁸. Each one had his own diary.

Juvenal’s diary, therefore, presents a revised and probably summarized version of classroom activities. Mathematics activities occupy an important place in the notebook because, according to him, “what she taught the most was mathematics, which we needed more, for in mathematics one has to show how to do it, how to set up the calculation.”

In the fourth grade, Gladis used four notebooks entitled “Arithmetic,” which comprised two sets; the presence of definitions and explanations in two of these indicates they were used to record class lessons, and the other two-notebook set was used for homework. The first records are from March 5, 1956, and the last is from December 4 of the same year.

The mathematics portion of Juvenal’s notebook and Gladis’ notebooks contain varied definitions, rules, types of exercises, and other tasks. We classify the activities that involve the recording and interpretations of verbal utterances containing numerical data, and the production of an answer obtained from calculations involving

⁸ In fact, Juvenal uses the expression “passar a limpo”, which could be translated as “writing a neat and revised copy of”.

those data as arithmetic problems. According to this definition, we include in the “arithmetic problems” category not only statements that mention extra-school contexts, but also those involving questions about arithmetic operations, geometric figures, or themes of other disciplines.

In the discussion that follows we focus on the text or the wording of arithmetic problems, and from them, we discuss the place of those problems in school activities.

First, we note that arithmetic word problems are common activities in Juvenal’s and Gladis’ classes. In Juvenal’s notebook, 81 problems are recorded over a period of 26 weeks; in Gladis’, 90 problems, over a period of 37 weeks. In Gladis’ notebooks, the statements are followed by detailed resolutions; in Juvenal’s notebook, the answers are, in some cases, accompanied by calculation algorithms.

It is possible to note that the inclusion of different types of numbers and units of measure in word problem statements correspond largely to the order in which they are presented and mobilized in other tasks.

In Juvenal’s and Gladis’ notebook recordings in March and April, nearly all problems involve whole numbers exclusively - including cases of monetary values represented by two zeroed decimal places.

In Juvenal’s notebook, the topic “ordinary fractions” appears at the end of April and the first problem involving fractions (a half and a quarter of certain amounts) appears in early May. The first computation algorithms involving decimal numbers appear at the end of April, and the problems involving decimal numbers appear most frequently beginning mid-May.

In Gladis’ notebooks, the topic “decimal numbers” appears in mid-June; the first problems involving decimals appear at the end of the month, and they are used frequently beginning in September. The metric system is introduced in mid-September with problems involving measures of length, volume or weight; hence decimals appear then most frequently.

The logic that organizes the approach of problems according to the types of numbers used in statements is interrupted here and there by commemorations of patriotic dates in compliance with the prescribed interaction between the study of mathematics and history. In Gladis’ notebooks, ten (10) problems are related to patriotic dates that recall the death of Tiradentes (considered a martyr of the struggle for the country’s independence), the arrival of German immigrants to Brazil, the birth of Santos Dumont (considered the patron of Brazilian aviation), and the proclamation of the Brazilian Republic. In these problems, only whole numbers appear.

Daily life in school arithmetic problems

Considering the precept that word problems should be plausible and have the possibility to be “experienced by the children,” what contexts do they evoke? What were the situations that the teachers considered worthy of being portrayed in these problems?

In both Juvenal’s and Gladis’ cases, most problems involve calculations with money - purchase and sale, salary, debt or savings - or measures of length (including distance, height), area, volume, capacity or weight expressed in units of the metric system. In Juvenal’s notebook, 70 of the 81 problems involve money or measures of length, area, volume or weight; in Gladis’ notebook, 57 of the 90 problems fall into these groups. Table 1 presents the distribution of problems according to this initial classification.

Table 1. Distribution of arithmetic problems in notebooks

	Juvenal’s notebook	Gladis’ notebook
Problems involving money, but not involving measures of length, area, volume or weight	46	18
Problems involving money and measures of length, area, volume or weight	3	18
Problems involving measures of length, area, volume or weight, but not involving money.	21	21
Problems that do not involve money or length, area, volume or weight	11	33
Total number of arithmetic problems	81	90

Among the problems that do not involve money, an important portion deals exclusively with Arithmetic or Geometry contents, with no reference to extra-school contexts. Table 2 presents the distribution of problems not involving money according to their connection with school contents, considering patriotic dates as school subjects.

Table 2. Distribution of problems not involving money

	Juvenal’s notebook	Gladis’ notebook
Problems involving only Arithmetic contents	0	10
Problems involving only Geometry contents	12	0
Problems involving patriotic or historical dates	3	10
Problems involving extra-school contexts	17	34
Total number of arithmetic problems not involving money	32	54

Problems involving money can, in turn, be classified into large groups: buying or selling food; purchasing items of clothing or materials necessary for their manufacture, such as fabric, buttons, thread, wool, ribbon; purchase of school supplies; rides, parties, and toys; receiving wages; variation of savings or debt; giving to charity or sharing of money. In Juvenal's notebook, we also find six (6) statements that we consider related to rural life, involving, for example, the purchase of wire for a fence, the purchase of a tractor or the sale of firewood obtained by the cutting of trees. Table 3 presents the distribution of problems involving money according to these groups.

Table 3. Distribution of problems involving money

	Juvenal's notebook	Gladis' notebook
Problems involving buying or selling food	11	3
Problems involving items or materials for clothing	9	14
Problems involving purchase of school supplies	5	1
Problems involving wages	3	2
Problems involving variation of savings or debt	4	2
Problems involving charity or sharing of money	3	2
Problems involving rides, parties or toys	5	5
Problems related to rural life	6	-
Other problems	3	7
Total number of arithmetic problems involving money	49	36

Most word problems involving monetary values use simple texts, from two to six lines. In most cases the problem involves a question of calculating change, profit, or the price of an amount of a certain product (beans, butter, milk, wine, cloth, wool, and others), given the price of another known amount, this is, under the assumption of proportionality. In the notebooks of Juvenal and Gladis, it is observed that the resolutions - certainly oriented or expected by the teachers - apply the strategy of calculating the unit value. For example, in the problem "7 meters of calico cost Cr\$ 84.00. How much is $\frac{1}{2}$ meter?," Gladis solved the problem in her notebook by calculating the price of one meter, and then the price of a $\frac{1}{2}$ meter of fabric.

As for problems not involving monetary amounts, the largest share contains measurements of length, area, volume, capacity or weight. As shown in table 1, these number 21 in each of the groups of notebooks.

The concern to reference situations likely to be "lived" by the children, as determined by the program, is evident in the word problems dealing with toys, parties, outings or school supplies.

In Juvenal's notebook, there are five (5) problems related to the traditional and popular feast of São João⁹, including the purchase of food, ornaments and various types of fireworks. The utterances of these problems are extensive; one of them holds two halves of pages including a price list, as illustrated in Figure 1.

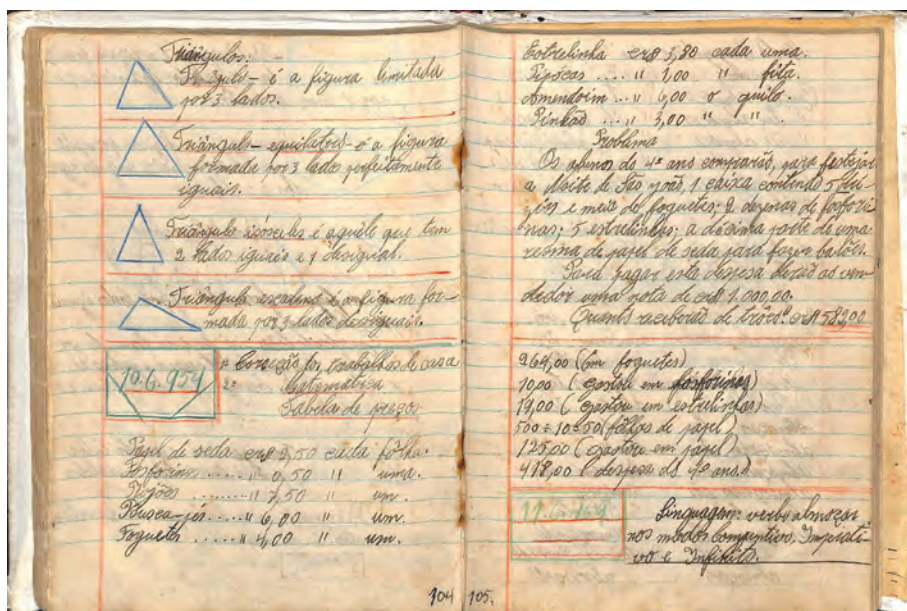


Fig. 1. Juvenal Nunes' notebook, fourth grade, 1954 (pp. 104-105).

In the following statement, one identifies the teacher's intention that the students identify themselves with the "characters of the problem," as the program recommended: "The students of the fourth grade will buy, to celebrate the Night of São João, one box containing 5 dozen and a half rockets; 2 dozen of fosforinas; 5 estrelinhas; a tenth of a ream of tissue paper to make balloons. To pay this expense, they will give the seller a note of Cr\$ 1,000.00. How much will they receive for the change?" In re-reading the notebook, Juvenal recalled the fireworks: fosforina was "a little bomb, like a thicker match, with gunpowder, which one would light in the matchbox and throw away; it was not too noisy." And estrelinha he described as "a stick that sparkled little stars, like sparks"¹⁰.

The richness of details, which coincide with Juvenal's memories, suggests that the problem was drawn up by the teacher, perhaps even with the participation of the students, seeking to explore an activity familiar to them.

9 Celebrated on June 24 around a bonfire, with songs and customs of Portuguese origin.

10 According to the testimony given to Nicolas Giovanni da Rosa in January, 2017.

In one of Gladis' notebooks, we find four (4) statements that reference a tour of the school pupils to the newly opened Railroad of Dreams. The numbers extol the greatness of the work, with no concern for presenting real data: "The engineers who designed the Railroad of Dreams needed 15,000 hours to complete that marvel. How many days were needed to accomplish this feat, calculating that they worked eight hours a day?"

Some statements try to combine the celebration of historical dates with references to the universe of children: "Grandpa says he was seven years old when the Brazilian Republic was proclaimed. How old is Grandpa now?"

In both groups of notebooks, we find references to the local context. In Juvenal's notebook, there are nine (9) problems that refer to activities practiced on small farms, which were common in his school environment: growing vegetables, poultry, logging, and cultivation of fruit trees or selling fruit common in the region - papayas, bananas, avocados, oranges, apples, pears, and quinces. In Gladis' notebook, we find references to names of banks and stores in Porto Alegre, the tram, and the urban bus: "For a Vila Jardim bus that makes 8 round trips to the market and carries 42 passengers each time, what revenue should be generated each day, if the fare is Cr\$ 3.50?"

The frequency of buying/selling word problems, references to local contexts, historical dates, and school activities as shown in Tables 1, 2 and 3 correspond to the guidelines published in the aforementioned Bopp's article (1954). From such connections, however, one can not conclude that statements originated from "situations arising in the life of the pupil or the class," as it was prescribed then by the current program (Decreto n. 8020, 1939, p. 85).

Some of the statements involve data implausible in real life, such as a problem that asks for the price of 1 centimeter of fabric in Gladis' notebook, and a statement in Juvenal's notebook that reports a purchase of 15 kilograms of sugar in the grocery store by a girl. Other statements present situations that do not constitute real-life problems, such as calculating the sum of teacher and pupil weights in Gladis' notebook, or the difference between the height of two fences in Juvenal's notebook. Furthermore, extra-school contexts largely refer to activities of adult life, such as shopping, work, saving.

Final considerations

As pointed earlier, our analysis of Juvenal's and Gladis' notebooks presupposes the practice of simultaneous teaching, in which a teacher addresses all students, teaching the same contents and proposing the same tasks. It also presumes the regular and daily use of the notebook for recording the pupil's writing activities. Under these

assumptions, which are historically dated (Ackerberg-Hastings, 2014), the study of the two notebooks shows a partial compliance with the official guidelines on the approach to arithmetic word problems.

The ordering of the problems according to the numerical data involved and the analysis of the statements indicate that the situations evoked are mainly those that can be conveniently expressed according to the known numerical universe and the notions already studied in class. To do so, teachers made use of a series of problems common to the school tradition, introducing in them local traits and colors, with the purpose of lending them verisimilitude and familiarity.

On the one hand, they are not merely exercises copied from books. On the other hand, they don't arise from pupils' experiences and don't require solutions to be carried out. In the two sets of notebooks of the 1950s, arithmetic problems mimic real life but are formulated and resolved within the school.

The observations on the two sets of notebooks can't be extended to other classes and schools of the same time and region. But their fidelity to a portion of the official guidelines and their correspondence with the findings of Moreira (1955) suggest that they are not isolated examples of contradictory practices.

Are such contradictions intrinsic to official guidelines? Are they related to the training of the teachers? Is it the school effectiveness criteria which imposes this chasm between real life situations and math word problems in the 1950s?

The verification of such hypotheses requires further investigations, including the access to more representative collections of notebooks and the crossing with other types of sources. What can be concluded in any case from the two notebooks is the fecundity of these sources in the identification of singularities and traits common to classroom practices in a given place and time.

Whereas similar guidelines have been introduced in the curricula of other countries such as the United States and France, at the same time, it would be interesting, also, to examine representations of daily life activities and school culture featured in arithmetic word problems in these different contexts through dialogue among researchers.

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Mathematical reasoning in trigonometric definitions, proofs, and calculations on early 19th century textbooks from Norway and Denmark

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Abstract

This paper presents the reasoning behind some definitions, proofs, and calculations from a textbook in trigonometry from 1834 written by Bernt Michael Holmboe from Norway and compares his reasoning with a Danish textbook in trigonometry written by Christian Ramus in 1837. There is focus on the understanding of trigonometric functions and magnitudes, the use of circle arcs and trigonometric lines, and the proofs of trigonometric statements. There are examples demonstrating various definitions, constructions, and calculations of trigonometric magnitudes.

Keywords: history of mathematics; Learned Schools; textbook; trigonometry; mathematical reasoning

Introduction

Bernt Michael Holmboe (1795–1850) wrote most of the textbooks in mathematics that were used in the learned schools in Norway between 1825 and 1860, and he was a very influential person in the development of school mathematics in this period. He wrote textbooks in arithmetic, geometry and stereometry, trigonometry and higher mathematics. Most of Holmboe's textbooks were published in several editions, but the textbook in plane and spherical trigonometry was his only textbook in basic mathematics that was published in one edition only, in 1834. As a young man, Holmboe became the teacher of Niels Henrik Abel, and he is today maybe most known to be the teacher that discovered Abel's genius, and became his first benefactor.

My focus is textbooks written for use in the learned schools. In Norway the pupils started at the age of 9–10 years, and the duration was normally eight years consisting of four two-year grades. The learned schools gave a classic education, meant to qualify for the university. The university in Christiania (Oslo) was established in 1811 and came in function in 1813. The only use of mathematics in the beginning was for the 'examen philologico-philosophicum' – a preparatory exam

for other subjects at the university. The lectures in mathematics were on algebra, geometry, trigonometry, stereometry, and later applied mathematics.

I will in this paper describe the mathematical reasoning in Holmboe's textbook in trigonometry (Holmboe, 1834) and, in some cases, compare with the mathematical reasoning in a Danish textbook in trigonometry (Ramus, 1837), written by a contemporary mathematician Christian Ramus (1806–1856). Both these textbooks are nearly 200 years old, and it is important to have in mind that mathematical conventions have changed during the period that has elapsed since they were published. I will here focus on the use of trigonometric objects and concepts. The aim of this paper is to describe some methods used by these two authors, and not to give a complete overview of the trigonometric textbooks in Norway and Denmark. Holmboe wrote the first textbooks in mathematics to be used in the learned schools in Norway, while Denmark has a longer tradition with textbooks in mathematics.

There is of course no absolutely correct way to present a proof, but whoever the reader is, the proof should present the logic of the proof as clearly as possible. It is therefore an interesting question if that is done best using an easily understandable algebraic proof or using a rather complicated geometric proof. What type of proof is best suited to answer the reader's question 'Why?'

Research on the textbooks by Holmboe may also be found in Christiansen (2009; 2010; 2012a; b; 2015a; b). All translations from Norwegian and Danish-Norwegian to English are made by me.

The textbooks

Holmboe starts by defining 'trigonometric lines' (trigonometriske Linier) to an arc, and with this definition, lines in spherical geometry also have trigonometric lines. Trigonometric values calculated by ratios of sides in right angled triangles are results, deduced from the definitions, and presented in his textbook as a theorem with proof. This theorem will be presented in the paper, but I will start by showing the construction of all the trigonometric lines. I will present the laws for sine and cosine of sums and differences of angles with proofs, and I will show the proof of the law of cosines. Furthermore, I will demonstrate Holmboe's method for calculating trigonometric values with required accuracy, with a specified number of correct decimals.

In some cases, Ramus had a completely different approach to proving trigonometric theorems that might seem more complicated to the pupils, but one may argue that Ramus' methods are truer to the discipline. Christian Ramus was professor in mathematics in Denmark and wrote several textbooks in mathematics between 1837 and 1855, the first one being his textbook in trigonometry from 1837. A Danish

author encyclopaedia from 1899 states that he had great influence on the progress of mathematics in Denmark. (Dansk biografisk Lexicon, 1899)

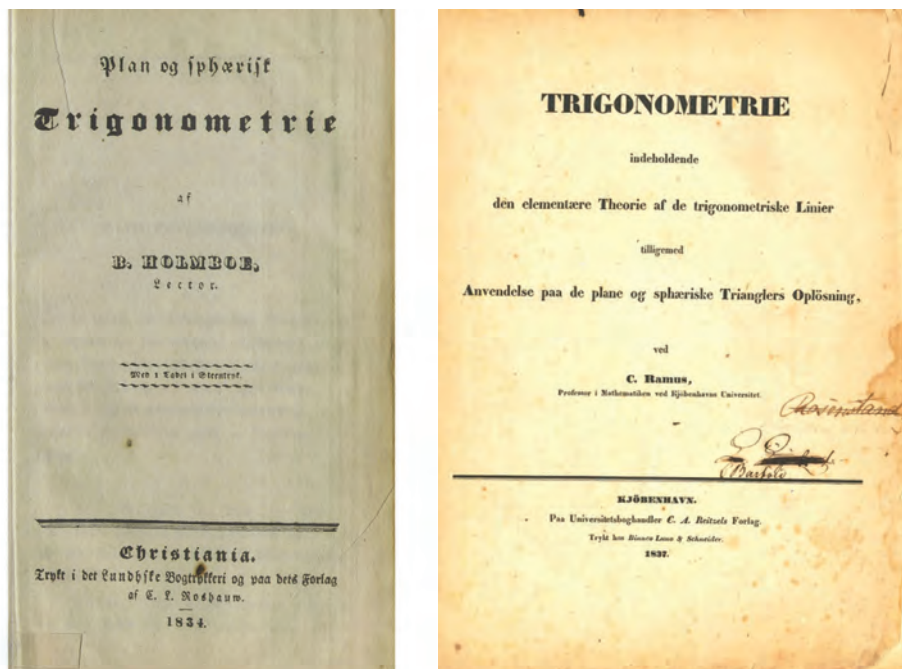


Fig. 1 Holmboe's and Ramus' textbooks in trigonometry

I will show that in some cases their ways of reasoning were quite similar, but in other cases, their ways of reasoning were very different. I will give two examples in detail, their proofs for the law of cosine, and for the sine and cosine of sums and differences of angles. Some of the proofs are purely algebraic, but we will see that Ramus uses a geometric reasoning in one of the proofs.

Trigonometric lines

Holmboe defines plane trigonometry (Holmboe, 1834, p. 3) to be that part of geometry that contains solutions for the following assignments: there are six magnitudes in a triangle, three sides and three angles, and when a necessary and sufficient number of these six magnitudes are known to unambiguously define the triangle. The task is to find the remaining magnitudes. There is a similar definition where Holmboe defines spherical trigonometry (Holmboe, 1834, p. 45) to be the task to unambiguously define a spherical triangle.

The dependencies between sides and angles in a triangle, or in general between straight lines and angles, or the circular arcs that measure the angles, may be expressed by certain lines, called trigonometric lines. (Holmboe, 1834, p. 3).

These lines are geometric objects shown on the constructions in figures 2 and 3, and the numerical values of their lengths are what a modern reader understand by sines, cosines, etc. With a current understanding of straight lines, we would say that the trigonometric lines were ‘line segments’. It was, however, not uncommon in the 19th century and earlier to use the word ‘line’ (Linie or linje) for a line segment, and we will continue that tradition for the rest of this paper.

There is also another important definition that implies that Holmboe is assuming Euclidean geometry. §2 defines that if the sum of two angles equals 90° , they are called complementary, and if the sum of two angles equals 180° , they are called supplementary. A corollary to §2 states that each of the acute angles in a right-angled triangle is the complement of the other acute angle, and that each of the angles in a right-angled triangle is the supplement of the sum of the other two angles (Holmboe, 1834, pp. 3–4). This corollary refers to §44, with corollaries 1 and 2, in his textbook in geometry (Holmboe, 1827, pp. 55–56).

Figure 2 is presented in the textbook (Holmboe, 1834, Final page, Figure 1). It is interesting to note that in Holmboe’s figure, quadrant 1 is ACH , quadrant 2 is HCK , and so on – mirrored from what we use today.

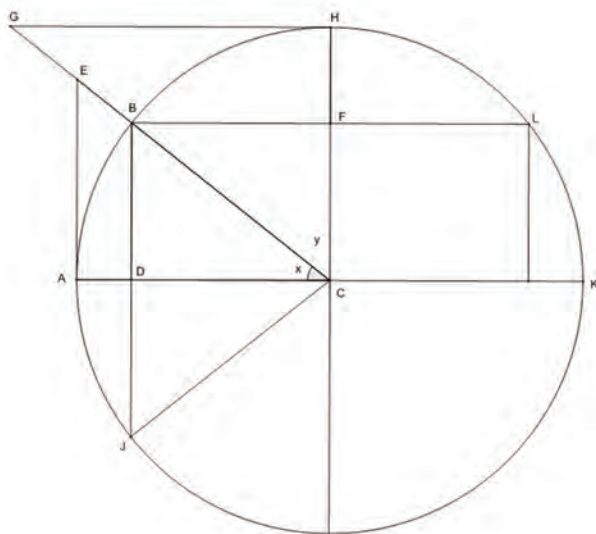


Fig. 2 Construction of trigonometric lines by Holmboe

The trigonometric lines to the arc AB are:

Sine – BD The sine of an arc, or of the central angle when the radius is 1, is the perpendicular from the end point of the arc (B) to the diameter through the other end point of the arc (D).

Cosine – DC The cosine of an arc, or of the central angle when the radius is 1, is the sine of the complementary angle.

$$BF = DC = \sin y = \cos x = DC$$

A consequence of this is that $\cos x = \sin(90^\circ - x)$

Tangent – AE The tangent of an arc, or of the central angle when the radius is 1, is the perpendicular on the diameter through one of the end points of the arc (A), to the point of intersection with the prolonged diameter through the other end point of the arc (E).

Cotangent – GH The cotangent of an arc, or of the central angle when the radius is 1, is the tangent of the complementary angle.

$$GH = \tan y = \cot x$$

Secant – EC The secant of an arc, or of the central angle when the radius is 1, is the distance between the centre (C), which is the angle vertex, and the end point of the tangent line of the arc, outside the periphery of the circle (E).

Cosecant – GC The cosecant of an arc, or of the central angle when the radius is 1, is the secant of the complementary angle. The distance between the centre (C) and the cotangent line outside the periphery of the circle (G).

$$GH = \tan y = \cot x \Rightarrow GC = \sec y = \csc x$$

Versed sine – AD The versed sine of an arc, or of the central angle when the radius is 1, equals $1 - \cos x$

Versed cosine – HF The versed cosine of an arc, or of the central angle when the radius is 1, equals $1 - \sin x$

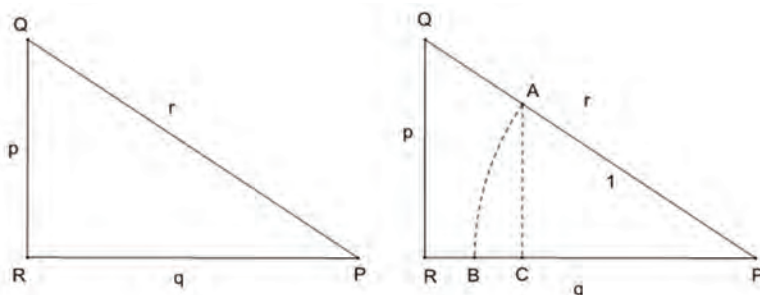


Fig. 4 Trigonometry in right-angled triangles

Theorem 1

Consider the triangle $\triangle PQR$, $\angle R$ is a right angle, and the sides are lower-case p , q , and r , as shown in Figure 4.

$$\begin{array}{ll} \frac{p}{r} = \sin P, & \frac{q}{r} = \cos P \\ \frac{p}{q} = \tan P, & \frac{q}{p} = \cot P \\ \frac{q}{r} = \sec P, & \frac{r}{p} = \csc P \end{array}$$

Proof

With radius $PA = 1$, draw arc AB . Construct AC perpendicular on PB .

$$\Rightarrow AC = \sin P \quad \text{and} \quad CP = \cos P$$

The ratios between the corresponding sides of the triangles $\triangle PQR$ and $\triangle PAC$ are the same, so we have

$$\begin{array}{l} r:1 = p:\sin P \\ \Rightarrow \sin P = \frac{p}{r} \end{array}$$

and correspondingly for all the other trigonometric values.

Sums and differences of angles

The four laws for finding the sine and cosine of sums and differences of angles are proven in §11 in Holmboe's textbook (Holmboe, 1834, pp. 18–21).

Theorem 2

For arbitrary arcs or angles, we have that

$$\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B \quad (1)$$

$$\cos(A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B \quad (2)$$

$$\sin(A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B \quad (3)$$

$$\cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B \quad (4)$$

Holmboe has a comprehensive proof based on trigonometric and algebraic reasoning, of which we here will cite the proof of (1) as an example.¹

Holmboe's proof

Holmboe has in a previous theorem (Holmboe, 1834, p. 18) proved that

$$c = a \cdot \cos B + b \cdot \cos A \quad (*)$$

and the law of sines stating that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

which gives us that

$$a = \frac{c \cdot \sin A}{\sin C} \quad \text{and} \quad b = \frac{c \cdot \sin B}{\sin C}$$

Inserting these values for a and b into $(*)$ gives us

$$c = \frac{c \cdot \sin A \cdot \cos B}{\sin C} + \frac{c \cdot \cos A \cdot \sin B}{\sin C}$$

$$\Rightarrow c \cdot \sin C = c \cdot \sin A \cdot \cos B + c \cdot \cos A \cdot \sin B$$

$$\Rightarrow \sin C = \sin A \cdot \cos B + \cos A \cdot \sin B$$

We have that $A + B + C = 180^\circ$, so C and $(A + B)$ are supplementary angles. Supplementary angles have the same sine values, therefore $\sin C = \sin(A + B)$ and we get that

$$\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

¹ Holmboe uses the sign “.” for multiplication in his textbook, while Ramus in most cases use no sign, and in some cases uses “.”. Therefore “.” or no sign will be used to indicate multiplication.

The two triangles $\triangle CIG$ and $\triangle BIF$ are both right-angled and have opposite angles in I , they are therefore similar. That gives us $\angle FBI = x$ and $\angle EFM = x$. We find the trigonometric lines for sine and cosine for the sum and the difference of x and y on the figure.

$$\begin{aligned}\sin(x+y) &= BG \\ \cos(x+y) &= CG \\ \sin(x-y) &= EK \\ \cos(x-y) &= CK\end{aligned}$$

We then decompose these trigonometric lines to lines that are sine and cosine lines to the angles x and y .

$$\begin{aligned}\sin(x+y) &= BG = BL + LG \\ \cos(x+y) &= CG = CH - GH \\ \sin(x-y) &= EK = FH - FM \\ \cos(x-y) &= CK = CH + HK\end{aligned}$$

That gives seven different lines that we need to identify, BL , LG , CH , CG , FH , FM and HK . The next step is to find which trigonometric values these lines represent, and Ramus uses only trigonometric definitions to do so. Assuming that $CB = 1$, we get

$$\begin{aligned}\sin y &= \frac{BF}{CB} = BF \quad \& \quad \cos x = \frac{BL}{BF} \Rightarrow BL = BF \cdot \cos x \\ &\Rightarrow BL = FM = \sin y \cdot \cos x \\ \\ \sin x &= \frac{LF}{BF} \Rightarrow LF = BF \cdot \sin x \\ &\Rightarrow LF = ME = GH = HK = \sin y \cdot \sin x \\ \\ \cos y &= \frac{CF}{CB} = CF \quad \& \quad \sin x = \frac{FH}{CF} \Rightarrow FH = CF \cdot \sin x \\ &\Rightarrow FH = LG = \cos y \cdot \sin x \\ \\ \cos x &= \frac{CH}{CF} \Rightarrow CH = CF \cdot \cos x \\ &\Rightarrow CH = \cos y \cdot \cos x\end{aligned}$$

We have found the seven lines, represented by sine and cosine to x and y , and we are thus able to conclude

$$\begin{aligned}BG = BL + LG &\Rightarrow \sin(x+y) = \sin x \cos y + \cos x \sin y \\ CG = CH - GH &\Rightarrow \cos(x+y) = \cos x \cos y - \sin x \sin y \\ EK = FH - FM &\Rightarrow \sin(x-y) = \sin x \cos y - \cos x \sin y \\ CK = CH + HK &\Rightarrow \cos(x-y) = \cos x \cos y + \sin x \sin y\end{aligned}$$

The law of cosines

Both Holmboe and Ramus demonstrate the law of cosines by using trigonometric argumentation. Holmboe starts by referring to a theorem in §10 in his textbook (Holmboe, 1834, p. 18). In an arbitrary triangle we have that

$$c \cdot \cos B + b \cdot \cos C = a$$

Holmboe states the law of cosines in §15 (Holmboe, 1834, p. 30), and proves it algebraic without any geometric reasoning.

Theorem 3. Law of cosines

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$

Holmboe's proof

$$b \cdot \cos C + c \cdot \cos B = a \quad (1)$$

$$a \cdot \cos C + c \cdot \cos A = b \quad (2)$$

$$a \cdot \cos B + b \cdot \cos A = c \quad (3)$$

Multiply (1) with , (2) with and (3) with , and we get

$$ab \cdot \cos C + ac \cdot \cos B = a^2 \quad (4)$$

$$ab \cdot \cos C + bc \cdot \cos A = b^2 \quad (5)$$

$$ab \cdot \cos B + bc \cdot \cos A = c^2 \quad (6)$$

(6) subtracted from (5) gives

$$ab \cdot \cos C - ac \cdot \cos B = b^2 - c^2 \quad (7)$$

and (7) added to (4) gives

$$\begin{aligned} 2ab \cdot \cos C &= a^2 + b^2 - c^2 \\ \Rightarrow c^2 &= a^2 + b^2 - 2ab \cdot \cos C \end{aligned}$$

Ramus also proves this statement using algebra but has a different approach. The statement giving the sine of the sum of two angles is already known. Ramus also does some clever algebraic operations, that gives us the result, operations that has no geometric meaning, but is easy to follow. In addition, his own presentation is very condensed.

Ramus' proof (Ramus, 1837, pp. 32–33)

$$\begin{aligned}
 A + B + C = 180^\circ &\Rightarrow \sin C = \sin (A + B) \\
 \sin C = \frac{c \sin A}{a} &\Rightarrow c \sin A = a \sin (A + B) \\
 &\Rightarrow c \sin A = a \sin A \cos B + a \cos A \sin B
 \end{aligned}$$

$$\begin{aligned}
 a \sin B = b \sin A &\Rightarrow c \sin A = a \sin A \cos B + b \cos A \sin A \\
 &\Rightarrow c = a \cos B + b \cos A \\
 &\Rightarrow b \cos A = c - a \cos B \\
 &\Rightarrow b^2 \cos^2 A = c^2 - 2ac \cos B + a^2 \cos^2 B \\
 &\quad + b^2 \sin^2 A = a^2 \sin^2 B \\
 &\Rightarrow b^2 = a^2 + c^2 - 2ac \cos B
 \end{aligned}$$

Calculation of trigonometric values

Holmboe (1834, pp. 25–27) describes how to calculate trigonometric values with the precision required. In a unit circle, the length of the full periphery equals 2π , and the problem is to find a value for $\sin x$ with an arbitrary x . He has already established in a theorem that every arc x between 0 and $\frac{\pi}{2}$ is larger than its sine and smaller than its tangent (Holmboe, 1834, pp. 24–25), as shown in Figure 3. Holmboe states in a theorem (Holmboe, 1834, pp. 25–26) that for every arc x between 0 and $\frac{\pi}{2}$, the sine must be greater than $\frac{x}{\sqrt{1+x^2}}$. The proof for this is purely algebraic, but my geometric understanding of $\frac{x}{\sqrt{1+x^2}}$ is shown in Figure 6.

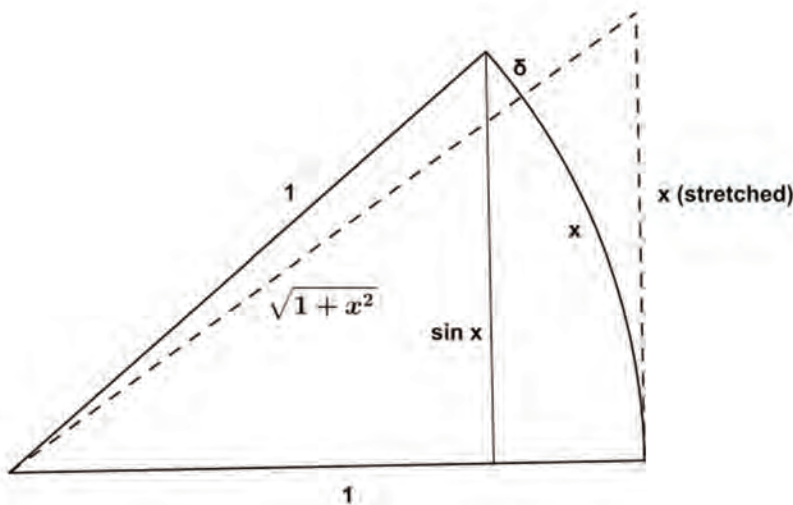


Fig. 6 A geometric understanding of $\frac{x}{\sqrt{1+x^2}}$

The magnitude δ is varying continuously and increasing between $\delta_{\min} = 0$ when $x = 0$, and δ_{\max} when $x = \frac{\pi}{2}$. From Figure 6 we then see that

$$\sin(x - \delta) = \frac{x}{\sqrt{1+x^2}}$$

and since $x > x - \delta$ we have that $\sin x > \sin(x - \delta)$. We may therefore conclude that

$$0 < x < \frac{\pi}{2} \Rightarrow \sin x > \frac{x}{\sqrt{1+x^2}}$$

The following assignment is then presented.

Assignment

To calculate the value of trigonometric lines to any given arc with a certain number of decimals.

Solution

$$x - \frac{x}{\sqrt{1+x^2}} < 1 \text{ decimal unit}$$

$$x > \sin x > \frac{x}{\sqrt{1+x^2}}$$

By decimal unit is here meant 'decimal unit of a certain order after the decimal point', and Holmboe gives the following example.

$$\begin{aligned} x &= 1 \text{ arcminute} \\ &= \frac{\pi}{180 \cdot 60} \\ &= 0.00029088 \\ \sqrt{1+x^2} &= 1.0000000423 \\ \frac{x}{\sqrt{1+x^2}} &= 0.00029087 \\ \sin 1' &= 0.00029088 \end{aligned}$$

The value of π was known with a sufficient number of correct decimals to make these calculations accurate, and $\sin 1'$ has the first eight decimal places in common

with 1'. The use of modern calculators shows, however, that $\sin 1'$ has the first eleven decimal places in common with 1'

$$\begin{aligned}x &= 0.0002908882087 \\ \sqrt{1+x^2} &= 1.000000042308 \\ \frac{x}{\sqrt{1+x^2}} &= 0.0002908881964 \\ \sin 1' &= 0.0002908882046\end{aligned}$$

The solution is that the difference $x - \frac{x}{\sqrt{1+x^2}}$ will be smaller than 1 decimal unit of the least value that one requires. Then $\sin x$ will be between x and $\frac{x}{\sqrt{1+x^2}}$ where x is the length of the arc. Holmboe's motivation for calculating sine for arcs as small as 1' is that

Sine of every arc less than 1' has the first eight decimals common with the arc;

the smaller the arc is, the smaller is the difference between x and $\frac{x}{\sqrt{1+x^2}}$, and $\sin x$ is always between these two magnitudes (Holmboe, 1834, p. 27).

If one needs an accuracy of eight correct decimals, and the arc is less than 1', one may calculate the length of the arc instead of finding the sine value. To calculate larger angles with the same number of accurate decimals, one may use angles where the sine and cosine values are known, and one may use the following formulas and others.

$$\begin{aligned}\cos x &= \sqrt{1 - \sin^2 x} \\ \sin(A+B) &= \sin A \cdot \cos B + \cos A \cdot \sin B \\ \cos(A+B) &= \cos A \cdot \cos B - \sin A \cdot \sin B \\ \dots &\quad \&c\end{aligned}$$

Ramus does not have the same method for calculating numerical values for sines in his textbook, but in a chapter describing how to make trigonometric tables (Ramus, 1837, pp. 22–24), he demonstrates a similar way of reasoning by comparing the numerical value of the sine with the length of the arc for small angles or arcs. He starts by stating that one may easily find the numerical value of $\sin 1'$ as exactly as required, and by using this numerical value one may find all other trigonometric values using known formulas. In the first quadrant, the numerical value of the sine is less than the length of the arc, which again is less than the numerical value of the tangent. By making the arc as small as necessary, these three values will approach each other

as much as required. When the arc is sufficiently small, one may by approximation consider the sine and tangent values equal to the length of the stretched arc.

Some concluding remarks

A question was formulated in the introduction regarding what type of proof is best suited to answer the reader's question 'Why?', and it is difficult to give a clear yes/no answer to this question. Various examples of both definitions and proofs have been given in this paper, where some are algebraic, and some are geometric.

Both textbooks have definitions where the dependencies between sides and angles in triangles are geometric magnitudes, i.e. trigonometric lines, and these lines may be constructed. A consequence of this could be that all trigonometric theorems ought to be proven geometric, not algebraic. This is, however, not the case, and the reason for this is probably diverse. The only example in this paper of a theorem that is proven purely geometric is Ramus' proof of sine and cosine of sums and differences of angles. Geometric proofs tend to be more convoluted than algebraic proofs. Even if the various steps of an algebraic proof of a trigonometric theorem may be difficult to justify geometrically, an algebraic proof may be easier for a student or pupil to follow than a geometric proof.

The textbooks by Holmboe and Ramus are for pupils before they enter courses at the university. It is therefore obvious that they are not meant for self-study, but need to be used by a competent teacher.

Norwegian pupils today are introduced to trigonometry during their three-year upper secondary school if they choose the courses preparing for further studies. They are introduced to trigonometry during the first year, and they may have more in-depth training in trigonometric equations and functions during the third year. There are several textbook works in mathematics used by the Norwegian upper secondary schools, and a textbook work called *Sinus* (Oldervoll et al., 2006) is used in this description. *Sinus 1T* has one chapter of approximately 30 pages called "Trigonometry" and is used in the first year. *Sinus R2* is used the third year for pupils choosing specialization in mathematics and has one chapter called "Trigonometric equations" of around 45 pages and one chapter called "Trigonometric functions" of around 50 pages.

The goal of the introductory training in trigonometry today is for the pupil to know the definitions of sine, cosine and tangent, and how to use trigonometry to calculate lengths, angles and areas of arbitrary triangles.²

² Mål for opplæringen er at eleven skal kunne gjøre rede for definisjonene av sinus, cosinus og tangens og bruke trigonometri til å beregne lengder, vinkler og areal i vilkårlige trekanter. (Oldervoll et al., 2006, p. 169)

The textbook starts by defining sine, cosine and tangent for angles between and by using the ratios between sides in right angled triangles, that is the length of one side divided by the length of another side, giving the sine, cosine or tangent as a numerical value. From these definitions, the formula for finding the area of triangles without right angles is derived. The definitions for sine and cosine are then extended to the range between and by studying supplementary angles. This way the pupils can work with both acute and obtuse triangles. The laws of sine and cosine are then described. At this stage, the unit circle is not introduced, but in *Sinus R2* together with trigonometric equations and functions.

The term ‘trigonometric lines’ is not used in the Norwegian schools today, neither are the terms cotangent, secant, and cosecant. For a modern student, sine, cosine and tangent are therefore only numerical values without any direct connection to geometric objects. The examples from the modern textbook and the textbooks from 1834 and 1837 are meant for pupils in the same age group.

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Mathématique moderne: A pioneering Belgian textbook series shaping the New Math reform of the 1960s

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Abstract

*In 1963 the Belgian mathematician and mathematics educator Georges Papy published the first volume of his groundbreaking textbook series entitled *Mathématique moderne* (MM) (in collaboration with Frédérique Papy), intended for students from 12 to 18 and based on several years of classroom experimentation. It marked a revolution in the teaching of mathematics and in the art of textbook design. Papy reshaped the content of secondary school mathematics by basing it upon the unifying themes of sets, relations and algebraic structures. Meanwhile, he proposed an innovative pedagogy using multi-colored arrow graphs, playful drawings and ‘visual proofs’ by means of drawings of film strips. During the 1960s and early 1970s, translations of the volumes of *Mathématique moderne* appeared in European and non-European languages and were reviewed in mathematics education journals of that time. Papy’s MMs influenced the national and international debates and became major guides for shaping the New Math reform in several countries.*

Keywords: Frédérique Lenger; Georges Papy; *Mathématique moderne*; New Math; textbook analysis

Introduction

In Belgium, the ‘New Math’ reform movement or ‘modern mathematics’, as it was commonly called in Europe, was inextricably linked with the personality of Georges Papy. Papy was born in Anderlecht, a municipality in Brussels, on November 4, 1920 and died in Brussels on November 11, 2011. He grew up in the pre-World War II period and studied mathematics at the Université libre de Bruxelles (ULB). During the war, he was an active member of the armed underground resistance forces in Belgium, serving in particular in the areas of intelligence and action, and teaching clandestine courses at the ULB and in a clandestine Jewish school (Gotovitch, 1991). After the end of the war, Papy obtained a PhD in mathematics (in 1945) and was granted an advanced teaching diploma by the Faculté des Sciences of the ULB in 1951. He was appointed lecturer at the ULB in 1956 and promoted to full professor in 1962, in charge of the chair of algebra which he occupied until his retirement in 1985 (Van Praag, n.d.). After a relatively short career in pure mathematical research, with several studies in the fields of algebra, topology, analysis and differential geometry, work being awarded at that time and still being recognized as important (Dieudonné, 1992), Papy reoriented his professional career. It was probably in 1959

that he became involved in the Belgian – and very soon also in the international – New Math educational reform movement. From then on, the modernization of mathematics teaching ‘from kindergarten to university’ became his life mission.

We briefly sketch Papy’s career in mathematics education. The impulse for Papy’s involvement in the New Math movement was a question by Frédérique Lenger and Willy Servais, two influential Belgian mathematics teachers at the time and both, from the early 1950s on, members of the Commission Internationale pour l’Étude et l’Amélioration de l’Enseignement des Mathématiques [International Commission for the Study and Improvement of Mathematics Teaching] (CIEAEM). In 1958, in the margin of 12th CIEAEM meeting in Saint Andrews (Scotland, UK), Lenger and Servais had developed an experimental program for the teaching of modern mathematics to future kindergarten teachers (who were at that time in Belgium trained from the age of 15 in a special kind of secondary schools) (Félix, 1985). The program was run during the school year 1958-59 in two such schools, but Lenger and Servais realized that they needed the assistance of a research mathematician for the further development of their experimental actions and consulted Papy to advise them. Papy not only accepted their invitation but immediately took charge of the project and in the school years 1959-60 and 1960-61 he started teaching experimental classes in Berkendael, a school for kindergarten teachers in Brussels (G. Papy, 1960). There he formed his ideas about the construction of the content and laid the basis of his innovative didactic approach. In this period Lenger and Papy married each other. Also professionally Frédérique and Georges became a team, permeated with the same ideas and with a firm determination to actually implement the reform (Noël, 1993).

In May 1961, Papy published his *Suggestions pour un nouveau programme de mathématique dans la classe de sixième* [Suggestions for a new mathematics curriculum in the first year of secondary schools] (G. Papy, 1961). This curriculum proposal was based on Papy’s experiences in Berkendael and already in September 1961, a new and large-scale experiment was set up in the first year of secondary school (and from then on gradually in the subsequent years). To coordinate and to develop meticulously these and other initiatives related to the upcoming reform, Papy founded on May 24, 1961 the Centre Belge de Pédagogie de la Mathématique (CBPM) [Belgian Center for Mathematics Pedagogy] of which he also became the chairman. The Centre, which brought together all main actors in the reform movement, organised, beside other meetings, the Journées d’Arlon [Days of Arlon], a series of large-scale teacher (re) education courses and, from 1968 on, also published its own journal NICO. The Centre’s actions finally led to a generalized introduction of modern mathematics in the first year of all Belgian secondary schools from September 1968 on (and from then on gradually in the subsequent years). At that time however, Lenger’s and Papy’s interests were already re-directed towards the primary level for which they started in September 1967 an extensive experiment in two classes of six-year-old

children (F. Papy, 1970). In September 1976 modern mathematics was introduced in the Belgian primary schools of the catholic network and two years later in the publicly run schools.

During the 1960s, Papy was also a prominent actor on the international mathematics education scene. As an expert he was invited to present his reform proposals at the major forums of that period, including the UNESCO symposium in Budapest (1962), the OECD conference in Athens (1963), the ICMI symposia in Frascati (1964) and Echternach (1965), the 15th International Congress of Mathematicians in Moscow (1966) and the UNESCO colloquium in Bucharest (1968). Papy also played a major role within the CIEAEM (as vice-president since 1960 and as president since 1963), but in 1970, due to strong disagreements about the Commission's future, Papy left the Commission (Bernet & Jaquet, 1998) and founded, with some loyal followers, the Groupe International de Recherche en Pédagogie de la Mathématique (GIRP) [International Group for the Study of the Pedagogy of Mathematics] (1971). However the meetings of the GIRP, as well as the other international activities of Papy during the 1970s, had limited impact on further developments in mathematics education. The tide had turned; modern mathematics was already on the way back in most countries.

The MMs

In 1963, Papy started with the publication of *Mathématique moderne* (in collaboration with Frédérique Papy) (G. Papy, 1963, 1965, 1966, 1967a, 1967b), a revolutionary textbook series, both in terms of content and layout. The series was intended for teaching modern mathematics to students aged 12 to 18 and was partly based on Papy's previous classroom experimentation in Berkendael (G. Papy, 1960). However, the series did not follow any specific program approved by the Ministry. In the next subparagraphs we briefly review the different volumes of this series to gain insight in Papy's viewpoints as revealed through this work.

MM1 – *The language of modern mathematics*

MM1 (G. Papy, 1963, see also Fig. 1), “particularly suited for students aged 12” (p. vi)¹, has 24 chapters with the following mathematical contents: (1-5) Algebra of sets (pp. 1–52), (6) First elements of geometry (pp. 53–87), (7-13) Relations, properties, composition (pp. 88–215), (14-15) Transformations of the plane (pp. 216–235), (16-18) Natural numbers (cardinal numbers), operations (pp. 236–295), (19) The binary system of numeration (pp. 296–315), (20) Integers (pp. 316–341), (21-23) Equipollence, translations, vectors, central symmetries (pp. 342–440), and

1 Unless otherwise stated, all translations were made by the authors.

(24) Groups (pp. 441–459). On the basis of this overview, it might be clear that this textbook proposed a completely new structure and foundation of initial secondary school mathematics of that time. In a note for students Papy explains the need for a renewed approach to mathematics education:

If you want to play an effective part in the world of tomorrow [...] you must master the mathematics of today. You will have to get to grips with the basic ideas of modern mathematics – which are used by all scientists – without wasting any time. And that is why we cannot teach you mathematics in the way your parents or your grandparents were taught – though you will be discovering (by a different route) all the basic things they were taught. (p. 45)

Papy's mathematical universe is built on (naïve) set theory. Sets are also the unifying element in this universe. The set-theoretical concepts are visualized by Venn diagrams. Venn diagrams are used for concept development, but also for reasoning and proof, e.g. of the properties of set operations. The algebra of sets is studied extensively because of its intrinsic value and its interesting new applications (which, however, are not discussed in the book). Moreover, because the algebra of sets resembles, but in some respects differs from the usual 'algebra of numbers', this study can also contribute to a better understanding of the latter. Implicitly the symmetric difference provides a first example of a group structure.

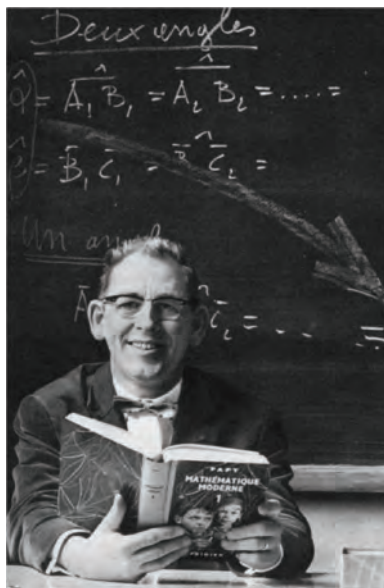


Fig. 1. Papy with a copy of the first volume of *Mathématique moderne*

The plane (an infinite set of points) is denoted by Π , and straight lines are subsets of Π . Two straight lines are equal sets or their intersection is the empty set – in both

cases they are called parallel – or their intersection is a singleton (a set with exactly one element). These possible mutual positions are illustrated with Venn diagrams (“The set diagrams provide intuitive support for the logical structure of the theory”, p. vi). In his first introduction to geometry Papy already brings in some basic topological notions, e.g. he differentiates between an open and a closed disk, and a circle (which only includes the perimeter). To visualize these notions the red-green convention (for parts that are ex/included) is introduced. Papy devotes ample attention to proving and logical-deductive reasoning. Therefore some initial propositions are selected as axioms. These axioms are not given all at once, but are released subsequently. In the chapters on geometry Papy proves some properties of parallelism and perpendicularity. These properties are simple and also intuitively clear, which makes them particularly suited for learning to reason correctly and for understanding the essence of proof. Noteworthy is Papy’s axiom $\Pi 4$, a reformulation of Euclid’s parallel postulate as “Every direction is a partition of the plane” (p. 74). It typifies his concise and abstract style.

Relations (sets of ordered pairs), their properties and composition, are discussed in great detail. Special attention is paid to relations of equivalence and order, functions and permutations. Papy further develops his pedagogical method based on arrow diagrams or papygrams (Holvoet, 1992). Papygrams made their entrance in Papy’s Berkendael course (G. Papy, 1960), but reappear here in multiple colours (up to six!) and in different geometrical constellations. New concepts are typically introduced with some simple and familiar situations with which “the student is encouraged to take an active part in building the mathematical edifice” (p. vi). Then this situation is abstracted to prepare a precise concept definition. Although the situations are common, they are especially designed for the above purpose and hence often a bit artificial. Papy never uses newly learned concepts to analyse (really) realistic situations although he argues that sets and relations are versatile and widely applicable instruments of thought.

The scope of the material studied in the first 13 chapters goes far beyond the boundary of mathematics. The student is initiated into types of reasoning constantly used in all spheres of thought, science and technology. (p. vii)

Functions and their graphical representations by arrow graphs reappear in geometry as ‘transformations of the plane’. Papy first discusses the constant and the identical transformation, the simplest – but not necessarily the most relevant – cases. For the third case, the parallel projection onto a line, again the way ‘from simple to more complex’ is followed: first points, then line segments and finally some other ‘sets of points’ are projected. In the last chapters on geometry, the concept of equipollence of ordered pairs of points is defined. It is proved that equipollence is an equivalence relation, of which the equivalence classes are called translations or vectors, and the set of translations forms a group under composition. In this section Papy introduces

the didactical tool of proof by film fragments: a sequence of suggestive images, from which a line of thought may be read easily, is presented and students are asked to add justifications.

The assimilation of a proof involves several stages which we should try to keep separate. The first step is for the pupil to understand the film so that he can explain it in informal language. Next he must be able to reconstruct the argument himself. After this comes the stage where more formal justifications are required. Only after all this do we turn our attention to the proper setting out of the proof. (p. viii)

Regarding algebra Papy first finds students' pre-knowledge about numbers and their operations in a set-theoretical framework. Natural numbers are defined as cardinal numbers of finite sets and the addition and multiplication of such numbers is related to, respectively, the union and Cartesian product of sets. The positional notation of numbers is revisited with the study of the binary system for which some kind of abacus is developed. This didactical tool, already prefiguring Frédérique's minicomputer for introducing numerical calculation at the primary level (F. Papy, 1969), is also used for the introduction of integers and their addition. For that Papy introduces a combat game with red and blue counters, representing oppositely signed numbers which 'kill' each other when coming in the same compartment. Properties of the operations with integers are strongly emphasized and lead to the discovery of a group and ring structure. MM1 concludes with a chapter on (abstract) groups, bringing together and systematizing several 'concrete' examples from the previous chapters.

MM2 and MM3 – *Real numbers and the Euclidean vector plane*

MM2 (G. Papy, 1965), subtitled *Real numbers and the vector plane*, provides a rigorous, but didactically elaborated construction of the field of real numbers and the vector plane. The textbook, "intended for 13-year olds" (p. vi) has 18 chapters and deals with: (1-2) The group $\Pi_0, +$ (Π_0 is the plane in which a point has been fixed so that each point represents a vector) (pp. 1–40), (3-5) Graduation of the line and the axiom of Archimedes (pp. 41–92), (6-7) Real numbers (pp. 93–154), (8) The theorem of Thales (pp. 155–177), (9) Homotheties (pp. 178–207), (10-13) The multiplication of real numbers (pp. 208–317), (14) Rational and irrational numbers (pp. 318–355), (15) Vector spaces (pp. 356–384), and (16-18) Equation of a straight line in the plane (pp. 385–434).

The major part of MM2 is devoted to a mathematically sound construction of the real numbers and to the equipment of the set of real numbers with order and with an additive and multiplicative structure. Papy uses a process of binary graduation of a straight line to build up the real numbers. By inserting the axioms of Archimedes and continuity he is able to establish a one-to-one correspondence

between the points on a straight line and the set of numbers, represented by terminating or non-terminating binaries, at a certain moment called ‘real numbers’. Then the order and additive structure of the points (vectors) on that line is transferred to the set of real numbers. For the multiplicative structure, Papy first defines multiplication of real numbers by means of homotheties (= homothetic maps): if h_1 and h_2 are homotheties with factors a and b , then $a \cdot b$ is the factor of the homothety $h_2 \circ h_1$ (the composition of h_1 and h_2). Finally, the basic properties of multiplication are deduced from the corresponding properties of composition of homotheties. The ordered field of the real numbers appears as the ultimate reward (on page 275). Papy believes that introducing the real numbers in this manner “enriches both the geometrical notions and the real number concept” (G. Papy, 1962, p. 6). The rational numbers are defined after the real numbers (!) – their structure appears to be an ordered subfield of that of the real numbers – and finally, some attention is paid to general vector spaces (exemplified with the principles of the vector plane) and to elements of vector-based affine analytic plane geometry.

In *MM3 – Euclid now* – (G. Papy, 1967a) the axiomatic-deductive building up of plane geometry is continued and will finally result in a contemporary vector-based exposition of Euclidean (metric) geometry for 14–15-year old students.

Euclid’s *Elements* exposed the basic mathematics of his time, about 300 years before J-C. The monumental work of Nicolas Bourbaki presents, at the highest level, the basic mathematics of today. The MMs want to expose the *Elements* of today’s basic mathematics for adolescents ... and people of any age and schooling who wish to initiate themselves in the mathematics of our time. (p. vii)

The 19 chapters cover the following topics: (1-3) Point reflections, (oblique and perpendicular) line reflections (pp. 1–46), (4-8) Isometries, classification (pp. 47–141), (9-13) Distance, circle, scalar product of vectors (pp. 142–281), and (14-19) Angles (pp. 282–441).

Transformations and groups which are generated by these transformations, play a key role in Papy’s construction of (Euclidean) geometry. Isometries are defined via the composition of a finite number of (perpendicular) line reflections. The different types – translations, rotations, reflections and glide reflections – and their possible compositions receive considerable attention. Colorful classification schemes based on Venn diagrams are presented and group structures are highlighted. Each time a group is discovered, it can provoke an Aha-Erlebnis: when a student recognizes a known abstract structure in a new setting, he might be able to apply all previously-learned knowledge and skills about this structure to that setting, an example of Ernst Mach’s ‘economy of thought’ principle (see, e.g., Banks, 2004).

Over the past half century, mathematics has switched from the artisanal stage to the industrial stage. The machine tools of our factories made it possible to save human muscular effort. The great structures of contemporary mathematics allow to save the human mind. (p. vii)

Transformations are also promoted as an alternative for traditional methods in school geometry in comparison with which they are much more intuitive and universal.

The outdated artisanal technique based on congruence of triangles must be abandoned in favor of translations, rotations and reflections, which are much more intuitive and whose scope goes far beyond the framework of elementary geometry alone. (p. ix)

Once the group of isometries is established, the fundamental concepts of Euclidean (metric) geometry can be introduced. The distance of a pair of points and the length of a line segment are defined by means of isometries (and from then on, isometries gain their etymological meaning of ‘length preserving transformations’). Definitions of the norm of a vector and the scalar product of two vectors follow. The natural structure for Euclidean geometry – a vector space equipped with an inner product – is thus created. Classical results, such as the Pythagorean Theorem – the cosine rule formulated in terms of vectors – can be proved easily within this structure.

Certain statements, once fundamental, are reduced to the rank of simple corollaries. That they now stop cluttering up the memory of our students. If necessary, they would be able to retrieve these results by routine use of one of the machine tools of modern mathematics. (p. ix)

Angles are equivalence classes of rotations (in Papy’s words “rotations that have lost their center”, p. 289). The sum of angles is defined by means of the composition of rotations and so the group of angles, isomorphic to the group of rotations, is created. MM3 ends with some (very) basic elements of trigonometry.

MM4 to MM6 – *The series’ closing in a minor key*

We can be short about *MM4*: the book remained unpublished. Late 1975, early 1976 Gilberte Capiaux, assistant at the CBPM, has worked on a volume about real functions referred to as *MM4*, supervised and guided by Papy, but the initiative has not yielded more than a hand-written draft (G. Papy, n.d.).

MM5 – Arithmetic – (G. Papy, 1966), intended for 14–18-year old students, presents a contemporary introduction to discrete mathematics. It only relies on contents that were exposed in *MM1* and *MM2*. The book has five major sections: (1) Combinatorics (pp. 1–50), (2) The arithmetic of integers (pp. 51–136), (3) The arithmetic of rational numbers (pp. 137–156), (4) An introduction to commutative

rings and fields (pp. 157–230), and (5) Arithmetical properties of groups and finite fields (pp. 231–280).

In combinatorics Papy deliberately avoids the “disused terminology” (p. vii) of variations, combinations, groupings with or without repetition, ... and consequently bases his exposition on the theory and language of sets and relations (in particular mappings), both for formulating counting problems and for developing and defining the necessary instruments. This is the only domain in the *MMs* in which some real problem situations are presented and discussed. The arithmetic of integers, dealing with divisibility, prime factorization and related issues, is embedded in the ring of integers (*MM1*). This structure and its substructures are studied in depth, and further abstracted and generalized to commutative rings and fields. In the last section, “particularly intended for students [...] preparing for mathematics studies” (p. x), one can find theorems as, e.g., ‘every transformation of a finite field is a polynomial function’ and ‘the multiplicative group of every finite field is cyclic’, results that normally go beyond secondary school mathematics.

In *MM6 – Plane geometry* – (G. Papy, 1967b), published in a somewhat more sober style than the other volumes, Papy takes up the geometric thread for 15–16-year olds. The 11 chapters cover the following contents: (1-3) Repetition/summary of *MM1*, *MM2* and *MM3* (pp. 9–82), (4-7) The (Euclidean) vector plane, linear transformations, matrices (pp. 83–183), (8-9) Orthogonal transformations, similarity (pp. 184–234), (10) The complex plane (pp. 235–258), and (11) Trigonometry (pp. 259–267).

Papy first retraces in brief the laborious path, from the original ‘intuitive’ (synthetic) axioms of geometry to the establishment of a Euclidean vector plane structure, the path that the students had followed from the age of 12 to 15. This summary, mainly clarifying the math-educational methodology of the first three *MMs*, must prepare these students for the second step which is described as a ‘psychological reversal’: the structure of a Euclidean vector plane is taken as a new and unique starting axiom for the further development of plane geometry (from p. 84 on). This approach also opens perspectives for the future study of higher-dimensional Euclidean spaces, in particular for building up solid geometry. Although it is possible, in principle, to rebuild all mathematics from previous years on the basis of the new axiomatic, this is not suggested.

This wonderful machine tool should not be used to rediscover what we already know. It should allow new conquests. (p. v)

Linear transformations of the vector plane play a key role in the continuation of the book. Special types, such as orthogonal transformations (= linear transformations that preserve the scalar product) and similarities, are studied in depth. As in *MM3*, the transformational aspect as well as the identification and classification of (sub)

structures receive ample attention and are illustrated with multicolored papygrams, playful drawings and Venn diagrams. In *MM6*, transformations are also algebraically typified by means of matrices. At the end of the book complex numbers are introduced as direct similarities. By relying on the structure of the latter and isomorphism, it is proved that the complex numbers form a field extending the field of the real numbers.

Shaping the New Math reform

Although conceived as textbooks the *MMs* have never been used for that purpose, except in experimental classes. When from 1968-69 on New Math was made compulsory in Belgian secondary schools, the official programs were different from and less ambitious than those developed by Papy. In response Papy and collaborators started the *Minimath* series, a ‘light version’ of the *MMs*, but only the first two volumes of that series ever appeared (G. Papy, 1970, 1974). Papy’s *MMs* have thus served primarily as a major source of inspiration, both in terms of content and style, for mathematics educators and textbook developers during the 1960s, the period in which the New Math reform was prepared and implemented in several countries. More specifically with respect to the reform in Belgium, Warrinnier (1984) noted that “all textbooks essentially go back to the remarkable series *Mathématique moderne* by G. Papy” (p. 120).

The *MMs* were also translated into several different languages, including Danish (Vols. 1, 2, 3), Dutch (Vols. 1, 2, 3, 5), English (Vols. 1, 2), German (revised version of the geometric chapters of Vols. 1 and 2), Italian (Vol. 6), Japanese (Vol. 1), Romanian (Vols. 1, 2), and Spanish (Vols. 1, 2, 3, 5).

The impact of Papy and his *MMs* on the international mathematics education debates during the 1960s can hardly be overestimated. As mentioned before, Papy acted as an uncompromising New Math ambassador at major international conferences of that period, reported about his successful experiments and defended, with verve and authority, his views on the modernization of mathematics teaching. Already at the 1963 OECD conference in Athens, Papy presented an extended sneak preview of the mathematical content and methodological approach of his first two *MMs* (G. Papy, 1964). Papy’s design of teaching modern mathematics was well received by the other OECD experts:

The example given by Mr. Papy [...] was stimulating as to what can be accomplished by a proper blend of modern mathematical ideas with very conscious psychological methods of presentation. When students are directed toward the discovery of mathematical patterns and the self-construction of mathematical entities (such as the real numbers), motivation and permanency of learning are greatly enhanced. (OECD, 1964, p. 296)

International debates on the teaching of geometry

Of particular interest is the influence of Papy's experimental approach, as documented in the *MMs*, on the international debate on the teaching of geometry that took place in Europe during the 1960s. This debate was launched by Jean Dieudonné who rejected, in combative terms, the traditional teaching of Euclidean geometry at the 1959 Royaumont Seminar. A group of experts that met at Dubrovnik in 1960 in order to work out a detailed synopsis for a modern treatment of the entire mathematical curriculum, only partially succeeded for the geometry part. Also at related international conferences specifically devoted to 'the case of geometry' (Aarhus, 1960 and Bologna, 1961), only compromises could be reached and the debate increasingly narrowed to the search for the most adequate axiom system for the teaching of geometry at the secondary level.

The debate heated up in 1964. That year Gustave Choquet published his *L'enseignement de la géométrie* [The teaching of geometry] in the Introduction of which he stated that the perfect "royal road to geometry is based on the notions of vector space and scalar product" (Choquet, 1964, p. 11). However, Choquet acknowledged that children benefit from an approach to geometry based on concepts drawn from the real world such as parallelism, perpendicularity and distance. To reconcile this pedagogical concern with the mathematically most valid method, Choquet set out intuitively clear synthetic axioms to demonstrate the algebraic structure of the plane. Then using the tools of linear algebra he developed his course of geometry. Also Jean Dieudonné published a book on geometry teaching that year, entitled *Algèbre linéaire et géométrie élémentaire* [Linear algebra and elementary geometry] (Dieudonné, 1964). Dieudonné uncompromisingly based his geometry course on linear algebra and made absolutely no concessions to synthetic methods. Moreover, in the Introduction he launched a violent attack on Choquet's more realistic and evolutionary axiom system which demonstrated "a remarkable ingenuity which shows the great talent of its author, but that I consider as completely useless and even harmful" (p. 17).

André Revuz tried to reconcile Choquet's and Dieudonné's points of view. Therefore he enlisted the help of Papy whose approach was very similar to that of Choquet². Moreover, Dieudonné himself paid tribute "to the remarkable and promising trials of our Belgian neighbours" (Dieudonné, 1964, p. 17), referring to Papy's experiments with 12–13-year olds as documented in *MM1*. So Revuz could authoritatively defend the systems of Choquet (and Papy) as 'intermediate steps' between students' intuition and the 'ideal' (purely linear algebra based) system proposed by Dieudonné.

² However, there are areas of difference. For example, whereas Choquet deliberately assumed the distance on a line and the structure of the real numbers, Papy gradually developed these concepts.

However, if one believes that geometry is not only a mathematical theory, but also a physical theory, if one thinks that the role of education is not only to know mathematics, but also to learn to mathematize reality, one can think about Choquet's system as an intermediate step, which will not only allow teachers to change their mentality, but perhaps also will enable any student to move easily from the intuitive space to the mathematical theory. (Revuz, 1965, p. 273)

On the initiative of Revuz, the disagreement between Choquet and Dieudonné was officially settled in April 1965 at the 19th CIEAEM meeting in Ravenna (Italy), with a statement about the role of geometry in the education of 10–18-year old students, agreed by all CIEAEM members present (but in the absence of the two disputants). In this statement the special place of geometry was recognized. More concretely, an approach in two stages, inspired by the ongoing experimentation of Papy and his CBPM, was recommended. The first stage (for 10–11- to 14–15-year olds) aimed at mathematizing the concrete space of the student by means of axioms and deductive reasoning, leading to the construction of a Euclidean real vector space structure. In the second stage (for 14–15- to 17–18-year olds) this structure was taken as a new axiomatic for the further development of geometry. The Ravenna manifesto was approved by Choquet and Dieudonné at the ICMI meeting in Echternach in June 1965 (Félix, 1985).

Contribution of the MMs to national reform debates

The contribution of Papy and his *MMs* to the New Math debates in the different countries that were involved in this reform movement, has not been investigated systematically. With respect to France and the Netherlands, we are, to a certain extent, informed about the reception of Papy's ideas by reviews of the *MMs* in journals of professional organizations of mathematics teachers.

Papy's *MMs* were enthusiastically welcomed in France. In the *Bulletin de l'Association des Professeurs de Mathématiques de l'Enseignement Public* (APMEP) [Bulletin of the Association of Mathematics Teachers of Public Education], Gilbert Walusinski, influential member and former president of the APMEP, reviewed the first two *MMs* in glowing terms (1963, 1966). For him, Papy's experiments, as documented in the *MMs*, demonstrated that it is really possible to teach modern mathematics in an active and non-dogmatic way, in real classes of 12–13-year olds, including students who are not necessarily gifted for mathematics. Walusinski characterised Papy's actions as encouragements and inspiring examples for future developments in France:

Papy's *MM1*, the printed testimony of his experience, it is the reform in act. [...] The time will come when a reconstruction of our public schools will be possible; it will not be a time of rest, but one of an action that should be

fast and efficient. We must prepare for it from now. Papy is helping us. Very friendly, I want to thank him. (Walusinski, 1963, p. 126)

Piet Vredenduin, a prominent mathematics educator from the Netherlands, reviewed Papy's experimental approach in global terms (1967a), as well as the different MM volumes upon publication, in *Euclides*, the journal of the Nederlandse Vereniging van Wiskundeleraren [Dutch Association of Mathematics Teachers] (1964, 1966, 1967b, 1967c, 1968). As a former Royaumont delegate, Vredenduin was favourably disposed to New Math. He acclaimed Papy's radical and uncompromising modernization efforts, and recommended all mathematics teachers to read and to enjoy the MMs. Nevertheless, he raised questions about six main differences with what was common practice in his country at that time: (1) Papy pushes mathematical rigour to the extreme, (2) from the outset, the emphasis is on the structure of the mathematical systems, (3) the training of mathematical techniques is neglected, (4) the interest in the triangle, criteria for congruence, the special types of quadrilaterals, ... has decreased, (5) the use of symbols is strongly enforced, and (6) the acquisition of a specific mathematical language is promoted to a central goal. He concludes that also in the Netherlands a modernization is desirable, but he is concerned that too many valuable things from the past will be dumped (Vredenduin, 1964, 1967a).

Vredenduin's compatriot Gerrit Krooshof, member of the editorial board of *Euclides* and leading author of a successful Dutch textbook series with the same title as Papy's MMs (*Moderne wiskunde* [Modern mathematics]), formulated his reservations in terms of a metaphor:

We can confidently say that this means a new building of secondary school mathematics. As a Le Corbusier of mathematics education, Papy has created, from pre-stressed concrete and glass, a robust and (at least for us) transparent structure. But for our students the windows are too high. (Krooshof, 1967, p. 194)

Papy also left his footprints in the Nordic countries. The Danish translation of MM1 and MM2, dating from 1971, was reviewed in *Nordisk Matematisk Tidsskrift* [Nordic Mathematical Journal] (Solvang, 1972). Ragnar Solvang praised Papy's audacity to present completely new materials for secondary school mathematics in an accessible, informal and 'entertaining' style, but raised the question whether the new topics can really defend their place in a mathematics syllabus for 12–13-year olds:

It's clear that this can be discussed and the answer one gives will depend on the objectives that are sets up for the subject. If one believes that the goal should be to provide the students with an insight into the subject's structures, then it is clear that some of these topics motivate themselves directly. And when it concerns adjusting to the age level, Papy has realized a nice piece of work. In the MMs, the students are guided through the individual substructures and finally end up with the group concept. His use of filmstrips in the

chapter on groups is methodology of the best class. (Solvang, 1972, p. 146, translated from Norwegian by Kristín Bjarnadóttir)

Moreover, Solvang wondered how the teaching, based on these books, can be organized practically, in particular how the practical arithmetic work and the applied side of mathematics can be integrated in the same spirit of what's already there. He nevertheless acknowledged the centrality of Papy's work in the debate about the new directions of lower secondary school mathematics.

Concluding remarks

The *MMs* were definitely a milestone in the history of the New Math reform movement of the 1960s, having inspired many mathematics educators worldwide. Papy did not present a cautious compromise by integrating new ideas in an existing tradition, but radically reshaped the content of secondary school mathematics by basing it upon the unifying themes of sets, relations, functions and algebraic structures. Meanwhile, he proposed a completely new teaching approach using multi-colored arrow graphs, playful drawings and 'visual proofs' by means of film fragments. However, these strengths were also weaknesses. Probably it was too ambitious to try to change at the same time both the content and the pedagogy of mathematics education. Although the approach proved to be successful in an experimental setting, directed by Papy or Frédérique, it went far beyond the range of competence of most common teachers.

Papy's voluminous books also did not offer a practical solution for actual classroom teaching. Although the *MMs* were conceived as textbooks, they didn't follow any official program. Moreover the series remained incomplete: the teaching of algebra and geometry for 12- to 16-year olds, is well elaborated in the first three *MMs* and in *MM6*, but the continuation to 16- to 18-year olds remained unclear. Only for the teaching of arithmetic, a convincing alternative in line with the other volumes was developed in *MM5*, but for several reasons, this topic never received a central position in secondary school mathematics. For the teaching of analysis, which constitutes the lion's share of the mathematical curriculum for 16- to 18-year olds, the *MMs* didn't provide the necessary material. This is strange as Papy (and Frédérique) definitively had some clear ideas about the teaching of analysis (see, e.g., G. Papy, 1968) and these ideas were already prepared, at least to a certain extent, in *MM1*. But unfortunately an integration of these ideas in the *MM* project was never realized. A possible reason is that by the end of the 1960s, Papy became discouraged about the actual implementation of the reform he had initiated. In 1972 he declared to the press:

The message of the promoters of the reform of the teaching of mathematics, all over the world, has been lost ... the ship of the reform is stranded. (Debefve, 1972, p. 7)

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3 There may occur some confusion regarding references to publications by Frédérique Lenger and Georges Papy, wife and husband from October 1, 1960. From the 1960s, Georges typically signed his publications as 'Papy', while Frédérique, after marriage, used her husband's surname (and signed 'Frédérique Papy') or signed with only her first name 'Frédérique'. The MMs mention 'Papy' as an author, but it is stated that the series was written 'in collaboration with Frédérique Papy'. Lists of publications from both authors can be found at <http://www.rkennes.be/>.

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John Leslie's (1817) view of arithmetic and its relevance for modern pedagogy

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Abstract

John Leslie in his book The Philosophy of Arithmetic (1817) presented a hypothesis that arithmetic didn't start with the process of counting, but with partitioning a collection of objects into two equal parts with or without a remainder. He described how our ancestors could have developed arithmetic not only without written language, but also without number words and other linguistic skills. They only needed manual skills to manipulate small objects on primitive counting boards. This paper shows how the method described by Leslie can be used to introduce numbers in early grades, and to teach basic arithmetic skills.

Keywords: counting boards, arithmetic tasks

Brief history of concepts presented

Sources

This paper develops the ideas of three mathematicians, John Napier (1550-1617), John Colson (1680-1760), and John Leslie (1766-1832).

John Napier, mostly known for his invention of logarithms, also designed an interesting counting board utilizing a geometric progression to carry out multiplication, computation of roots, and other more complex arithmetic operations efficiently. This work was described in one chapter of his book *Rabdology* (1617/1990), and remained mostly unknown until an article about it was published by Martin Gardner (1986).

John Colson introduced the concept of 'negative digits', which simplified arithmetic computations. They were not considered to be negative numbers, but positive numbers marked to be subtracted instead of being added. This work also was mainly forgotten until Donald Knuth quoted it in his *The Art of Computer Programming* (1997). The same concept was used in ancient China (Hart, 2011) where computations were carried out with bamboo sticks, before the invention of the suanpan (the Chinese abacus).

John Leslie in his book *The Philosophy of Arithmetic* (Leslie, 1817) provided two parallel versions of elementary arithmetic: ‘palpable’, where computations are carried out on a counting board, and ‘figurate’, where computations are carried out in writing. His book contains two original ideas. He takes ‘halving with a remainder’, which is partitioning a set into two almost equal subsets, instead of ‘counting’, for a basic arithmetic operation. This allows him to develop the arithmetic processes and his notation for numbers without any reference to a person’s linguistic skills. According to Leslie, number words are optional and not a prerequisite for learning arithmetic. Also, no writing skills are required during palpable computations with counters. He also uses ‘negative digits’ in all computations, in a manner that was proposed by Colson (Colson, 1726). Leslie introduced his own terminology for negative counters used on boards, and for written digits, calling them ‘empty’ or ‘deficient counters’ and ‘deficient figures’.

In his book Leslie displayed a very impressive knowledge of the history of mathematics, including non-European traditions, but because he did not provide any specific references, we may only assume that he took negative digits from Colson and the name “palpable” from Nicholas Saunderson (1740).

Leslie’s approach, combined with concepts from Napier and Colson, adapted to the modern school setting, can be used in early grades. It may be useful in multilingual classrooms and when some pupils have inadequate writing skills¹.

Counting boards

Counting boards are often considered to be precursors of more modern mechanical computing devices, and of modern calculators and computers. This is a rather misleading analogy. Counting boards played the same role as modern tablets, white- and blackboards, or even paper, when one writes on it with a pencil and not with a pen. A counting board provides an external ‘memory’ that allows one to record, modify, and erase partial results of computations and other information that is necessary during the process of computing, but may be forgotten when the final result is obtained.

Historical sources indicate that all societies that developed advanced arithmetic used some counting boards, even after full written systems of arithmetic were introduced. Unfortunately, information regarding how the boards were used and what algorithms were carried out exist only for post-medieval Western Europe about computation ‘on lines’ (on a ‘Roman abacus’), and in Eastern Asia about computation on a ‘Chinese abacus’.

¹ Biographies of J. Napier, J. Colson, N. Saunderson, and J. Leslie are available on the internet; see References.

There are also data indicating that arithmetic was developed very early and spread among prehistoric societies. But it seems that at present the internet is the best source for finding the current state of knowledge and a variety of different hypotheses concerning counting boards².

The counting boards described by Leslie are inadequate for modern school use, but they can be replaced with more 'modern' counting boards (Baggett & Ehrenfeucht, 2016) designed on the principle described by John Napier.

Modern counting boards

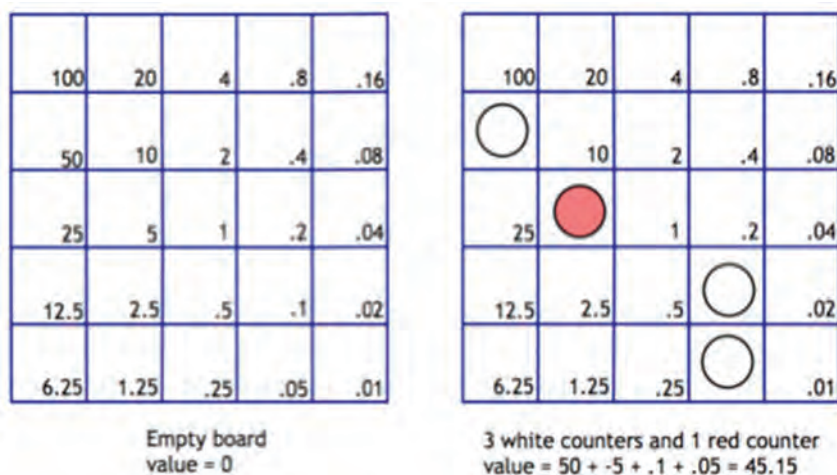


Fig. 1. A typical modern counting board for computation with decimals

Numbers in each column (a column is also called a rod) form a geometric progression with ratio 2. (On this board, rows also form a geometric progression with ratio 5, but this will not be the case on some other boards.)

Counters are two-colored; one side is white and the other is red. A white counter on a board has the value of its location (which is positive) and a red counter has the opposite value (which is negative).

The value of the whole board is the sum of the values of all counters.

One can stack several counters on one square.

² See History in the References.

There are three rules of regrouping counters on this board that are sufficient to transform a configuration of counters into any other configuration of the same value.

One can extend this board in all four directions, left, right, up, and down, as long as numbers in rows and columns form the geometric progressions, because the rules of regrouping remain unchanged.

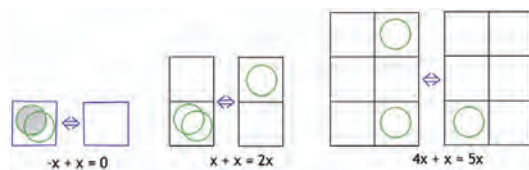


Fig. 2. Three rules of regrouping corresponding to three algebraic equalities, $x + -x = 0$, $x + x = 2x$, and $x + 4x = 5x$.

In Figure 2, a white circle represents either a white or red counter. And a gray circle represents its opposite.

Arithmetic operations performed on a board

Addition and subtraction is straightforward. One puts both numbers on the same board and regroups the counters to a desired final format.

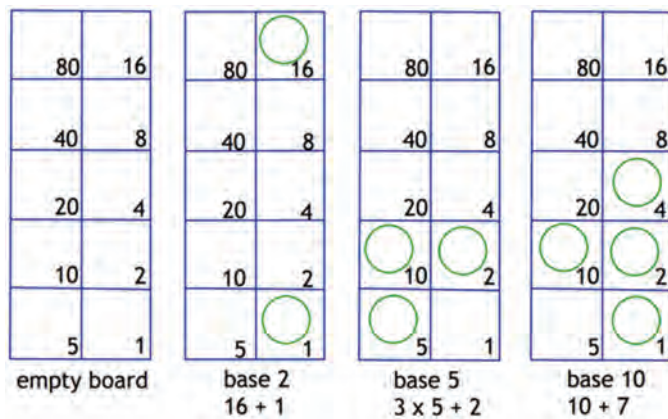


Fig. 3. The number 17, represented in bases 2, 5 and 10, on a decimal board.

Multiplication of decimals is also easy. Multiplication and division by 5 correspond to shifting all counters one square to the left or one square to the right. Multiplication and division by 2 correspond to shifting counters up and down. Thus, the product of any two numbers can be computed by shifting and adding patterns of counters.

Introducing the concept of numbers in grades kindergarten, 1st, and 2nd

In US schools learning arithmetic starts in kindergarten and elementary grades with verbal counting, and children need to master this skill to at least 20. This is followed by addition and subtraction of whole numbers, multiplication and division of whole numbers, common fractions and operations on them, decimals and finally negative numbers. These topics strongly overlap, so for example, addition often starts as counting up, multiplication starts as repeated addition, and common fractions start with fractions having small denominators.

Here is a different sequence of material for kindergarten and the first two grades. It starts with addition and subtraction of whole numbers on the board with two-color tokens (leaving open the question whether they represent negative numbers or positive numbers marked to be subtracted). It is followed by counting, and by multiplication, and the set of whole numbers can be extended by binary fractions.

During these three years, children would use only boards built from four rods of length at most 6, consisting of at most 4 rods containing the numbers 1, 3, 5, and 15 (and possibly binary fractions). The order in which the rods are arranged to form a board may vary.

4	12	20	60
2	6	10	30
1	3	5	15
$\frac{1}{2}$	$1\frac{1}{2}$	$2\frac{1}{2}$	$7\frac{1}{2}$

Fig. 4. Examples of rods used in grades K, 1st, 2nd.

The only rules needed for regrouping on these boards are rules for the decimal board and a rule corresponding to the algebraic equality, $2 + 1 = 3$.

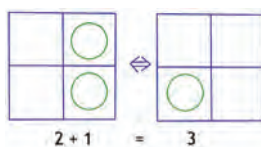


Fig. 5. The new regrouping rule; it corresponds to $2 + 1 = 3$.

Reading location numbers

Numbers labeling locations on a board are written as positive decimals or as mixed numbers that combine a counting number with a proper binary fraction.

Reading number-words, even without understanding, is a difficult task that children acquire rather slowly in their native language, and it is even more difficult for children in multi-lingual classrooms. So a different reading method, that is simple and easy to learn, is needed in each classroom.

On each board there is one location that holds the number 1. Any other location can be reached by moving some number of steps left or right, followed by some number of steps up or down.

So for example on the board in Figure 1, instead of saying “white counter on square five hundredths”, one can say, “white, one right, two down”.

This reading has one big advantage. If you already know some fractions and that the label .2 means one fifth, you would know that .05 stands for one twentieth (because numbers in every column form a progression with ratio 2).

Three small boards and their ranges



Fig. 6. Three small boards for grades kindergarten, 1st, and 2nd

The range of a board is the set of all numbers represented on it with at most one counter for each location. (Because we use two-color counters, the range of each board always contains negative numbers, which we don't list here.)

Only integers are represented on these boards, and their ranges are 0 to 12, 0 to 28, and 0 to 42.

The first board can be used to introduce all arithmetic information about numbers up to 12 that children may need later, providing that they already know the numbers 1, 2 and 3.

Examples

Question: What is the number 5?

Answer: Make a stack of five white counters on the square labeled 1; regroup two counters on 1 to one counter on 2; regroup a counter on 1 and a counter on 2 to a token on 3; and finally, regroup two counters on 1 to one counter on 2. So we found that 5 equals 2 plus 3.

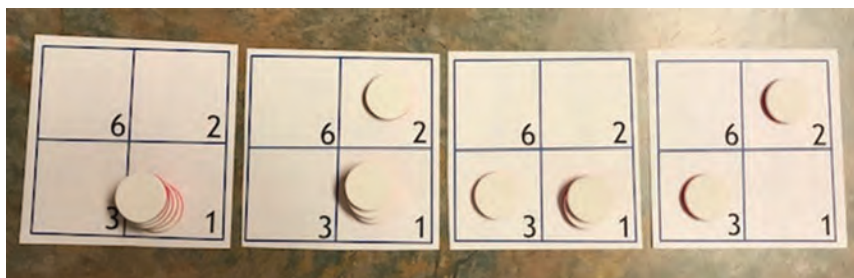


Fig. 7. Representing 5 on the board

Question: How to represent 0 with exactly 3 counters? (Find all the answers.)

The two other boards are used for actual calculations. Children should be shown how to represent, on each board, numbers from 0 to 12, with at most 2 tokens. (Notice that it requires the use of both white and red counters.)

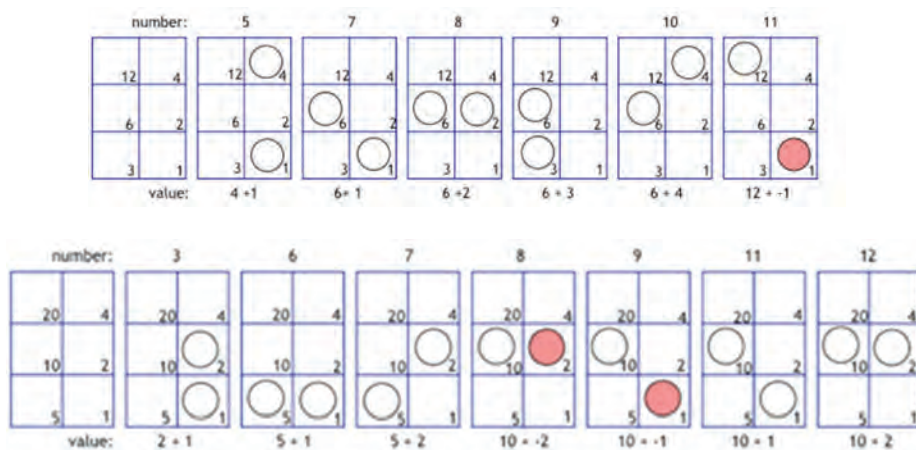


Fig. 8. Numbers up to twelve on two small boards

The board with the smaller range is easier to use, but the other board is necessary for introducing the concept of base 10.

General comments

- If one wants to extend the range of whole numbers that are used, each board can be extended up.
- Using four-column boards and boards with fractions is optional.
- Many tasks require using more than one board. They are easy to make, and there is no restriction on how many of them are used at one time.

- Writing numbers should be done only for keeping records and communication; all computations can be done mentally or on counting boards.

Examples of tasks for use in grades K, 1st and 2nd

These tasks are not lesson plans because they do not include stories that usually accompany and justify the tasks, or address pedagogical issues such as whether children should work in groups or individually, or how to teach them procedures and how to assess their work.

First task

Students learn how to compute all products of whole numbers up to 12×15 (multiplication ‘facts’). Each product is computed by adding (at most) two numbers on one counting board. But three boards, called multiplier, multiplicand, and product are used. How the boards are laid out on a table is a matter of convenience.

Each number from 1 to 12 is represented by at most two tokens on the multiplier board. But 4, 8 and 9 require the use of red counters, $4 = 5 + -1$,

$8 = 10 + -2$, and $9 = 10 + -1$.

When a number has multiple representations with the same number of tokens, students may use any one of them, depending on which one is most useful in a given context.

The numbers 1 to 15 are represented on the multiplicand board with white counters only. The number 15 requires 4 counters, $15 = 8 + 4 + 2 + 1$.

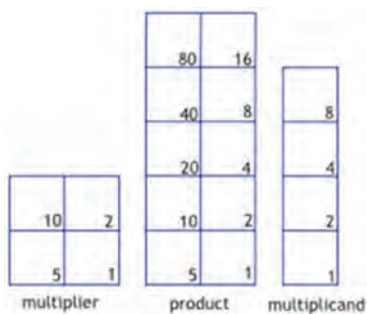


Fig. 9. Boards for multiplication

Algorithm for multiplication

Call the multiplicand m , and start with an empty product board.

To add $1 \cdot m$ to the multiplier, just copy m onto the product;

to add $2 \cdot m$, copy m , and shift it one square up;

to add $5 \cdot m$, copy m , and shift it one square to the left;

to add $10 \cdot m$, copy m and shift it one square left and up.

When multiplying by a negative number, flip over all counters of the multiplicand.

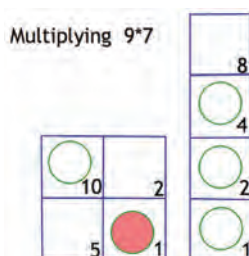


Fig. 10. Multiplying nine times seven

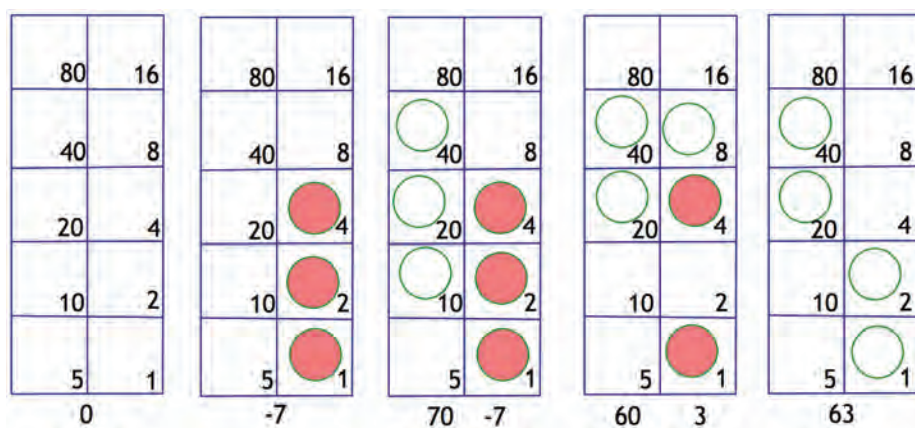


Fig. 11. Multiplying nine times seven step-by-step

Second task

A student is given a cup of beans or other small objects such as river pebbles, pennies, or non-rolling marbles. The number of objects, unknown to the child (but known to the teacher), can vary from 2 to 63.

The child is asked to count the objects and write down the result in Hindu-Arabic notation. All computations have to be done mentally and on a counting board, having only one rod as shown in Figure 12.

The child has to use the algorithm ‘Counting by halving’, which does not require knowing number words in any language.

Algorithm for counting by halving

Initial step:

Put one white counter on square 1 on the board. Empty the cup, making one pile of beans on the table.

Next step:

Use both hands to partition the pile into two equal parts. Notice if one bean is leftover, or no bean is leftover. (This is important!)

Case 1: If no bean is left, move the top counter on the board one square up.

Case 2: If there is one leftover bean, put one new white counter on the square above the top counter on the board.

Put one of the two piles of beans from the table and the leftover bean (if any) back into the cup.

Look at the pile that is left on the table.

If it contains 2 or more beans, repeat this step.

If it contains only one bean, put it into the cup, and you are done.

Writing the result

Mentally compute the total value on the board and write it down.



Figure 12. Four steps in counting nine beans

Third task

In this task each student works alone. But a student who has already finished the task should help anyone who needs help.

Each student is given an irregular flat piece of modeling clay and a plastic knife.

The sharing-task instructions: Cut the ball of clay into pieces so that each student in the class can get one piece. In each step you are going to cut one piece of clay into two or three parts. Try to make them approximately of the same volume. Their shape is unimportant.

Algorithm the students need to learn:

Count the number of students in the class and represent it with counters on a one-column board of length 6. Now you will be working with “piles” of pieces.

Start: Put the piece of clay on a rather large flat surface. It is the first ‘pile’, consisting of only one piece.

There are two kinds of moves, even and odd.

Even move: Cut each piece from the pile into two, approximately equal, parts..

Odd move: Cut one piece (preferably the biggest one), into three, approximately equal parts; and all other pieces into two parts, as in an even move

You need to make the right number of moves in the right order to make the required number of pieces; and you get this order from reading the number from the counting board from the top down. When there is a counter on the location, you make an *odd* move. When the location is empty, you make an *even* move.

Example: 37 pieces are needed.







	Move:	Number of pieces:
	<i>start</i>	1
	<i>even</i>	2
	<i>even</i>	4
	<i>odd</i>	9
	<i>even</i>	18
	<i>odd</i>	37

Figure 13. Seven stages of a sharing task

Remark.

This algorithm is easy to use, but rather difficult to learn. So before individual students would try it, they should be quite familiar both with representing the numbers

on this board and with cutting clay. Also the teacher may demonstrate to the whole class how to do it.

Final comments

1. More classroom materials using counting boards are available at *Breaking Away from the Mathbook* under *Number Boards*. Most of the available materials have been tested with current and practicing teachers taking college math courses. But so far we have no data showing how a proposed curriculum for early grades would work as a whole. So we present it only as a possible alternative approach and not as one that is recommended.

2. We have avoided discussing pedagogical issues involving learning arithmetic in early grades. (For example: In a bilingual classroom, should all children learn spoken number words in both languages?) In material for teachers we try to present only the mathematical content, the task, and the organization of student work that is appropriate for it, leaving all other decisions to the classroom teacher.

3. None of the ideas presented in this paper concerning mathematical concepts and methods of computation is original or new. All of them, as shown at the beginning of the paper, were either used before or at least considered, but somehow they were never fully developed and they never broadly spread.

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Pedagogical value of the Russian abacus and its use in teaching and learning arithmetic in the 19th – early 20th century

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Abstract

The paper is devoted to the history of use of various forms of the beads abacus in mathematics classrooms. The authors briefly present the history of the Russian beads abacus (счёты / schyoty), discuss its transmission to Europe, and focus on the didactical applications of various types of the beads abacus in mathematics classrooms in Western Europe and North America in the 19th and early 20th century. They claim that while counting instruments designed as arrays of sliding beads were relatively well known in Western Europe even before the 19th century, the introduction of manipulations with beads abacus in mathematical curricula was a rather complex process related to the emergence of new educational theories and practices.

Keywords: teaching and learning arithmetic, bead abacus, pedagogical innovations, 19th century Europe, abacus in Russian and North American schools

Introduction

Under different forms and names, the abacus remained a useful computational tool in many countries and cultures over the history of humanity and recently has gained popularity in the context of school reforms. Historians of science are well aware of the fact that abaci of various types appeared in numerous cultures (Greece, Roman Empire, Medieval Europe, China, Japan, Russia, etc.) during certain historical periods; these instruments were used to perform a variety of calculations, starting from the simplest ones, such as adding up costs of items sold on the market, to quite complex and sophisticated ones, for instance, extraction of square and cube roots. However, the didactical value and implications of the use of these abaci for mathematics instruction still remains underexplored, even though it is obvious that the history of the abacus is closely related to the history of didactical approaches devised to teach basic arithmetic operations. Indeed, during certain periods, instruments of various types appeared as the mandatory teaching and learning tools in various countries, sometimes only to completely disappear later being replaced by other tools.

The variety of counting devices is too large to be treated here, this is why we chose to focus only on one type of the instrument, namely, on the so-called ‘Russian abacus’ and its modifications that became popular among mathematics educators in Western Europe, North America, and in Russia/USSR in the 19th and early 20th century. This short paper, however, cannot provide a detailed account of the history of the instrument and of its applications in all the mentioned didactical contexts. This is why we decided to limit ourselves to a short discussion of a relatively small selection of cases and provide their detailed discussion in future publications.

Context and rationale of the present study

The present preliminary study of the Russian abacus is related to our ongoing investigation of the history of development of mathematics education in Russia. It is well known that the so-called Russian abacus (‘schyoty’) was widely used in Russia no later than the 17th century (Volkov, 2018 forthcoming); however, when working on L.F. Magnitskii’s (1669–1739) *Arithmetic* (1703), the first Russian printed arithmetic textbook, we found no references to this instrument. Instead, the author focused on written computations with clear reference to the Western arithmetical manuals of the 16–17th centuries (Freiman and Volkov, 2012; 2014; 2015). While working on a forthcoming book on the history of computing and computational devices (Freiman and Volkov, 2018, forthcoming), we clearly realized the important role that the bead abacus played in the mathematics instruction of Russia and Western Europe in the 17th–19th century (Karp, 2018, forthcoming; Volkov, 2018, forthcoming). This prompted our special interest in the history of the device and of its use in mathematics classroom.

Some authors suggested that the massive introduction of the bead abacus (also known as ‘numeral frame’ or ‘boulrier’) in (mainly) elementary school arithmetic classrooms in Western Europe and North America started with Jean-Victor Poncelet (1788–1867) who supposedly brought its Russian prototype to France (Guzevitch and Guzevitch, 1998). This hypothesis, which is directly related to the history of mathematics education as well as to the present-day debates concerning the role of counting instruments in the mathematics classroom, still deserves a further investigation (Volkov, 2018, in press).

The history of the didactical use of counting instruments and, in particular, of various forms of bead abacus in Europe and North America in the 19th and early 20th century, arguably, remains underexplored. This history is associated with the names of Johann Heinrich Pestalozzi (1746–1827), Samuel Wilderspin (1791–1866), and Marie Pape-Carpantier (1815–1878), who, among others, suggested various didactical applications of counting instruments and elaborated their own types of didactical devices (Kidwell et al, 2000; Régnier, 2003; Bjarnadóttir, 2014). Russian

connections were explicitly mentioned by numerous authors (for one of the earliest publications mentioning this point see Curie (1846)), yet it has not been often noticed that the very idea of the use of the bead abacus as ‘didactical tool’ was brought to Russian educators *from* Western Europe in the 19th and early 20th century. The concluding part of the present chapter deals with this particular element of the history of Russian abacus.

The origin of the Russian abacus

A definitive history of the Russian abacus (счёты; hereafter, ‘schyoty’) still has to be written. The earliest available materials in Russian language describing the construction and operations with the device are dated of the mid-17th century (Spasskiĭ, 1952). The first mentions of this instrument appeared in publications of West-European travelers who visited Russia in the late 17th century. One of them was Nicolaas (or Nicolaes) Witsen (1641–1717), a Dutch traveler who visited Russia in 1664 and in 1692 published his book titled *Noord en Oost Tartarye* [North and East Tartary] containing a very brief mention of the schyoty.¹ The origin of the instrument suggested by Witsen (the Golden Horde) was later identified by a number of authors with the Mongol Empire, and, subsequently, with China. This hypothesis about the origin of the Russian instrument was energetically rejected by Spasskiĭ (1952).²

The instrument described in the Russian manuscript mathematical manuals of the 17th century differed considerably from the one used in Russia starting from the early 18th century till the late 20th century. Unlike its later descendant, it contained two counting fields; the lower sections of each of these two fields were subdivided into two parts used for operations with common fractions of two types. The beads in the left section of the lower part represented fractions with denominators 2^n , $n = 2, \dots, 7$, while the beads in the right section represented fractions with denominators $3 \cdot 2^n$, $n = 0, \dots, 5$; see Figure 1.

The aforementioned manuscript manuals contained a table of identities that allowed conversion of fractions with denominators 2^n and $3 \cdot 2^n$ represented with the schyoty. The table was followed by a list of identities that allowed the operator reduce combinations of fractions of these two kinds to units or to relatively simple fractions, or combination of both; each identity was accompanied with a picture representing the respective configuration of beads on the schyoty. For example, the

1 Witsen (1692, p. 472). We would like to express our gratitude to Professor Jan van Maanen who kindly helped one of the authors (A. Volkov) obtain access to the 1692 edition of Witsen’s book preserved in the Library of Utrecht University.

2 For a detailed discussion of Witsen’s statement and the possible origin of the Russian instrument see (Volkov, 2018 forthcoming).

configuration shown in Figure 2 corresponded to the identity $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{6} + \frac{1}{12} + \frac{1}{48} + \frac{1}{96} = 1 - \frac{1}{4}$.



Fig. 1. Russian “Counting Board” (“Дщица счётная”) of the 17th century (Spasskii, 1952, p. 325).

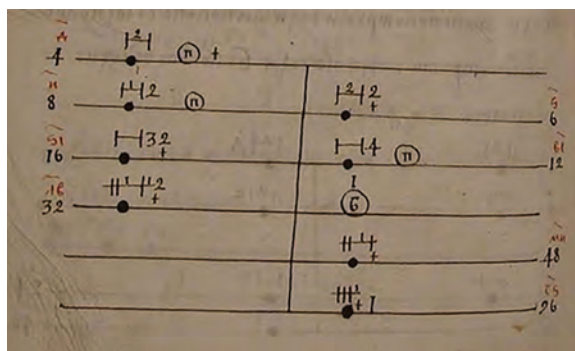


Fig. 2. Identity $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{6} + \frac{1}{12} + \frac{1}{48} + \frac{1}{96} = 1 - \frac{1}{4}$ represented with the Russian abacus ([Anon.], 1865, p. 110v.).

Poncelet and the introduction of the Russian schyoty to France

Michel Chasles (1793–1880) conjectured that a specimen of the Russian schyoty was brought to France by the French mathematician Jean-Victor Poncelet from his captivity in Russia that started during Napoleon’s invasion in Russia (1812) and lasted till 1814 (Chasles, 1843). According to Chasles, the Russian instrument was used for mathematics instruction in the schools of Poncelet’s hometown, Metz, after his return from Russia and later became popular in numerous schools across France. Modern historians did not find supportive evidence for Chasles’ statement; however, Gouzevitch and Gouzevitch (1998) argued that this hypothesis can be safely accepted since Poncelet most likely knew about the publication of Chasles of 1843 and yet never contested the latter’s conjecture. Even if Chasles is right, the Russian abacus was known in Europe well before the captivity of Poncelet: besides the abovementioned report of Witsen, there existed other rather detailed descriptions of the instruments, the earliest of which, provided by Jacob (Iacobus) Reutenfels in his *De Rebus Moschoviticis*, was published in 1680. This and other descriptions of the Russian schyoty and of the Chinese ‘suanpan’ widely circulated in continental Europe and in the United Kingdom prior to the early 19th century; see, for instance, the description of suanpan by John Barrow (1764–1848) (1804, pp. 295–297). These publications suggest that the beads abaci of Russian and Chinese type were known in France even before Poncelet’s captivity. It remains unknown how exactly the French teachers introduced this new teaching tool in their classrooms, and what was the role played by Poncelet in bringing the Russian abacus to French schools; we hope to return to these topics in a later publication. As far as the beads abacus is concerned, such an instrument was indeed used for instruction in the first half of the 19th century in some European countries and in North America (see below), yet it remains unclear how much the use of beads abaci in schools were influenced by the specimen supposedly brought by Poncelet. In this respect, possible connections of the use of abacus in schools to the pedagogical innovations introduced by the Swiss educator Johann Heinrich Pestalozzi are particularly interesting. Indeed, Jules Thurmann (1804–1855), in his *Principes de pédagogie* (1842), refers to the beads abacus (Fr. ‘boulrier’) as “Pestalozzi’s abacus” ([boulrier de Pestalozzi]) and states that it had been used in German and French educational institutions called “salles d’asiles” to provide young children with basic arithmetic instruction. The claim about Pestalozzi’s use of the abacus (or of *an* abacus) thus certainly deserves a further investigation.

Pestalozzi’s tables and the Russian abacus

The name of Pestalozzi is often cited when talking about didactical ideas of using particular ways to visualise numbers that may help grasping the concept of number in intuitive way. However, to the best of our knowledge Pestalozzi himself never

mentioned the Russian or Chinese abaci in his work; instead, for teaching arithmetic he used tables. Not a mathematics teacher himself, he conceptualized some (apparently, believed to be innovative at that time) teaching approaches; among them was the use of table of units supposed to enhance the development of mental faculties based on visualization (*Anschauung*) emphasizing sense-impression, observation, perception, and intuition (Bjarnadóttir et al., 2013, p. 30).

1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9	9
10	10	10	10	10	10	10	10	10	10

Fig. 3. Pestalozzi's table representing natural numbers from 1 to 10 in 10 columns (Galanin, 1912, p. 9, Figure 1).

The table shown in Figure 3 is divided into ten rows; each row contains ten cells with vertical line segments, all of the same length. The first row has ten cells each containing one vertical line, the second has ten cells with two vertical lines, and so on. Daniel-Alexandre Chavannes (1765–1846) who visited Pestalozzi's school, describes how the instructors taught students to count lines following, for example, the first row: one, two times one, three times one, etc. For the second row it would be one time two, two times two, three times two, and so on. The same way of counting was used for the rest of the rows. Further on, the table was used to help students visualize how many fives are contained, for example, in 37: the student would examine the fifth row to see that seven times five would be equal to 35; this product plus two units would altogether result in 37 (Chavannes, 1805, p. 32).

Beads abacus in French 'salles d'asile'

The use of abacus in French schools for infants [*salles d'asile*] is another interesting case. During the 19th century special schools for children of 2-6 years of age were created in France. Their development was guided by increasing emphasis on the importance of education and schooling, which included basics of arithmetic. A

large-size abacus was a part of mandatory equipment of every classroom and was supposed to be used to perform demonstrations of manipulations and operations with (integer) numbers. The earliest instruction of the Ministry of Education concerning the use of this instrument that we were able to find is found in an official letter sent to all the directors of the ‘salles d’asile’ in April 1836 and titled *Instruction relative à l’établissement et à l’organisation des salles d’asile* [Instruction concerning the establishment and organization of the schools for infants]. It prescribed that each classroom should have a beads abacus “un boulier-compteur ayant dix rangées de dix boules chacune” [a beads calculator having ten rows of ten balls each] (Duruy 1865, p. 440). Later, “des exercices avec le boulier-compteur” [exercises with beads calculator] were mentioned in the program of examinations of the applicants for position of “directrice de salle d’asile” of 1856 (Duruy 1867, p. 22). The quality of the exercises with the abacus was one of the evaluation items of the candidate (*ibid.*, p. 23). The abacus is again mentioned in one of the documents of 1899 as one of the classroom accessories (Duruy 1902, p. 698).

The abacus of Marie Pape-Carpantier

Marie Pape-Carpantier is known as the founder of a system of kindergartens whose curriculum included arithmetic (Cosnier, 2003); she claimed that she invented her own abacus (which she called “boulier numérateur”) in 1868 and recommended it to be used to support teaching and learning of arithmetic (Pape-Carpantier, 1878, pp. 17-18; Régnier, 2003). The instrument was supposed to be used in classroom; it consisted of a large vertical wooden frame with bars and sliding beads. The main difference between Pape-Carpantier’s abacus and its precursors consisted of the presence of ‘broken’ bars, that is, bars having horizontal and vertical parts; see Figure 4.

The *Manuel de l’Institutrice* [Manuel of the Educator], co-authored by Pape-Carpantier, provides a description of the instrument (Pape-Carpantier, Delon, & Delon, 1869, pp. 199-200); the instrument apparently was a modification of the bead abacus [boulier-compteur] that had been used for some time in France (see the previous section). The modified ‘boulier’ invented by Pape-Carpantier consisted of a wooden frame 70 cm of height and 60 cm of width crossed by 11 wires, two of which (on the top) were horizontal (and contained, according to Figure 4, 15 beads each), while nine other bars comprised vertical and horizontal parts. To set a digit, the operator was supposed to place the given number of beads on the vertical section of the respective bar (Pape-Carpantier, Delon & Delon, 1869, p. 199).

The extant specimens of Pape-Carpantier that we inspected had the ‘broken’ bars organized in three groups of three bars each; the color and size of beads in each group was different from those of the beads in two other groups. The difference of sizes was insisted upon by Pape-Carpantier in her description; it was supposed to

inform the learner about the relationships between units (that is, powers of 10) in different positions. Designed to introduce the basics of decimal system and elementary operations, the abacus could have been used to show, for example, that 9 units on the first wire on the right could not form ten in other way than by using one bead on the second wire representing tens, and so on. However, while Pape-Carpantier and her co-authors mention that while being an excellent tool to introduce counting and number system to beginners in a visual and exact way (“boulrier ... parle aux yeux et ne permet ni incertitude, ni erreur”, that is, “the abacus... speaks to eyes, and does not allow either uncertainty or error”), the abacus *per se* does not guarantee an understanding of the structure of numbers and could become just a tool for counting beads from 1 to 9 without grasping the concept of decimal position. Hence its use, according to them, should be combined with other counting tools to count different types of objects.

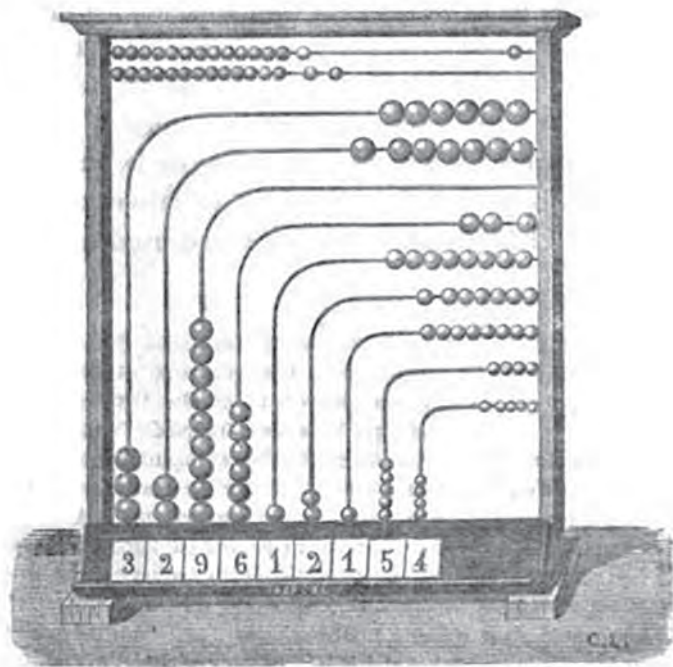


Fig. 4. “Boulrier-numérateur” of Marie Pape-Carpantier (Pape-Carpantier, Delon & Delon, 1869, p. 199).

Interestingly enough, Pape-Carpantier did not know that the abacus that had been used in France originated from the Russian schyoty and believed instead that it came from Ancient Greece:

L'instrument employé au début dans les Salles d'Asile françaises pour l'enseignement de la numération est le boulier-compteur, imité de l'abaque des Grecs. On en a imaginé un plus en rapport avec la numération décimale qui n'existait jamais dans l'antiquité. L'ancien boulier, comme l'abaque, offre à l'enfant des boules enfilées par des tiges horizontales, et ne peut servir que pour la numération par unités. En disposant les tiges verticalement, et en y plaçant des boules de grosseurs graduées comme les ordres d'unités dans la numération décimale, l'instrument se prête à toutes les opérations et démonstrations du calcul [...]. (Pape-Carpantier et al., 1887, p. 211)

Abaci in the United Kingdom

In the late 18th and early 19th century descriptions of beads abaci of various types (esp. of the Chinese suanpan) widely circulated not only in continental Europe but also in the United Kingdom; see, for instance, the description of suanpan by John Barrow (1764 - 1848):

Their [i.e., Chinese. – V. F. & A.V.] arithmetic is mechanical. To find the aggregate of numbers, a machine is in universal use, from the man of letters, to the meanest shopman behind his counter. By this machine, which is called a Swan-pan arithmetical operations are rendered palpable. [...] [A detailed description of the Chinese suanpan follows. – V. F. & A. V.] This is clearly a system of decimal arithmetic, which, for the ease, simplicity, and convenience of its operations, it were to be wished was generally adopted in Europe, instead of the endless ways in which the integer is differently divided in different countries, and in the different provinces of the same country. The Swan-pan would be no bad instrument for teaching to a blind person the operations of arithmetic. (Barrow, 1804, pp. 295-297)

It is important to stress that Barrow does not mention education of young children and instead suggests that the abacus may be used for teaching blind persons. The idea that a tangible representation of numbers may have been used for such teaching was advanced by other authors before him; see, for example, the writings of Nicholas Saunderson (1682–1739).³ In English it was called “palpable arithmetic” (this name was given by John Colson, 1680–1760). The instrument of Saunderson was quite enthusiastically presented in the textbook of Christian Wolff (1679–1754).⁴

In 1819 Samuel Wilderspin, whose fundamental principle was to introduce arithmetic to children on the basis of experience with material objects prior to introducing the abstract concept of a number, experimented with buttons moving along a screen to help children with counting. Spending years in inventing a numeral

3 See Saunderson, 1741, pp. xx-xxvi. See also Tattersall, 1992, p. 358.

4 Wolf refers to Saunderson's *Elements of Algebra*, published in 1741; see Wolf, 1747, vol. 1, pp. 71-79; 1777, vol. 1, pp. 145-154.

frame, he designed two original versions of beads abaci to be used in the “infant schools” for children of one to seven years of age established across the United Kingdom. The first device called “Transposition frame” was designed as a beads abacus with 12 bars with 1, 2, ..., 12 beads of the same color; the second device called “Arithmeticon” was designed as a large size vertical wooden frame with twelve horizontal wires each having 12 black and white sliding beads.

Pestalozzi’s work drew the attention of English educators of the mid-19th century. For instance, F. Curie (1845; 1846) analyzes methods of teaching the whole numbers by comparing Tables of Units (from Pestalozzi’s treatise *Exercises of Arithmetic for Elementary Schools*) with what he calls “Russian Balls Frame” (that is, a beads abacus resembling the Russian schyoty) and which, according to him was “little known to this country” (Curie, 1845, p. 131). While comparing it to Pestalozzi’s table, the author finds that the device is “still more mechanical and tangible apparatus;” yet, without diminishing Pestalozzi’s merits, the printed textbooks and diagrams accompanying them would hardly “arrest and engage children’s attention for any length of time” whereas the infant’s mind would be “immediately interested and impressed” with the “Ball Frame” and its coloured balls and wires. The work with the abacus thus would help the instructor to introduce to the learners “any collection of units, tens, hundreds, or thousands” as well as to rapidly and clearly show “the relations and various factors in multiplications.” Curie’s abacus, the “Russian Ball Frame,” is shown in Figure 5. The author mentions the possibility of placing the frame vertically or horizontally, while in his figure the instrument is placed in such a way that the bars are vertical; this orientation of the instrument is similar to that of the Chinese abacus and not of the original Russian one. With nine balls on each wire, the nine digits in each position can be represented. (In Figure 5, position *a* is that of units, *b* of tens, etc.; wire *f* thus represents hundreds of thousands whereas the wires to the right of wire *a* represent the decimal fractions 10^{-1} , 10^{-2} , etc.).

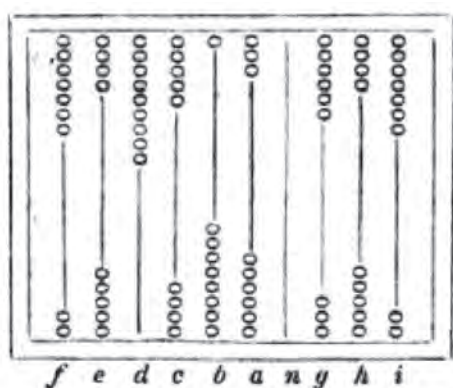


Fig. 5. “Russian Ball Frame” according to Curie (1845, p. 132).

North American experience

In the United States, William S. Phiquepal (1779–1855), a follower of Pestalozzi, used an abacus with 10 wires and 10 beads on each wire in 1825. Some authors noticed differences between his abacus and the instruments designed by British educators (Wilderspin and Wilson): the numbers of the wires, of the beads, and the shape of the beads were all different from those of the British instrument. It was suggested that Phiquepal brought the instrument from France (Kidwell, Ackerberg-Hastings, and Roberts, 2008), but it is unknown whether he knew about the Russian abacus (or its modifications) used in France. According to L. Dunton (1828–1899),

the best kind [of abacus. – V.F. & A.V.] is composed of a wooden frame about four feet long and two feet wide, in which, running horizontally from end to end, are fastened ten brass or steel rods; on each of these rods are ten easily moved wooden balls about an inch and a half in diameter. The whole is supported at a convenient height by means of upright standards attached to bars running crosswise at the bottom. It is well to have the balls painted different colors, say three red ones at the left on each wire, then three yellow ones, and four green ones at the right ; or, two and two, black, red, yellow, green, and white. One-half the frame should be covered with a board, so as to conceal all the balls that are not used in any example. (Dunton, 1888, p. 10)

As particular advantages of the tool, Dunton listed (*ibid.*):

- Balls can be easily seen across the classroom;
- Different colors used for the balls help student (even the farthest) see their number;
- It can be used in introducing first steps of counting, and perhaps later on during exercises (on arithmetic operations).

Although the author mentioned that different (and more sophisticated) forms of bead abacus had been invented, he argued that the simplest ones were the best to assist teaching.

Schyoty as computing device in Russia in the 19th and 20th century

It appears plausible to conjecture that the interest of Western authors in the Russian abacus (schyoty) expressed in their publications of the early 19th century soon became known to Russian readers and drew their attention to the instrument. An important role was played by a retired General of Russian Army, Fedor Mikhaïlovich Svobodskoï (b. in 1780s or early 1790s; d. 1829). The instrument of Svobodskoï was a combination of several “counting fields” (usually 12, but sometimes even 30) each of which was a set of horizontal bars with beads, that is, a specimen of traditional schyoty. Two “fields” were used to set the initial data, and the other fields

were used for intermediate results. A special commission of the Russian Academy of Sciences explored the instrument of Svobodskoĭ and approved its use for educational purposes in all Russian universities. Numerous manuals, for instance Orlitskiĭ (1829a; 1829b), Tikhomirov (1830), explained the operations performed with the instrument of Svobodskoĭ (Spasskiĭ, 1952). However, in the present paper we shall not discuss the use of the beads abacus and its modification in Russia and the Soviet Union, since the instrument was used primarily for actual computations and not for educational purposes (hence the university courses devoted to its use); the old traditions of operations with it were presumably followed by the authors of the manuals of the 19th and early 20th century. There exist numerous pieces of evidence suggesting that the manipulations with schyoty were studied by merchants, accountants, salesmen, clerks and office workers before and after the Socialist Revolution of October 1917. Despite the innovations of Svobodskoĭ, the instrument used the most was the traditional schyoty with only one counting field.

Teaching tools in Russian schools: a possible German influence?

While the bead abacus was long time used in Russia in everyday life, the history of its didactical application for supporting teaching and learning arithmetic had not been explored till the beginning of the 20th century. Dmitriĭ D. Galanin in his history of the use of visual manipulatives in teaching and learning arithmetic in Russian schools (1912) suggested the attention of Russian educators of the 19th century to didactical applications of the beads abacus was drawn by Russian educators under a clear influence of their German counterparts; however, it remained unclear how and when the abacus should have been used in Russian classroom (Galanin, 1912, p. 23).

Galanin claims that the Russian beads abacus (schyoty) was brought to Germany in 1812 by Russian soldiers (and not by Poncelet mentioned above),⁵ and that later on, German educators understood their didactical value as visual support to be set vertically in the classroom, so all students could see it (1912, pp. 22-23). On the contrary, Galanin found it meaningful to return to the primary (i.e. directly related to the real-life computations) role of the bead abacus: according to him, it was the device for counting and calculations to be put “in the hands of every student” (1912, p. 25). However, despite all the presumed usefulness of the bead abacus for teaching arithmetic, Russian educators saw it as only one of many other tools invented or used abroad, in particular in Germany where, according to Wilhelm August Lay

5 This claim of Galanin is not supported by any evidence and, most likely, is purely conjectural. Another conjecture of Galanin (1912, p. 22), namely, that a form of beads abacus (a board with wires and rings sliding on them) was mentioned as early as 1522 in a work of Adam Riese (1492 or 1493–1559), is equally questionable: the instrument mentioned by Galanin is apparently the “counting on lines”, the well-known medieval counting device using round tokens placed on a counting board (or any flat surface) with drawn parallel lines.

(1862-1926) cited in (Galanin, 1912, p. 13), more than 200 different visual manipulatives were used. Among the tools most frequently mentioned in the literature devoted to the use of manipulatives in arithmetic teaching, Galanin (1912, pp. 19-20) mentions the invention of Ernst Tillich (1780–1867) called “arithmetic box,” as well as graphical figures ‘Zalenbilders’ introduced by the German mathematician and physicist Friedrich Gottlieb von Busse (1756-1835) in 1786 (Smith, 1903, p. 77) and further developed by other educators in the 19th century (Figure 6).

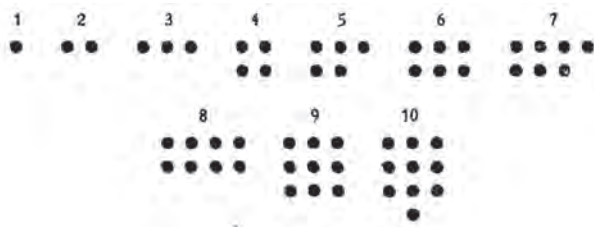


Fig. 6. Arithmetical figures of F. G. von Busse (from Galanin 1912, p. 25).

Galanin also analyses in details Pestalozzi's tables which were apparently not very efficient in doing calculations mentally without visual support; these difficulties, according to Galanin, led the Pestalozzi's followers (among them Johann Joseph Schmid, 1785–1851, a pupil and later a collaborator of Pestalozzi) to look for other visual tools to help students more efficiently. The use of manipulatives was also discussed by a renown Russian educator P.S. Gur'ev (1807–1887) whose didactical ideas are worth mentioning. In the context of Russian schools of the 19th century, the use of visual manipulatives in general, and of the bead abacus in particular, were subjects of numerous debates among educators. It remains not clear what was the place of the bead abacus in the portray of school arithmetic in the beginning of the 20th century and more detailed investigation is therefore needed.

Conclusions

The fact that the abacus was enthusiastically embraced by the educators of Western Europe and North America in the first half of the 19th century leads us to a deeper look into this case from the didactical perspective which was not necessarily related to the simplicity and efficiency of the instrument itself but rather to the new (at that time) didactical theories that stressed the importance of manipulations with physical objects in learning basic mathematical concepts and operations. Some of these theories, in particular, those found in the works of Pestalozzi, especially his concept of object-teaching method based on the intuitive, visual and hands-on pedagogical perspective had (and still has) a great influence on educational systems worldwide.

We would like to stress that the interest of Western educators in the abaci of Russian and Chinese type and their attempts to use them in schools in the 19th century were not related to the discovery of these instruments which, as we have shown in the opening sections of this chapter, had been described and were relatively well known much earlier. We argue that their reappearance (this time, in educational context) was related to certain shifts in didactical theories which implied development of new practices, in particular, manipulations with tangible (and not abstract) objects. At that moment the old and relatively well-known abaci (Russian, Roman, Chinese) came into focus and started to be explored and used by the educators. Interestingly, this was done not in the countries of origin of these instruments (Russia and China) where they were still used as mere tools for calculations, but in the countries where they were considered ‘exotic’ and ‘unusual’. It is also worth noting that not only the functions of the instruments were modified (from computing device to educational tool) but also the very operations with them were redesigned: the original operations with the instruments used in Russia and China most likely remained unknown to the educators who had to develop their own ‘software’ mainly dealing with the most elementary operation to be performed on the abacus.

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Jacob de Gelder (1765-1848) and his way of teaching cubic equations

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Abstract

Jacob de Gelder was a mathematics teacher who wrote many textbooks, inspired by his teaching experience. In his books you can almost hear him talking to, discussing with or questioning his pupils. He combined two styles, mathematical correctness and emphatic expression just like he did in his classroom.

Keywords: cubic equations; Jacob De Gelder; didactics; Socratic approach

Introduction

Jacob de Gelder (1765-1848) was a remarkable mathematics teacher with an intriguing career in the Netherlands in pre- and post-Napoleonic times. He wrote textbooks about mathematics for his pupils and books about mathematics and didactics for teachers and mathematicians.

These works are still of interest, as I will illustrate by the way in which De Gelder taught the solution of cubic equations. To modern eyes, the mathematics of solving cubic equations algebraically may seem old-fashioned, but his insights in didactics and pedagogics are still worth remembering.



Fig. 1. Jacob de Gelder, portrait by Berghaus, collection Universiteit Leiden - Academisch Historisch Museum

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Biography

De Gelder wrote in Dutch for a wide audience. Nowadays his works are studied by those who take an interest in Dutch history of mathematics education. Books and articles about De Gelder have been written in Dutch (van der Aa, 1862; Beckers, 1996b, 1996c; Smid, 1997), and in English (Beckers, 1996a, 2000). The following biography is mainly based upon these articles and books unless indicated otherwise.

Jacob de Gelder was born in 1765 near Rotterdam in a lower-middle-class family. He went to a so-called French school, which served as a good education and also prepared for a career in, for example, business, administration or engineering (Kruger, 2014). Latin was not taught, so unlike the Latin school, a French school did not give access to university. After finishing this school De Gelder worked as an assistant teacher at the same school. A few years later, he opened his own school in Rotterdam and recruited pupils who prepared for a naval career. In 1793 he published his first book *Grondbeginsels der Cijfferkunst* [Principles of Arithmetic] (Gelder 1793).

After the upheaval of the French Revolution, things changed in the Netherlands too. In 1795, the Batavian Republic replaced the old government and many institutions were reformed. Between 1806 and 1810, Louis Napoleon, brother of the emperor Napoleon, was king. After 1810 the Netherlands were annexed by Napoleon. In 1815 the Netherlands became an independent kingdom, including Belgium and Luxemburg. In 1830 Belgium separated and became independent, but it took the Dutch king nine years to accept that fact. Jacob de Gelder witnessed all these political events.

De Gelder had to close his school in 1795 and had difficulty in getting employment elsewhere. Thanks to his friend, the Amsterdam professor Van Swinden, De Gelder was invited to work for the Dutch triangulation project, mapping the Netherlands. Possibilities were also provided by cultural and scientific societies, which were very popular in those days. Some societies owned prominent buildings where their members met, stimulated each other in reading and writing articles and discussed current topics in science, art, and politics. De Gelder taught mathematics to the members of the society *Diligentia* in The Hague and their children. These lessons resulted in a new book on algebra, *Wiskundige lessen* [Mathematical lectures] (Gelder, 1808). During the reign of King Louis Napoleon, De Gelder took care of the mathematics education of his royal staff. In that period, De Gelder was offered a position at a newly planned Military Academy, but this project was not realized until a Military Academy was founded in 1814 by King William I, with De Gelder as professor of mathematics.

At first, the commander of the Military Academy, Voet, praised the didactics of De Gelder, but later he changed his mind. Conflicts arose about how much

mathematics should be taught to non-mathematicians and how rigorous. Eventually, in 1819, De Gelder was dismissed. Later in his life, De Gelder was also embroiled in other conflicts, mainly concerning his ideas about teaching mathematics. Although De Gelder was not explicit in his books in the early years, we now know that he wanted his pupils to start reasoning for themselves. For a modern reader this idea might be common, but in those years, it was innovative to write that you want pupils to learn why things are true or not true.

An interesting difference of opinion arose in 1821 when De Gelder was asked to teach mathematics at the Leiden Latin school. In Latin schools, mainly Greek and Latin were taught, and De Gelder ran into conflict with his principal, Bosse, concerning the question of how much mathematics should be taught. According to De Gelder, mathematics was necessary, but Bosse completely disagreed. According to him, De Gelder was too ambitious and too demanding, reaching far beyond what his pupils needed. Beckers (1996a) puts it this way:

De Gelder thought that mathematics was indispensable to anyone: to him, mathematics gave gateway to a way of thinking which prevented one from making mistakes. Mathematics gave certainty, mathematics provided a way of dealing with problems, which was far more powerful than any other science or knowledge. To him it was inconceivable not to educate people in mathematics: for De Gelder mathematics had a propaedeutic function in the education of people.

To Bosse and Voet however, mathematics was a more or less convenient way of solving a limited number of problems. To them, mathematics was important for people who were able to put it to immediate practical use. Mathematically proven truth was not worth more to them than any other kind of truth.

These conflicts show that De Gelder possessed a strong opinion on mathematical education. Although he lost his job in these conflicts, they did not destroy his career. In 1819, he became a professor by special appointment at Leiden University and in 1824 he was appointed a regular university professor of mathematics there. In 1826, the Dutch government decided to require the teaching of mathematics at all Latin schools and the books of De Gelder were recommended. Finally, his son started the *Pedagogium*, a pre-university boarding school, in Leiden. Thanks to the sketches and the diary of one of his son's pupils, we have an idea of how pupils experienced life at such a school (Bervoets, 1985; Bervoets & Chamuleau, 1985). Jacob de Gelder died in 1848. He had seen many changes in his country and with regard to mathematics education, he wanted things to change and was one of the actors.



Fig. 2. Jacob de Gelder around 1835-1840 by Alexander Verhuell, collection: Gelders Archief

How to teach mathematics

De Gelder wrote his ideas about education and mathematics in his book *Verhandeling over het verband en den samenhang der zedelijke en natuurlijke wetenschappen* [Treatise on the relationship and the coherence of the moral and natural sciences] (Gelder, 1826). The fifth chapter deals with his way of teaching. De Gelder wrote that he was teaching young pupils aged 12 to 16 and warns us not to overload these young children with too much homework. He wrote that some teachers complained that these young children cannot learn anything at all. De Gelder replied that these teachers forgot the playful and inventive monkeyshines of their pupils in which they display their shrewdness and prove that they do learn a lot (but outside the official curriculum).

De Gelder distinguished between two ways of teaching. One is the perfect demonstration by a well-prepared teacher who knows everything and explains all details in the right order. The pupils hear him talking and feel that the teacher is right, but would not know how to find a mistake. In this situation, the pupil accepts the teacher's reasoning because the teacher is their master. Pupils can repeat what their teachers have said, but do not know how to say it in their own words. Pupils cannot make small modifications because the reasoning is not their own reasoning. The other way is the Socratic discussion where the teacher keeps on asking questions, forcing the pupils to think for themselves, to internalize the reasoning, to find words of their own. When a pupil is really sure about the logic, when he masters the topic, he is able to explain things himself. Now, a pupil is in a different position during a lecture, because he is able to detect (small) errors during a not-so-perfect demonstration by the teacher, or even correct these errors.

Critics may say that De Gelder presented examples of successes only, suspecting that he quoted his best-performing pupils and that he made such small steps during the Socratic discussion that he was almost feeding them the right answers. Indeed, his examples are too good to be true according to modern standards, however, one cannot say that De Gelder is hiding his belief in the power of the Socratic way of discussion.

Solving cubic equations

Solving cubic equations is a good example to demonstrate De Gelder's way of teaching. From the seventeenth through the early twentieth centuries, the algebraic solution of polynomial equations of degree 3 and 4 was treated extensively in many textbooks; see for example *De gebeele Mathesis of Wiskunst* [The whole mathematics] (Graaf, 1676), *Wiskundige lessen* [Mathematical lectures] (Gelder, 1808), *De theorie en de oplossing van hoogere magtsvergelijkingen* [Theorie and solution of higher degree equations] (Ven, 1864) *Lobatto's lessen over de Hoogere Algebra* [Lobatto's lessons on advanced algebra] (Rahusen, 1892) or *Lessen over de Hoogere Algebra* [Lessons on advanced algebra] (Schuh, 1924).

In *Beginselen Der Stelkunst* [Fundamentals of algebra] (De Gelder 1819), Jacob de Gelder starts chapter 9 of Section 2 with an introduction on the Italian mathematician Cardano (1501–1576). He explains Cardano's approach for solving the cubic equation in modern notation and derives the single real solution, which Cardano found. De Gelder then refers to a previous chapter in which he states that a cube root of any (non-zero) number always has three possible (real or complex) values. By doing so, he raises the question concerning the remaining two solutions of a cubic equation with three real roots (De Gelder uses the word “bestaanbaar”). Patiently he explains how to construct the other two solutions of the general cubic equation. He urges the reader to try this himself. He stimulates the reader to write down the general case so that the reader is really aware that the construction is valid in general. This example makes clear what De Gelder is up to: he wants his students to accomplish true general mathematical reasoning without numerical examples. One can imagine that this way of doing mathematics was much more demanding for his pupils and colleagues than simply doing numerical examples and exercises.

De Gelder continues by asking questions about the existence of real solutions and wants his readers and pupils to investigate the mathematical expressions of each intermediate step. He shows that mathematical reasoning will tell you under which circumstances all three solutions are real, and also when there is only one real solution. He does not present a prescription but tries to explain his reasoning. One of the particularities of the cubic equation is that attempts to simplify its solution algebraically might result in a new cubic equation which means that you get stuck in an

endless loop without getting any further. De Gelder explains that this is sometimes unavoidable. See below for an example.

Audience

De Gelder wrote a number of textbooks and revised some of them. In the preface of *Wiskundige Lessen I* (Gelder, 1808, pp vii) one can read that this book was meant for young pupils, aged 13 or 14 years or older. According to the title page, the book had to be used by teachers in the classroom. De Gelder often wrote sentences like “the pupil should ...”, and this book is no exception. In the preface of *Wiskundige Lessen II* (Gelder, 1809, pp vi) one can read that this book was meant for pupils, aged 15 years or older. In the preface of *Beginselen der Stelkunst* (Gelder, 1836, pp v), the book is addressed to the young people starting to learn mathematics.

Language

It is characteristic for De Gelder and his contemporaries to use emotional expressions such as “zwarigheid” and “voor altijd moet wanhopen”. In English, one would say “severe difficulty” and “despair forever” or even “despond”. Here you see that De Gelder is talking to pupils, non-mathematicians, young or old, who might get stuck in their attempts to learn mathematics. De Gelder’s textbooks reflect his preference for Socratic reasoning, consisting of a dialogue with questions and answers. In my opinion, this chapter on the cubic equation is a good example of how to guide pupils through this topic.

Mathematics

We will now look at an example of De Gelder’s explanation of what he calls the severe difficulty and what he calls the unavoidable endless loop. In *Wiskundige Lessen* (Gelder,

1828), De Gelder introduces the equation $x^3 + px + q = 0$. He mentions that, ac-

cording to Cardano’s formula, $x = \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}} + \sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}}$ is a root of that equation. He pays much attention to the case where $\frac{1}{4}q^2 + \frac{1}{27}p^3 < 0$

In that case, he points out that the square root $\sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}$ does not exist because square roots of negative numbers do not exist. De Gelder does not use the modern terms real or complex. Instead of i he writes the square root of minus one, $\sqrt{-1}$. On the one hand, the square roots of negative numbers do not exist, but on the

other hand, the solution of the cubic equation does exist. He calls this situation a severe difficulty.

De Gelder did not present a numerical example, but in the classroom, he might have given an equation such as $x^3 - 6x + 4 = 0$ to specify the difficulty. It is quite obvious that $x = 2$ is a solution. According to the formula, the solution is $x = \sqrt[3]{-2 + \sqrt{-4}} + \sqrt[3]{-2 - \sqrt{-4}}$ and it is not obvious at all that this is the same as $x = 2$. At first sight, this sum of roots looks like a negative number in combination with a complex number. I give three more examples to underline the importance of the invention of complex numbers. Understanding the roots of complex numbers, one can easily prove that $\sqrt[3]{-2 + \sqrt{-4}}$ is the same as $1 + \sqrt{-1}$ and that $\sqrt[3]{-2 - \sqrt{-4}}$ is the same as $1 - \sqrt{-1}$. This example may give the impression that it is always possible to rewrite the Cardano solution. However, there is no general formula which for given a and b , and for unknown c and d provides the simplification of the expression $\sqrt[3]{-a - b\sqrt{-1}}$ to $c + d\sqrt{-1}$ in which c and d are stated in a and b . De Gelder would have stimulated his readers to solve the equation $\sqrt[3]{-2 + \sqrt{-4}} = c + d\sqrt{-1}$ by means of algebra. Do you think you will find $c = 1$ and $d = 1$? One may also experience De Gelder's severe difficulty when trying to solve the equation $x^3 - 6x + 2 = 0$ and simplifying its root $x = \sqrt[3]{-1 + \sqrt{-7}} + \sqrt[3]{-1 - \sqrt{-7}}$. In this case, there is no integer solution. For the general equation $x^3 + px + q = 0$, De Gelder investigates the possible simplification of a root $x = \sqrt[3]{h + k\sqrt{-1}} + \sqrt[3]{h - k\sqrt{-1}}$ where $h = -\frac{1}{2}q$ and $-k^2 = \frac{1}{4}q^2 + \frac{1}{27}p^3$. He tries to do this by raising this cubic root to the third power.

Here are the successive steps in his reasoning.

$$\begin{aligned}
 x &= \sqrt[3]{h+k\sqrt{-1}} + \sqrt[3]{h-k\sqrt{-1}} \\
 x^3 &= \left(\sqrt[3]{h+k\sqrt{-1}} + \sqrt[3]{h-k\sqrt{-1}} \right)^3 \\
 x^3 &= \left(\sqrt[3]{h+k\sqrt{-1}} \right)^3 + 3 \left(\sqrt[3]{h+k\sqrt{-1}} \right)^2 \cdot \left(\sqrt[3]{h-k\sqrt{-1}} \right) + 3 \left(\sqrt[3]{h+k\sqrt{-1}} \right) \cdot \left(\sqrt[3]{h-k\sqrt{-1}} \right)^2 + \left(\sqrt[3]{h-k\sqrt{-1}} \right)^3 \\
 x^3 &= (h+k\sqrt{-1}) + 3 \cdot \sqrt[3]{h+k\sqrt{-1}} \cdot \sqrt[3]{h-k\sqrt{-1}} \cdot \left(\sqrt[3]{h+k\sqrt{-1}} + \sqrt[3]{h-k\sqrt{-1}} \right) + (h-k\sqrt{-1}) \\
 x^3 &= 2h + 3 \cdot \sqrt[3]{(h+k\sqrt{-1})(h-k\sqrt{-1})} \cdot \left(\sqrt[3]{h+k\sqrt{-1}} + \sqrt[3]{h-k\sqrt{-1}} \right) \\
 x^3 &= 2h + 3 \cdot \sqrt[3]{h^2 + k^2} \cdot \left(\sqrt[3]{h+k\sqrt{-1}} + \sqrt[3]{h-k\sqrt{-1}} \right) \\
 x^3 &= 2h + 3 \cdot \sqrt[3]{h^2 + k^2} \cdot x \\
 x^3 - 3 \cdot \sqrt[3]{h^2 + k^2} \cdot x - 2h &= 0
 \end{aligned}$$

In the sixth line, there are still two cube roots. One of them is the unknown x . The final line is therefore a cubic equation, but things are even worse. Because

$h = -\frac{1}{2}q$ and $-k^2 = \frac{1}{4}q^2 + \frac{1}{27}p^3$ the equation $x^3 - 3 \cdot \sqrt[3]{h^2 + k^2} \cdot x - 2h = 0$ can be rewritten as $x^3 + px + q = 0$. Note, this is the equation De Gelder started with. So, this is the unavoidable loop. Although a real solution exists, which can be very simple in some cases, there is no general way to simplify the formula. It is not possible to get rid of the sum of cube roots and in the case of three real roots, it is not possible to get rid of the complex numbers.

Therefore, De Gelder calls the case $\frac{1}{4}q^2 + \frac{1}{27}p^3 < 0$ the irreducible case. He stresses that irreducible does not mean that you cannot solve the equation. It rather means that you cannot express the result of the formula, the sum of two cube roots as one number or one single cube root.

De Gelder was aware that his readers might despair because it might seem to them that they got stuck in an endless loop of rewriting expressions, always returning to a similar cubic equation with a square root that did not exist.

When we elaborate on the example of the equation $x^3 - 6x + 4 = 0$ we will get the same unfortunate result. We start with the Cardano solution, raise it to the third power and rewrite the expression. So $p = -6$ and $q = 4$ therefore $h = -2$ and $k = -4$. In the fifth line, at the right, we recognize our x , the Cardano solution. So we end up with the original equation and there is no trace of the simple solution

$x = 2$. See the following deduction and check that the final line is identical to the equation we started with.

$$\begin{aligned}
 x &= \sqrt[3]{-2+\sqrt{-4}} + \sqrt[3]{-2-\sqrt{-4}} \\
 x^3 &= \left(\sqrt[3]{-2+\sqrt{-4}} + \sqrt[3]{-2-\sqrt{-4}} \right)^3 \\
 x^3 &= \left(\sqrt[3]{-2+\sqrt{-4}} \right)^3 + 3 \left(\sqrt[3]{-2+\sqrt{-4}} \right)^2 \cdot \left(\sqrt[3]{-2-\sqrt{-4}} \right) + 3 \left(\sqrt[3]{-2+\sqrt{-4}} \right) \cdot \left(\sqrt[3]{-2-\sqrt{-4}} \right)^2 + \left(\sqrt[3]{-2-\sqrt{-4}} \right)^3 \\
 x^3 &= (-2+\sqrt{-4}) + 3\sqrt[3]{-2+\sqrt{-4}} \cdot \sqrt[3]{-2-\sqrt{-4}} \cdot \left(\sqrt[3]{-2+\sqrt{-4}} + \sqrt[3]{-2-\sqrt{-4}} \right) + (-2-\sqrt{-4}) \\
 x^3 &= -4 + 3\sqrt[3]{(-2+\sqrt{-4})(-2-\sqrt{-4})} \cdot \left(\sqrt[3]{-2+\sqrt{-4}} + \sqrt[3]{-2-\sqrt{-4}} \right) \\
 x^3 &= -4 + 3\sqrt[3]{8} \cdot x \\
 x^3 &= -4 + 6x \\
 x^3 - 6x + 4 &= 0
 \end{aligned}$$

De Gelder stimulated his pupils and readers to investigate mathematical truth themselves. He might have given another example, like the equation $x^3 - 6x + 2 = 0$ in order that they experience the severe difficulty, the endless loop and the despair. You may try to find all three real solutions by means of algebra.

After this algebraic approach, De Gelder continued with numerical approximations and with a trigonometric solution in order to present a complete mathematical overview of cubic equations. It is almost unbelievable that young pupils aged 13 - 15 years were able to master this kind of mathematics.

Influence

After De Gelder, other Dutch authors, like Lobatto (1797–1866), Van der Ven (1833–1909), Rahusen (1860–1917), Schuh (1875–1966) and Wijdenes (1872–1972) followed a similar mathematical approach, but words such as “zwarijheid” and “wanhopen” were not used, except by Van der Ven (Ven, 1864). These textbooks contained short monologues instead of Socratic reasoning. Every next generation of textbooks paid less attention to the step from Cardano’s single real solution to the three cube roots and the problem of the endless loop. Van der Ven stated in his preface that he wrote his book *De theorie en de oplossing van hoogere magtsvergelijkingen* [Theory and solution of higher degree equations] (Ven, 1864) after he had heard many complaints about the excessive shortness of other authors when dealing with higher order equations.

De Gelder wrote in the preface that his books were intended for young pupils and their teachers. The books mentioned above are mostly addressed to an older audience, like students, teachers and those who wanted to become a teacher.

In the twentieth century, the cubic equation was no longer considered to be of much interest. After the fifties, the curriculum changed many times, but the interest in the algebraic solution of third-degree polynomial equations did not return. Nowadays, in the Netherlands, pupils use graphical calculators to solve cubic equations numerically. The story of Cardano and Tartaglia might be told by teachers who take an interest in the history of mathematics, but there is no time to explore the topic as extensively as De Gelder did. Nevertheless, mathematical reasoning is still part of the Dutch secondary school curriculum, as De Gelder would have appreciated.

Discussion

In my opinion, De Gelder's pedagogical ideas are still valid and his approach of solving cubic equations is of interest for modern mathematics teaching, because of the combination of its mathematical richness and its didactical approach. It would be worthwhile to investigate how teachers, students, and pupils would gain more insight into mathematical reasoning and problem solving when they attend a course based on De Gelder's ideas and approach. Within the ordinary Dutch secondary school curriculum, there is not much room for such a lesson series, but there is room in the extended math curriculum (labeled math D in the Netherlands). Pupils attending this program do have an interest in a broader and deeper mathematical knowledge. If I ever will teach such a group of pupils about complex numbers, I would like to develop a lesson series about the difficulties of solving cubic equations through the centuries. For mathematics students and those who want to become a teacher in mathematics, this topic would be of interest too.

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An overview of publications of Jacob de Gelder is on my website: see <http://www.fransvanschooten.nl/Gelder.htm>

¹ A transcription of *Verhandeling* is available at www.jphogendijk.nl/sources/gelder.html

The mathematics teachers' journal *Euclides* in the Netherlands in change, 1945-1976

Martinus van Hoorn

Former chief editor (1987-1996) of the journal *Euclides* in the Netherlands

Abstract

*After World War II, one mathematics teachers' journal still existed in the Netherlands. Its principal editor was Pieter Wijdenes, who was dominating in many discussions on mathematics education. The content of the journal emphasized a traditional way of teaching. Major curriculum changes were not stimulated, and organizations of teachers were never consulted. But things were changing. The publisher wanted a safer financial base, and, of course, Wijdenes was becoming older. Therefore, the publisher reached an agreement with the organizations of teachers at pre-university schools, ensuring their support. Soon, in the 1950s, a newly appointed editor, Johan Wansink, even brought the journal into the hands of the organizations. Wansink also wanted an editorial team, with editors collaborating for the journal. He succeeded in this, while maintaining his leading role as the chief editor. He modernized the content of the journal, with articles on developments in other countries, and he stimulated necessary curriculum changes. Wansink remained the chief editor until 1968. His successor was Gerrit Krooshof, who had to make the journal accessible for all mathematics teachers, at any secondary school. He had to do so because of a major change in the Dutch educational system. Boundaries between the different kinds of secondary education, pre-university or not, were erased. Krooshof also had to deal with a new programme containing some New Math. He made the journal *Euclides* more open indeed, and provided discussions when suitable or needed. So, within 20 or 30 years, the journal *Euclides* underwent major changes, it had become the journal of the organizations, its editors worked in a team, its content had been made suitable for many more teachers, and discussions were stimulated. There were no worries about surviving.*

Keywords: 1945-1976, teachers' journal, changes, editors, content

Introduction and method

The purpose of this contribution is to make clear the how and why of the changes in the editorial policy of the Dutch journal *Euclides* during the years 1945-1976. Hopefully, this article will give a view on developments that otherwise might have remained unknown. The research for this article was based on archives and literature. Both of these sources are, of course, manmade, so their reliability may vary, as is quite normal. Fortunately, the archives and literature contain data, propositions and views which turned out to be, in a satisfactory way, consistent. If, in some case, serious doubt is possible, this will be discussed. More problematic is that some of the sources do not completely cover the period under study. This holds for the archive of the secretary of the editorial team of the journal *Euclides*; this archive was started in 1959. From the years before, there is no such archive, and before 1956, there even was no secretary. For this reason, the period after 1959 is better

documented. Hopefully this did not generate an unbalance in attention with respect to certain periods. Additionally, the subject of the present article has not challenged many authors so far. So, in the present case, the literature references are restricted in number. The archives used, and most of the books and articles, are in Dutch. This may be a serious difficulty for perhaps a majority of the readers of the Proceedings of ICHME-5, although some passages will be translated or otherwise made clear. One of the archives mentioned is available online. This is the complete archive of all issues of the journal *Euclides*; only the outside and inside cover texts are often missing, and also several portraits of persons were left out. The story of the journal *Euclides* is told chronologically. As a consequence, the article is not organized around particular themes. Important themes will be considered when they turn up. In this article, much will be said about the years before World War II; this may enlighten the background of the developments from 1945. When content changes with respect to the journal are carried through, just the differences in content need to be emphasized¹.

Background: the journal *Euclides* from its start until 1945

The journal *Euclides* was founded in 1924. It was a private initiative of the publishing company P. Noordhoff in Groningen in the Northern part of the Netherlands, together with the very successful schoolbook author Pieter Wijdenes (1872, December 22–1972, February 17), living in Amsterdam. Wijdenes had stopped working as a teacher, which was possible because of the high yields of his textbooks (Henneman & Spek, 1971).

Wijdenes had a clear motive to start a journal: a publication had appeared pleading for major changes in mathematics education, especially concerning the school geometry. The author of this publication, Tatyana Ehrenfest-Afanassjewa (1876–1964), wanted an intuitive introduction to geometry, greatly analogous to the ideas of Felix Klein (1849–1925) and his Erlanger Programm (Ehrenfest-Afanassjewa 1924). She was answered in the first issue of the new journal by Eduard Jan Dijksterhuis (1892–1965), who strongly accentuated the importance of the classical axioms. The name *Euclides* – although only chosen in 1928 – can be seen as a mission statement of the journal. Further details about the start of *Euclides* can be found in the literature (Wansink, 1974; Berkel, 1996).

Wijdenes preferred a traditional treatment of mathematics; moreover, major curriculum changes might urge him to revise his textbooks. Indeed, only few articles in *Euclides* contributed to the knowledge about new insights into learning mathematics.

1 In this contribution the term chief editor will be used, although this was not always the official title.

Most subscribers of the journal were mathematics teachers at pre-university level. Possibly, many of them were not looking for changes either. Remarkable is that the journal hardly paid any attention to the activities of the two organizations of mathematics teachers at pre-university schools. Sometimes a lecture given at a meeting of one of those organizations found its way into the journal. But Wijdenes wished to remain independent from the teachers' organizations. In some other magazines and journals, serious attention was paid to the organizations, and also some discussions among their members became apparent, mainly concerning the contents of curricula (Smid, this publication). The organizations were named Wimecos and Liwenagel. Wimecos was an acronym for 'wiskunde' [mathematics], mechanics, cosmography; these were the subjects being taught by many teachers. Members of Wimecos were working at the school type HBS, which could be compared to the Realschule in Germany. Wimecos was a corporate body. The name Liwenagel was another abbreviation, not relevant here. Its members were working at the school type gymnasium. Liwenagel was not an official corporation, but part of a larger body of gymnasium teachers. The school type lyceum also existed, a combination of an HBS and a gymnasium. Mathematics teachers at lyceums could join the group Liwenagel.

Hence, the world of mathematics teachers, which was small, was still divided. The numbers of members of Wimecos and Liwenagel over the years are not precisely known, but did not exceed a few hundred (Wansink, 1976). As a consequence, the same holds for the numbers of subscribers to *Euclides*. Unfortunately, there are no accurate data concerning the numbers of subscribers.

The publishing company P. Noordhoff observed that *Euclides* had few subscribers, which might make long-term survival uncertain. Therefore, the publisher started talks with Wimecos and Liwenagel. The organizations, wanting to preserve the journal and knowing many of their members were already reading it, liked to have the journal as a medium for their announcements. The result of these talks was an agreement, fixed at the end of 1939, implying that *Euclides* would be the mouthpiece of the organizations, and their members would receive it, while, conversely, the organizations would pay a certain amount to the publisher. The agreement would apply from August 1940.

Wijdenes did not take part in the talks, and he will not have been enthusiastic when hearing about the agreement. He had been ignoring the organizations Wimecos and Liwenagel deliberately. Formerly, he had been an HBS teacher, but he could not remain a member of Wimecos since he was no longer a teacher. But now, he had to accept the agreement. The publisher may also have taken Wijdenes' age into account; Wijdenes was almost seventy, and it was not known how long he wanted to stay on as an editor. But nothing had been arranged about an editorial change.

Wijdenes had one colleague, who had been a co-editor from the beginning in 1924. This was Johannes Herman Schogt (1892-1958), who also lived in Amsterdam, where he was an HBS teacher. Schogt was a very precise man and had written some textbooks, known as very rigorous. One can discuss his influence on the content of *Euclides* in general; but this is not the purpose of this article. Wijdenes formulated his opinion about Schogt's role on an occasional post card to the editors of *Euclides* in 1961: he claimed that Schogt's significance had been hardly 1%. But perhaps this quote is more revealing about Wijdenes himself. He could be quite aggressive.

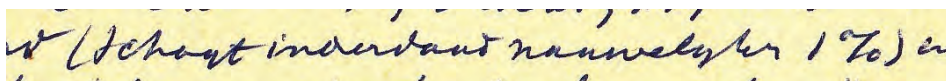


Fig. 1. Written text by Wijdenes: 'Schogt indeed hardly 1%'. Fragment of postal card Wijdenes to Koldijk, 1961, September 21. Archief van de secretares van *Euclides* [Archive Secretary *Euclides*].

From 1940, the Netherlands were involved in World War II. Already in 1940, new regulations were proclaimed, especially against Jewish people. During the wartime, living conditions deteriorated. In the last winter (1944-1945), in the western part of the Netherlands thousands of people died of hunger. Obviously, the wartime was not a time for implementing any new agreement about *Euclides*. During the last year of the war even schools were closed, and *Euclides* did not appear. Two, from many, details may be reported here. Firstly, the president of the organization Liwenagel, Christiaan de Jong (1893-1944), was shot; this happened in Leiden, where he was the deputy principal of the municipal gymnasium. De Jong was, so to say, imprudent in his statements, and the authorities at that time became informed about that. Secondly, the editor Schogt was hiding Jewish families in his private house, as revealed by his son (Schogt, 2003). These two single histories may depict the wartime atmosphere sufficiently.

***Euclides'* final Wijdenes years**

In September 1945, *Euclides* reappeared. The first issue after World War II was a thin one, and many of its pages were dedicated to mathematics teachers not having survived the wartime. There seemed to be no new views on didactical problems. But a lack of paper made it a priori impossible for any longer article to appear in print. Shortly after the war, there was not only a lack of paper; much had to be repaired and restored. But one thing had not changed: the editors of *Euclides* were Wijdenes and Schogt. There were no indications that they would resign. Because of the agreement, the organizations Wimecos and Liwenagel had paid for *Euclides* since 1940, and they could be certain their announcements, and minutes of meetings, were published in the journal.

After some time, life became more normal. Already by the end of 1945, *Euclides* started to publish didactical and philosophical articles. In fact, there was a growing variety of contributions in the journal, written by several teachers and other people concerned with mathematics education. In 1945, G.A. Janssen contemplated teaching: teachers should believe in their subject, they should be humble, and their attitude should be serving. In 1946, L.N.H. Bunt analyzed the didactics of integral calculus, and also in 1946, G. Wielenga asked whether alphas needed to learn (some) mathematics. In 1948, P.M. van Hiele tried to design directives for mathematical didactics. In those and other articles, the readers could feel stimulated to think about their own attitudes and methods.

Wijdenes had his preferences, as usual. He apparently wanted to accentuate that *Euclides* had a close connection to the universities. He had invited about fifteen mathematicians, many of them professors at universities, to be collaborators of *Euclides*. He also tried to achieve an international standing, by choosing some collaborators working in Belgium (also in Wallonia) and South-Africa. However, hardly any of these collaborators were actually publishing in *Euclides*. Of the professors, only one, Oene Bottema (1901-1992), working at the Technical High School - nowadays Technical University - in Delft, frequently contributed to *Euclides*.

Wijdenes also published the full texts of the inaugural addresses of newly appointed professors in mathematics. For instance, he published the inaugural address of J. C. H. Gerretsen (1907-1983), a new professor at the University of Groningen, about mathematics and esthetics, a topic not strongly related to mathematics education. Gerretsen's predecessor had died in 1945 in Amsterdam as a hunger victim (Berkel, 2005, p. 500), demonstrating that chairs could have become empty due to the war – which may explain the remarkably large number of inaugural addresses in *Euclides*. Wijdenes did not forget his personal interests. One instance may be mentioned here. Wijdenes wrote an article about the way parallel lines were presented in several textbooks. Then he wrote there was a better way, which he also described. It is hardly surprising that in his textbooks this – in his view – preferable method was used (Wijdenes, 1949). In other words, his textbook was the best. *Euclides* contained few discussions about such points of view.

Suddenly, Schogt left as an editor. He had asked to be relieved from his position from January 1, 1949. This was unexpected. Official reasons were not given. Wijdenes said Schogt had an aversion to some new spelling rules (Berkel, 1996, p. 587). Very interesting is that Wijdenes immediately found a successor for him. Wijdenes knew, of course, about Schogt's stepping down before the end of 1948. But the organizations Wimecos and Liwenagel were not informed. It was only in February 1949 that the board of Wimecos discussed the responsibilities for finding a successor when an editor might leave (Archief van de Nederlandse Vereniging van Wiskundeleraren, notulen bestuur Wimecos 1949 [Archief Dutch Association of

Mathematics Teachers, minutes Wimecos board 1949]). But at that time, the leaving of Schogt and the name of his successor had already been published in *Euclides*. On the Wimecos board, especially one new member felt uneasy about the rapid appointment of a new editor by Wijdenes alone. This new board member was Johan Hendrik Wansink (1894-1985). Wansink asked why the organizations had not had a say in the appointment of a new editor.

Schogt's successor, asked by Wijdenes, was Hendrik Streefkerk (1904-1985), a teacher in Hilversum. Hilversum is close to Amsterdam, so Wijdenes and Streefkerk could easily meet each other. Certainly, in Wijdenes's view, he and Streefkerk could co-operate just like he and Schogt had done for so many years.

However, the organizations were increasingly displeased about Wijdenes' rapid action. Should not they have a firm say in appointing new editors? (Archief van de Nederlandse Vereniging van Wiskundeleraren, notulen bestuur Wimecos 1949, 1950 [Archive Dutch Association of Mathematics Teachers, minutes Wimecos board 1949, 1950]). These discussions led to the appointment of two extra editors during the schoolyear 1949-1950, so that *Euclides* now had four editors. Wijdenes and Streefkerk had to accept this as a consequence of the new role of the organizations from 1940.

Then, in 1950, Wijdenes resigned. This was not unexpected, since he was 77 years old. It will always be unclear whether his stepping down was hastened by the new role of the two organizations. Wijdenes had been the chief editor for more than 25 years. Although he left his seat, he remained active, and sent articles to *Euclides* during the fifties and sixties. But many of these articles were rejected. Clearly, Wijdenes was outside the mainstream – at the most, he represented a minority. Anyhow, in his lifetime he was active for about 80 years, beginning in 1891, when he became a primary school teacher. In this article, Wijdenes' activities after 1950 will not be considered, since he did not play a significant role anymore.

Wansink coming onto stage

After Wijdenes' leaving, Hendrik Mooy (1900-1982) was appointed editor. Mooy was an undisputed authority. He was the first in the Netherlands whose thesis was dedicated to mathematical didactics, though partly (Goffree, 2002). Mooy was a gymnasium teacher in Amsterdam. Streefkerk and Mooy, together with the two representatives of the organizations, now could constitute an editorial team for the journal *Euclides*. In practice however, decisions were made by Streefkerk and Mooy. Probably it was Streefkerk, wanting to be Wijdenes' true successor, who had the lead. He was the one summing up news from the editors in the journal, if there was any. But formally there also were two editors representing both organizations.

In 1953, Wansink became an editor, as the representative of Wimecos. He immediately started with a series called 'Didactische revue' [didactical review], about foreign journals, treating developments and visions abroad. Wansink turned out to have great knowledge about mathematics education in other countries. He continued this series for many years. At that time, Wansink was the deputy principal of an HBS in Arnhem, in the east of the Netherlands. Later on, he would become principal of this school. More information about him is given by Smid (Smid, 2017).

EUCLIDES

**TIJDSCHRIFT VOOR DE DIDACTIEK DER EXACTE VAKKEN
ONDER LEIDING VAN Dr H. MOOY EN Dr H. STREEFKERK,
Dr JOH. H. WANSINK VOOR WIMECOS EN J. WILLEMSE VOOR
LIWENAGEL**

MET MEDEWERKING VAN

PROF. DR. E. W. BETH, AMSTERDAM
DR. R. BALLIEU, LEUVEN - DR. G. BOSTEELS, ANTWERPEN
PROF. DR. O. BOTTEMA, DELFT - DR. L. N. H. BUNT, UTRECHT
PROF. DR. E. J. DIJKSTERHUIS, BILTHOVEN - PROF. DR. J. C. H. GERRETSEN, GRONINGEN
DR. R. MINNE, LUIK - PROF. DR. J. POPKEN, UTRECHT
DR. O. VAN DE PUTTE, RONSE - PROF. DR. D. J. VAN ROOY, POTCHERSTROOM
DR. H. STEFFENS, MECHELEN - DR. J. J. TEKELENBURG, ROTTERDAM
DR. W. P. THIJSSEN, HILVERSUM - DR. P. G. J. VREDENDUIN, ARNHEM

Fig. 2. The first time Wansink's name is on the title page of an issue of *Euclides*.
Euclides 29 (1953, September).

So, on the one hand, Wansink played his role as an editor by giving useful information to the teachers. On the other hand, he expected to have a say about the whole content of the journal. But this seemed to be out of question. He received little information about the content, and there were hardly any regular editors meetings. For Wansink, all this was highly unsatisfactory (Archief van de Nederlandse Vereniging van Wiskundeleraren, notulen bestuur Wimecos 1953-1955 [Archive Dutch Association of Mathematics Teachers, minutes Wimecos board 1953-1955])

At the end of 1954, Wansink became the new president of the organization Wimecos. For years, very few members of Wimecos seemed to have had aspirations to become president of this organization, and since the end of World War II, already three members of Wimecos had fulfilled this role. Wansink used his new position to change the policy with respect to *Euclides*. Wansink himself assumed the role of chief editor. He succeeded in implementing this change with full commitment of Wimecos, and, undoubtedly, also of Liwenagel (Archief van de Nederlandse

Vereniging van Wiskundeleraren, notulen bestuur Wimecos 1954-1956 [Archive Dutch Association of Mathematics Teachers, minutes Wimecos board 1954-1956]). In 1956, Wansink introduced a completely new phenomenon: for *Euclides* an editorial team was formed². In Wansink's team, there were six editors. Wansink was president of the team, but there were also a vice-president and a secretary (Wansink, 1956). The president consulted his colleagues about the submitted contributions, and the secretary wrote to the authors about possible improvements. So, the secretary was given a central role in the team. Regularly, there were meetings, where discussions could take place about articles, about the whole content, about the task the journal should fulfill, and about vacancies in the team. The team was working in a transparent way, especially for its members and for the boards of the organizations (Archief van de secretaris van *Euclides*, vanaf 1959) [Archive Secretary *Euclides*, from 1959]).

Streefkerk, who had been carrying out, more or less, Wijdenes' role from 1950, stepped down in 1956. He clearly could not agree with the new structure. Mooy stayed as one of the six editors. Some years later, he also stepped down, but this was for another reason: he went to Liberia for a year, to build up the mathematics education there. In 1962 he became the principal of a lyceum in Amsterdam.

Resuming, Wansink had taken full power with respect to *Euclides*. But he did this in a transparent way. Everybody could feel satisfied about this move. Both organizations had their say. No longer did one single editor make decisions concerning the content, and the editors truly operated as a team. Wansink claimed *Euclides* had become an "onafhankelijk tijdschrift" [independent journal] (Wansink, 1956). By this he meant that the editors were free in making their decisions. They were responsible only to the organizations Wimecos and Liwenagel, and not to the publisher. It should not be forgotten that Wansink had the leading role; he was the president of Wimecos and also the president of the editorial team. The journal was still published by the same company, P. Noordhoff in Groningen. The independency meant especially that the publisher had no official say in appointing the editors or in the editorial policy.

Twelve years with Wansink

Wansink stayed on as chief editor until 1968. He strengthened the position of *Euclides*, which remained the one journal for mathematics teachers at pre-university schools. The number of issues in one school year increased from 6 to 10, and simultaneously the number of yearly pages increased from about 300 to about 360; this was realized immediately in 1956.

² From 1949 there were four editors, Wijdenes, Streefkerk and two editors appointed by the teacher's organizations, but they did not form a team.

Already in the first issue of the school year 1956-1957, Wansink exposed his ideas about the content (Wansink, 1956). In his view, *Euclides* should contain:

- a. Articles about higher mathematics, if relevant for teachers;
- b. Didactical articles;
- c. Articles concerning teacher training;
- d. Information about foreign countries and foreign contacts;
- e. Recreation;
- f. Book announcements and reviews;
- g. Notifications about meetings, conferences, and minutes of meetings.

So, Wansink immediately chose a professional approach. As a consequence, inaugural addresses were no longer needed. Wansink did not propagate certain didactical principles or methods of teaching; in his opinion the one who teaches is the most important factor for pupils in their learning process. The teachers should therefore be trained thoroughly, they should always be informed very well, and their voice had to be heard when curricula were to be changed. Under these conditions, good teachers would take care of good education.

Wansink, who was a central figure in this period, had also chaired a committee proposing a new curriculum, containing differential and integral calculus. This new curriculum would be the same for the two school types HBS and gymnasium. This was something new, not only because of the introduction of differential and integral calculus, but also because it was a first trial to demolish the border lines between the two school types, both having long traditions, especially in distinguishing one from another. Wansink gave much space to discussions about the new subject matter. He also invited two school inspectors to answer questions of teachers about the new curriculum. He clearly wanted the curriculum to be accepted broadly. Therefore, in *Euclides* discussions were published which were interesting for teachers at both types of pre-university schools. *Euclides* became indispensable for them all.

Wansink also chose new editors carefully. When the first secretary left in 1959, Wansink asked a teacher he knew very well. This new secretary was Albert M. Koldijk (1917-2005), who remained secretary until 1973 (Hoorn, 2002). Koldijk fulfilled his task very well, and he also built an archive - several data in this article have been found in this secretarial archive. Wansink did not forget to ask the organization Wimecos to appoint Koldijk officially. This was a mere formality, since Wansink was still president of Wimecos. In 1956, the six editors were officially appointed by the organizations, four of them by Wimecos, the other two by Liwenagel. This was a

result of the so-called independency – Wansink’s term. One of the Liwenagel editors was the president of Liwenagel, Pieter G.J. Vredenduin (1909-1996), who taught at a gymnasium in Arnhem, the town in which Wansink was teaching. Wansink always carefully respected the formal structures – which were largely designed by himself. He was a strategist knowing precisely what to do and what not.

The procedure for appointing new editors had been maintained many years. But, while Wansink himself was the president (till the end of 1961) of the largest organization (Wimecos), this procedure, especially within Wimecos, did not have any unexpected outcome. In other words, in fact the editors were choosing their new colleagues. This practice continued, as is apparent from Koldijk’s archive (Archief van de secretaris van *Euclides*). Therefore, a long-term result of the appointing procedure could be that, practically, the editorial team became independent. During most of the 1960s, Wansink stayed in his position as the chief editor. He had resigned from his presidency of Wimecos, possibly because he was over 65 and he was no longer teaching at a secondary school. In this period, two main developments became of special interest. Firstly, the New Math movement began to gain influence, also in the Netherlands. Secondly, professional mathematical didactics were developed. Wansink did not show much interest in the New Math, but he always stayed very interested in all aspects of the teaching. In *Euclides*, several contributions appeared about didactical insights.

In 1968, Wansink stepped down as the chief editor. He was 74, and he knew there was an experienced successor. This was Gerrit Krooshof (1909-1980), who had already been a member of the editorial team for four years. His principal work *Didactische oriëntatie voor wiskundeleraren* [Didactical orientation for mathematics teachers], containing contributions by several others, appeared in three volumes during the years 1966-1970. In the Netherlands, it was the first work on didactics covering the whole field of mathematics teaching in over one hundred years. Wansink remained active as an author and he also published articles. Among these, one is about the journal *Euclides* in the twenties (Wansink, 1974). In this article, he also gave his opinion on the content of the journal *Euclides* before 1940. Wansink observed, among other things, that educational matters in foreign countries, except Belgium and Germany, had only rarely been discussed in those early years of *Euclides*, and almost no attention had been paid at the time to the very important reform proposed by Felix Klein. An article written in Dutch, about Klein’s ‘Meraner Vorschläge’, had appeared elsewhere (Smid, this publication).

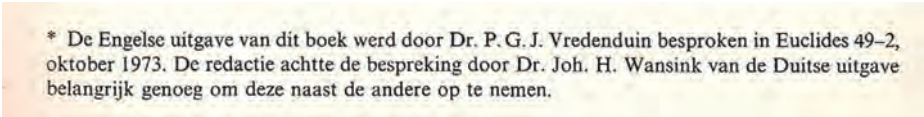
Wansink also observed that, in the twenties, the history of mathematics was frequently discussed in *Euclides*; one may call this topic over-accentuated. On the other hand, historical views differing from Dijksterhuis’ views were rarely seen. Dijksterhuis was a well-known Dutch teacher and historian of science who had rigorous ideas about how to teach mathematics, especially geometry, at school level

(Berkel, 1996). Wansink observed that, during the twenties and thirties, new textbooks had sometimes not been mentioned in *Euclides*. There seemed to have been a conflict of interests, since *Euclides*' publisher, the company P. Noordhoff, also was an important textbook publisher. Wansink stated that every important schoolbook should receive attention, notwithstanding its publisher. But this was not always the practice in the Wijdenes years. Finally, Wansink saw few contributions on major problems in mathematics education in the twenties and thirties. After World War II, Wansink himself published about the activities of the Wiskunde Werkgroep [Mathematics Working Group], founded in 1936 as the Dutch branch of the New Education Fellowship. In this Working Group, discussions took place on questions like: is mathematics a subject merely to be trained, or an essential part of general knowledge? In the early years of *Euclides*, hardly any principal discussion had been given space.

So, Wansink found that in the pre-war years, the journal had many deficiencies. He must have been motivated to improve its content when he started as the chief editor in 1956. Indeed, Wansink, when he was chief editor, took all his objections concerning the first *Euclides* decades seriously into account. Nevertheless, one can remark that in Wansink's years, a number of developments taking place in the Netherlands, were somewhat ignored. There were no great contemplative articles about the value of New Math, and also the activities of Hans Freudenthal (1905-1990) were not discussed broadly. Freudenthal was a distinguished mathematician, participating actively in the Mathematics Working Group. He organized a conference in Utrecht (1967), which may be said to have been a forerunner of the present International Conferences on Mathematics Education, and he established the international journal *Educational Studies in Mathematics* (1968). Interestingly, Wansink himself attended Freudenthal's conference. Wansink and Freudenthal greatly disagreed on the teacher's role in the learning process, and it seems Wansink did not want to provoke a major discussion on that theme in *Euclides*. On the other hand, Wansink and Freudenthal respected each other very much (Freudenthal, 1974; Smid, 2017). Wansink might have been avoiding a possible disturbance. He always was a strategist. What he did was to bring *Euclides* into the hands of the organizations and giving it a permanent role for each mathematics teacher at a pre-university secondary school.

Later on, in 1973, Freudenthal published his didactical masterpiece, *Mathematics as an educational task*. In *Euclides*, this work already had been reviewed by Vredenduin, who may have written his review without consulting the chief editor - still Krooshof at that time. Vredenduin always worked very fast; he was also one of *Euclides*' editors, and when he had an idea for an article, he wrote that article immediately, and sent it to the chief editor, who thus had no time to think about other possible authors on the same theme. Vredenduin had been an editor from 1956, so Wansink knew this style very well.

In this case, Wansink seemed to regret having missed a chance to expose his ideas on Freudenthal's book. Fortunately, a German edition of Freudenthal's book appeared, named *Mathematik als pädagogische Aufgabe*, and Wansink was able to review this so-called new work in *Euclides*. Wansink's review was an article of eleven pages (Wansink, 1975). A footnote made clear the appearance of the German edition was the reason for another review – by Wansink.



* De Engelse uitgave van dit boek werd door Dr. P. G. J. Vredenduin besproken in *Euclides* 49–2, oktober 1973. De redactie achtte de bespreking door Dr. Joh. H. Wansink van de Duitse uitgave belangrijk genoeg om deze naast de andere op te nemen.

Fig. 3. Editorial footnote below Wansink's review of the German edition of Freudenthal's book. *Euclides* 50 (1975), 401.

Another argument may have been relevant for Wansink. In 1970, when the third part of his own didactical work had appeared, this work received an extensive review, in the form of an article written by Krooshof (Krooshof, 1971). Therefore, perhaps in Wansink's eyes, also Freudenthal's masterpiece deserved a thorough review.

Krooshof, the natural leader

In 1968, Gerrit Krooshof became *Euclides*' chief editor. Wansink knew Krooshof very well. The two had met in the fifties, in the Mathematics Working Group. Krooshof was the main editor of the information periodical of the Mathematics Working Group. In 1961, after extensive preparation by Wansink and Freudenthal, a Dutch mathematics journal for secondary school students appeared, named *Pythagoras*. Krooshof and another teacher were the first editors of this student journal. Krooshof played the leading role as editor and made this journal very successful (Hoorn & Guichelaar, 2018). Thus, Krooshof had shown he could edit periodicals very competently. Some years after 1960, the information periodical of the Mathematics Working Group was abolished and its editors became editors of *Euclides*. This is the way Krooshof entered the editorial team of *Euclides*.

Krooshof was a teacher and deputy principal at a HBS for girls in Groningen. His school had a special department with an easier kind of secondary education, officially giving no entrance to universities, but nevertheless closely related to the HBS. There was no final exam. Krooshof was involved with the pupils there, and in November 1953, he gave a lecture about the mathematics education desirable for them at a conference of the Mathematics Working Group. In this lecture, he considered almost every aspect of the mathematics education for his pupils. He had been asking his colleague teachers about their opinions on the theme. This lecture, together with the subsequent discussion, was published in *Euclides* (Krooshof, 1954).

Some years later, Krooshof, together with another mathematics teacher, published a textbook, just for this school type.

From 1966, Krooshof had also been the leader of the team publishing a new textbook series for the secondary schools, which was based on a Scottish series, named *Moderne Wiskunde* [*Modern Mathematics*]. This textbook series appeared from 1968, because the transformation of the school system, together with the start of a new curriculum, took place from 1968 on. The editorial team of *Moderne Wiskunde* included some reputed mathematics teachers, but Krooshof was its undisputed leader, as is told by several people concerned with *Moderne Wiskunde* (Hoorn & Guichelaar, 2018).

As Wimecos and Liwenagel were replaced by one organization, which was accessible for almost all mathematics teachers at secondary level, Krooshof had to make *Euclides* really accessible for those teachers. Of course, he should also take into account the new curriculum, for which new didactical tools were needed. He was well aware of these tasks (Krooshof, 1969). Krooshof had to do so without favoring his textbook series *Moderne Wiskunde*. That did not pose a problem for Krooshof. He always gave space to criticism (Archief van de secretarissen van *Euclides*, passim). In his opinion, things would improve just because of serious criticism. He indeed took care to publish contributions about the practice of the new curriculum, and he certainly made space for articles for teachers at non pre-university schools. Also, in 1969, the first teacher at a non pre-university school joined the editorial team. It is debatable whether all this was completely successful or not, but apparently Krooshof made *Euclides* readable for a very broad group of teachers. In the same year, 1969, the list of collaborators disappeared. There is a letter from Bottema (mentioned earlier), who wondered why this list had disappeared; were his contributions no longer appreciated? To the secretary of the editorial team, he was friendly about it. Of course, he could remain as a collaborator. Only Wijdenes, whose name had been added to the list by Wansink, in December 1962, was not satisfied (Archief van de secretaris van *Euclides*, 1969).

Krooshof gave priority to forming a really cooperative editorial team. This may seem normal within any organization, but it was Krooshof who was realizing this in practice. He was not, like Wansink, a man asking for formal procedures, although he respected these. Krooshof will have observed the problem that many of the teachers at non pre-university secondary schools did not habitually write articles. But he published articles suitable for them, and found some of them willing to write about their experiences. Krooshof was also keen on developments concerning the development of (parts of) new curricula, as was done at the newly established institute IOWO at Utrecht University, with Freudenthal as its director. In 1972, a special double issue of *Euclides* appeared, dedicated to the expected activities of this new institute, with all articles written by its collaborators. In 1974, another special

appeared, on the occasion of *Euclides*' beginning fifty years before. It was largely dedicated to geometry, since in 1924 especially geometry had been discussed, and now, in 1974, new ideas, which were principally based on ideas rejected in 1924, were being given space. This special contained contemplative articles as well as practical ones.

Finally, one can observe that Krooshof published discussions about every topic, great or small, in mathematics education. So, *Euclides* went on to be a platform for all mathematics teachers. Simultaneously, its number of readers had increased strongly, up to over 2,000 – which was not mainly Krooshof's merit, but due to the increased number of members of the new Dutch Association of Mathematics Teachers, compared to the numbers of members of the former organizations Wimecos and Liwenagel.

However, Krooshof was not completely satisfied. In the annual editorial report about the schoolyear 1972-1973, Krooshof and Koldijk stated they wanted more contributions of present interest, and they wondered whether the subscribers really liked the various sections. Krooshof and Koldijk would like to have more interaction with the readers to know such things, but, in general, they found interaction was insufficient. Moreover, promised articles had sometimes not been submitted; this holds especially for reports about the international conference on mathematics education in Exeter, held in 1972. Finally, the information to the members of the association of mathematics teachers, from the board and committees within the association, still was far from complete. This last observation also suggests that the distance between the board of the association and the editors of their journal was increasing. Many things went well, but Krooshof always wanted to look in a mirror (Krooshof & Koldijk, 1974).

De redactie is van mening dat er behoefte is aan meer actualiteit. Zij vraagt zich af of op de min of meer geregelde rubrieken veel prijs wordt gesteld. De interactie met de lezers is op dit punt en eigenlijk in het algemeen veel te gering. Het spijt de redactie zeer dat bepaalde toegezegde artikelen, in het bijzonder die over het congres te Exeter, niet ontvangen werden. De voorlichting van de leden binnen de Nederlandse Vereniging van Wiskundeleraars door bestuur en commissies laat nog steeds te wensen over.

Fig. 4. Part from the annual editorial report about the season 1972-1973, undersigned by Krooshof and Koldijk; the Dutch text is – roughly - translated in the current article. *Euclides* 49, 201.

Final statement

In the year Krooshof resigned, 1976, *Euclides*, when compared to the journal as it was in 1950, had become a completely different journal, still thorough and sound, but focused on teachers' needs and taking modern developments consequently into

account, without avoiding discussions on great or small aspects of mathematics education, and readable for teachers at any secondary level. All these achievements were due to two successful chief editors, Wansink and Krooshof.

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Dmitry Chizhov and the examination of mathematics textbooks in Russia during the 1820s-1830s

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Abstract

The late 1820s and early 1830s were a time when the requirements placed upon education in Russia became stricter (or in some cases were established), including the requirements placed upon mathematics education. A significant role in this was played by the so-called Committee for the Examination of Textbooks, in which St. Petersburg University Professor Dmitry Chizhov was responsible for mathematics. The present article, which relies on surviving documents, is devoted to Chizhov's work on reviewing existing and planning new textbooks.

Keywords: mathematics textbooks, Ministry of Education, regularization, Committee for the Organization of Educational Institutions, Committee for the Examination of Textbooks

Introduction

This article describes the regularization of the requirements placed on mathematics textbooks that occurred in Russia during the 1820s and 1830s. This regularization occurred within the context of a broader regularization and stiffening of the requirements placed on the teaching process in general and in particular on the teaching of mathematics. A certain degree of liberalism, which had until then prevailed in privileged civic educational institutions (gymnasias or boarding schools for the nobility), thanks to which students not wishing to study mathematics were free not to pay too much attention to it, gradually gave way to a more exacting arrangement. The memoirist Nikolai Markevich, who attended the Noble Boarding School from 1817 on, left memoirs in which he vividly described how rude he was to his teacher Dmitry Chizhov, who was unhappy that Markevich was not doing any work. Meanwhile, the head of the St. Petersburg educational authority (and later minister of education) Sergey Uvarov, who, as the same memoirs make clear, visited the Boarding School quite often, sided with Markevich, and not with his teacher (Karp, 2007). Gradually, such a thing became impossible. Emperor Nicholas I was inclined to promote rigid discipline in general; according to surviving archival documents he once ordered that a teacher be dismissed only because a student of his had allowed himself to lean on his elbow in class (Karp, 2015). It was within the context of this “establishment of order” that the government practice of examining textbooks

developed, and not a small part in this development was played by the aforementioned Chizhov, as will be discussed below. The present article is based on materials from the Russian State Historical Archive, which, to the author's knowledge, have not been previously studied¹.

On the state of affairs prior to 1826: Legendre's textbook as a case study

In Russia, the role of the government in education had always been considerable. One episode will suffice to illustrate this fact. On October 3, 1810, the academician Semyon Guryev was sent a letter from the Ministry of Education, informing him that "Petrushevsky, Senior Teacher of Physics and Mathematics at the Pskov Gymnasium, has translated Mr. Legendre's Geometry and Trigonometry into the Russian language" (Po predpisaniyu, 1810, p. 1). Guryev was asked to evaluate whether this work deserved "to be published, both in terms of the quality of the book and in terms of the quality of the translation."

Foma Petrushevsky (1785-1848) left his mark on Russian history with his other works as well (Polovtsev, 1902). At a later time, after moving to St. Petersburg, he served as director of a home for destitute children, and after that, as director of the Institute for the Blind. Petrushevsky wrote books on measurement science and translated the works of Euclid and Archimedes, as well as *Traité élémentaire d'arithmétique* [Elementary arithmetic] by Lacroix (which he published in 1817).

The archives contain a draft of the letter to Guryev, which reveals that Petrushevsky's translation was originally going to be sent for review not to him, but to Nicolas Fuss (1755-1825), who had once been Euler's secretary, and who later became a Russian academician and textbook author. Evidently, it was found that inviting Fuss for the role of reviewer presented a certain inconvenience, and this, as we will see, is understandable.

Guryev responded directly to the minister, concisely and point by point. First, he explained that:

Mr. Legendre's work... although in a strict sense it possesses infirmities and deficiencies, for example, in its manner of ordering or arranging the topics, their proofs, the omission of certain rather important ones, and conversely in its inclusion of others that do not at all belong here; nonetheless, this work of a man so well known in the scientific world may be considered, as a whole,

1 Those interested can find more information about the history of Russian mathematics education as a whole in the book by Karp & Vogeli (2010).

2 All translations from Russian are by the author.

to be one of the good works of this kind that have been published (Po predpisaniyu, 1810, p. 2).

Subsequently, however, he goes on to say that this work assumes that its readers already possess considerable knowledge, and consequently that “it can be useful only for adults and sufficiently prepared young people, not for children” (pp. 2-3). This remark is followed, finally, by his main thought:

It is known that the Main Directorate of Schools³ is publishing a complete Course in Pure Mathematics by the academician Fuss, which contains also the foundations of geometry, published already in the year '98 in the French language and shortly thereafter in the Russian; which foundations, in my opinion, will be more useful for children than Legendre's, since in terms of their order and even the strictness of their proofs, they are not inferior to the latter, while in terms of brevity and convenience for teaching, they incomparably surpass them (p. 3).

In addition, Guryev discovered flaws in the translation, which made immediate publication impossible. Consequently, Petrushevsky was informed that “he may use his work as he sees fit” (p. 4). Petrushevsky's translation, as far as we know, was never published, while the first Russian edition of Legendre's Geometry appeared only in 1819 (Legendre, 1819), in Matvey Sakharov's translation, having been published, as the book indicated, “for the use of the cadets of the Imperial Military Orphanage.”

This episode just described displays the situation quite clearly: at that time, books were quite expensive, and the publication of a textbook that was not explicitly recommended for use in schools could not come close to paying for itself; consequently, without government support, it would have been difficult to publish a book on geometry (although no one prohibited Petrushevsky from doing so). Certain books were undoubtedly recommended to educational institutions (Fuss's books, for example), but given the shortage of textbooks, on the one hand, and the variety of types of educational institutions, on the other, the latter ended up using a great variety of texts. To repeat, this episode clearly shows that diversity and freedom in the choice of textbooks were not unlimited; nonetheless, the Russian Emperor Nicholas I was not satisfied with the existing state of affairs. In May 1826, he wrote:

“Reviewing with especial attention the organization of the educational institutions in which Russian youth is educated to serve the state, I regret to observe that they lack that necessary and indispensable uniformity which must be the

3 Main Directorate of Schools (Glavnoe pravlenie uchilishch) – a branch of the Ministry of Education that was responsible at that time for providing textbooks to educational institutions, among other things.

foundation of both child rearing and instruction” (Perepiska, 1826-1828, p. 1⁴).

The Committee for the Organization of Educational Institutions and the Committee for the Examination of Textbooks

The words of Nicholas I quoted above are taken from his rescript to the then-Minister of Education, Shishkov, announcing the formation of a new Committee: the Committee for the Organization of Educational Institutions. The following were listed among the Committee’s objectives:

Article 4. To define in detail all courses of study for the future, indicating likewise the texts that must henceforward be used in the teaching of them.

Article 5. In connection with this, to decide which of the existing texts are good, and also to make arrangements for providing what is missing, selecting to this end the appropriate professors and academicians, subject to Your approval and my confirmation, with a view to proscribing thereafter all arbitrary teaching based on arbitrary books and notes. (Perepiska, 1826-1828, p. 1)

In his reply, Minister Shishkov proposed to form a special Committee for the Examination of Textbooks, which would focus on achieving these two objectives and which would be chaired by one of the members of the Committee for the Organization of Educational Institutions. He also proposed various members as possible candidates. One of these was Dmitry Chizhov, clearly intended to oversee mathematics. In response to this, the Sovereign wrote: “[I am] in complete agreement, but would request that the matter not be delayed, since this sometimes happens with scientists” (p. 3). And the Committee was formed. Long transcripts of the Committee’s meetings have survived, which include materials dealing with the examination of numerous handbooks, written both in Russia and abroad (Perepiska, 1826-1828; Zhurnaly, 1828-1835). The account below is based on these transcripts. But first a few words must be said about Dmitry Chizhov.

Dmitry Chizhov

Dmitry Chizhov (1784-1852) played a significant role in the formation of Russian mathematics education (Karp, 2014). He began his education at the so-called Kashin

⁴ The document’s pages were numbered in two ways: including and excluding the table of contents. We find it more convenient to use the second method. According to the first method, this page is number 12.

clerical gymnasium, continued it (beginning in 1792) at the Tver Seminary⁵, and then in 1803 was transferred to the Teachers' Gymnasium (later called the Pedagogical Institute, which subsequently became the basis of St. Petersburg University). After completing his education there in 1807, he, along with a number of the other top students, was sent in 1808 to study abroad, where his teachers included Pfaff during the year and a half that he spent in Germany, and then, in France, D'Alembert, Lefevre-Gineau, Lacroix, and especially Poisson, under whose direction Chizhov studied integral calculus. Upon returning to Russia in 1811, he was appointed adjunct professor at the same Pedagogical University, assisting his former teacher Matvey Rezanov. Later, Chizhov became a professor at this university (whose name in the meantime had changed – the university opened in 1819). At the same time, he also taught at other educational institutions, both at the higher level (Institute of Railway Engineers, Main Engineering Academy) and middle level (at the aforementioned Noble Boarding School). He repeatedly served as dean of the physics and mathematics faculty, becoming a distinguished professor (an honorary title) in 1841. Chizhov was an associate member (from 1826) and a full member (from 1828) of the Imperial Academy of Sciences. He retired in 1846. Chizhov was a recipient of various orders of merit and an actual state councillor (from 1842) (*Ob utverzhdenii*, 1842).

The works and writings that Chizhov left behind are rather few in number, but he clearly devoted a great deal of attention to teaching and administrative work. Thus, for example, among his other functions, he was appointed visitor (as it was then called) to private educational institutions. His duties in this connection were formulated quite clearly: to ascertain whether these institutions met the government's requirements, since "the government must not permit even the existence of such Institutions, if their prevailing orientation is different from the one that the government wishes to give to public education in general" (*O naznachanii*, 1833, p. 4).

He was also sent to government gymnasia. Thus, he attended an exam at the Third Gymnasium in St. Petersburg and punctiliously reported that the title of the father of one of the students was not indicated in the school documents, and that the document lacked an official seal, while another student's certificate had been printed on ordinary rather than watermarked paper and also lacked the proper official seal (*Perepiska popechitelya*, 1831-1835, p. 25).

The Russian Biographical Dictionary (Polovtsev, 1905) notes, however, that in 1821, during what may be described as an ideological purge of the University,

5 The word "seminary" in different countries and at different times has been used in different senses; thus, it must be noted that, at that time, a "seminary" in Russia was understood to refer to an educational institution at what today would be considered the pre-college level (insofar as such terminology can be used with reference to the eighteenth century at all).

Chizhov was one of the few professors who refused to censure those whom the administration accused of disseminating subversive ideas (the accused included his fellow students from the Pedagogical Institute and his trip abroad).

As a member of the Committee for the Examination of Textbooks, Chizhov regularly reviewed various textbooks – both Russian and foreign, printed or still in manuscript. He also created something like a program for equipping the country with textbooks and facilitated their writing or translation. We will describe these aspects of his work in greater detail.

Chizhov's textbook reviews

From the surviving documents, it is not clear how it was decided which textbooks should be reviewed. Sometimes the textbooks that had to be reviewed were inaccessible to private individuals, as Chizhov's writes, complaining at the very beginning of his work as a member of the Committee in August 1826 that he was unable to acquire the books printed for the schools of so-called military settlements⁶ at their headquarters' printing office and requesting that they be sent to him (Perepiska, 1826-1828, p. 96). One may assume that such difficulties were subsequently ironed out.

It is likely that the books published by the printing office of the military settlements' headquarters that Chizhov had to review included the manual in arithmetic *Uchebnaya kniga* (1825). The archive contains a review of certain arithmetical tables for cantonists⁷ based on the monitorial method of education (no more precise information about them exists), and of all known publications, the manual just referred to best matches this description. Chizhov's review of it is indicative of his style: he does not merely provide a discussion of the contents of the book, but considers it more important to express his general opinion. After praising the reviewed book for its gradual approach and clear exposition, he adds:

Indeed, the monitorial method of education in my view can be useful only in the teaching of reading, writing, and the elementary rules of arithmetic, but nothing more. For this method, even with the mechanism and clarity connected with it, while making the study of the aforementioned subjects easier, can constrain the imagination in the study of higher subjects and by doing so hinder its further development (Perepiska, 1826-1828, p. 89).

6 A form of military organization that existed in Russia during the period 1810–1857, which combined military service with productive labor, first and foremost, agriculture.

7 Cantonists -- students who, due to their origins, were required subsequently to serve in the army (in particular, children of residents of military settlements).

This was by no means the only book on arithmetic reviewed by Chizhov and he was not always so restrained and benevolent. Several years later, for example, he reviewed *New Arithmetic* (Teriukhin, 1827). In this case, he did not mince words:

This book cannot be put into general use as a textbook in schools for the very numerous and pernicious defects with which it is replete. It contains much that does not accord with the subjects addressed in it, which may give rise to false notions among the students. [...] Proofs are either unsatisfactory or not offered at all. [...] I found nothing new, despite the fact that this book is titled 'New'. [...] On the whole, the presentation of the subject is far from comprehensible either for children or for adults (Zhurnaly, 1828-1835, p. 121).

Even here, however, specific observations are few in number; Chizhov writes:

A detailed written analysis would have pointlessly taken up too much time and therefore I have confined myself to notes on the margins in pencil in only a few places in the book.

Indeed, he is not so much a reviewer as a judge, authorized to make decisions, briefly stating his reasons for them.

Chizhov also had to review foreign textbooks. Their selection remains a mystery. It is easy to understand why *Traité élémentaire de statique* [An elementary treatise on statics] by Gaspard Monge attracted attention. Praising it for the clarity of its presentation, Chizhov writes: "in France, this book in general is considered to be the best textbook" (Perepiska, 1826-1828, p. 111). It is far more difficult to understand why it was necessary to discuss the book *Geometrische Anschauungslehre: Eine Vorbereitung zum leichten und gründlichen Studium der Geometrie* [Geometrical apprehension: A preparation for the easy and thorough study of geometry] by Johann Josef Ignaz Hoffmann, published in Mainz in 1818. Schubring (1993) notes that Hoffmann (1777-1866), professor of mathematics in Aschaffenburg, was "quite influential in his own day (though since fallen into oblivion)" (p. 46). And yet, neither Hoffmann himself, nor the aforementioned book, can be placed alongside of the books and authors that were most popular in Western Europe. As it happened, Chizhov did not endorse the book: after discussing the importance of geometry and the difficulties involved in its teaching, he asserted that the book contained too many questions and repetitions, while the tedious details, "instead of arousing the students' attention and interest, can breed revulsion in them" (Perepiska, 1826-1828, p. 110).

In other instances, Chizhov could be more favorable. After finding numerous shortcomings in a book by Kroymann (1807), he nonetheless points out that the book could be taken into consideration in the writing of a similar book in the Russian language, and moreover, that even apart from the writing of such a similar book, the present collection of problems

could be beneficially used in our schools if it were translated into the Russian language and adapted throughout to Russian units of measure. It could especially lighten the work of teachers in collecting problems of this type (Zhurnaly, 1828-1835, p. 146).

In reviewing books, Chizhov not only recommended or did not recommend books for translation and publication, but also assembled lists of books recommended for acquisition by the libraries of gymnasia. Thus, in 1828 he prepared a list of books “indicated in the order of their usefulness for Libraries⁸” (Zhurnaly, 1828-1835, p. 120). The first place on this list is occupied by Bellavène’s course (*Cours de mathématiques: à l’usage des écoles impériales militaires*), the second place by Francoeur’s course (for example, see Francoeur, 1809). It is not clear whether the books being recommended are already existing translations—for example, Bellavène (1824-1825) – or French originals. Then comes the Small Mathematics Encyclopedia [*Ruchnaya matematicheskaya entsiklopediya*] (1826-1827). The fourth place is occupied by Legendre’s *Geometry* (in French), and so on. There were not enough Russian books or even books in Russian. Chizhov’s greatest efforts were aimed at increasing their numbers.

The planning and preparation of new textbooks

In response to his superiors’ queries, Chizhov submitted opinions, as they were then called, about which books and other teaching manuals were lacking, and which of the existing ones he considered to be the best. The best text for gymnasia, in his opinion, as has already been noted, was the “Course in Pure Mathematics (known under the name of Bellavène), translated from the French, with addenda and published as a textbook for the Artillery Academy” (*Perepiska*, 1826-1828, p. 341). Chizhov proposed republishing this book with certain abridgements and changes (as was subsequently done). At the same time, he quite realistically understood the difficulties connected with the transition to a new textbook, suggesting that until the new books had been introduced, everything should be left as it was, because to change textbooks several times and to buy all of the new books would be difficult for parents without sufficient means. To show the diversity of textbooks in use he lists the books used only in St. Petersburg school district – these include both original Russian textbooks by Fuss and Osipovsky and translated texts by Bellavène

8 It is noteworthy that Fuss’s books are missing from the list. A “Report by the School Committee of St. Petersburg University” prepared in 1825, in the writing of which Chizhov clearly took part, contains many critical remarks concerning them: concerning Fuss’s book on algebra, the report states that its “arrangement of topics is extremely difficult for students”; concerning his book on geometry, that it is “inadequate in terms of the strictness of its proofs”; and so on (Shmid, 1879, p. 196).

and Lacroix, and much else besides, even including books published in the Polish language (Perepiska, 1826-1828, p. 343).

Somewhat later (in May 1828), Chizhov submitted a more elaborate statement about educational texts “recognized as indispensable for the mathematical sciences” for parish schools, uyezd (district) schools, and gymnasia. For the first of these, he names arithmetical tables (for the monitorial method of education), and for those cases where such tables cannot be used due to the small number of students, he recommends another brief handbook (without giving any specific titles). For uyezd schools, brief handbooks in arithmetic and geometry are called for (again, there are no specific suggestions), as well as various visual aids – as we would say today – and instruments, models for drawing, a compass, an astrolabe, a proportional compass, and so on. To this, Chizhov adds a remark:

As the aim of these schools is to educate children who in time will devote themselves for the most part to the trades and to manufacturing, the methods of education must as far as possible be adapted to this aim.

To this end, he recommends that at least teachers should be able to make use of translated texts by Dupin (1826) or Bergery (1825). Lastly, for gymnasia, Chizhov notes the need for a handbook on the foundations of pure mathematics, and also for instruments, models of machines and geometrical objects, and models for drawing (*Zhurnaly*, 1828-1835, p. 106).

Under Chizhov’s supervision, new textbooks were prepared; he oversaw both the organizational and the methodological sides of the work. The textbooks that probably deserve to be mentioned first are those of Fyodor Busse (1794-1852), written under Chizhov’s supervision (as he himself wrote). Busse prepared textbooks in arithmetic and geometry for uyezd schools and a problem book in arithmetic to complement his textbook (in subsequent decades, these books were reissued – for example, Busse (1829-1830; 1832; 1835)).

Presenting Busse’s problem book in January 1831, Chizhov explained that problems were arranged in it “in the same order as the topics in *Textbook in Arithmetic for Uyezd Schools*,” published in 1829, and that the collection of problems was modeled on Gremilliet (1826), but in a different order and with different measures, monetary and other units, to adapt them to Russian schools. Chizhov wrote about this book: “I have repeatedly examined it and for my part I approve it for publication, and recommend that the proofreading be assigned to Mr. Busse himself” (*Zhurnaly*, 1828-1835, p. 404).

Chizhov himself explained to his supervisors why one or another delay in publication occurred – “necessitated by the coordination of the parts and the

correspondence (for which Mr. Busse made considerable financial outlays)” – (Zhurnaly, 1828-1835, p. 179) – or kept various financial accounts. In 1831, he wrote:

With the completion of this book, Mr. Busse has successfully carried out all of the instructions of the Committee related to the writing of textbooks by him under my direction and supervision. For this reason, I make so bold as to ask whether it may please Your Excellency to petition the higher authorities for a fitting reward for Mr. Busse for this (not visible, but nonetheless) useful and time-consuming work (Zhurnaly, 1828-1835, p. 404).

Subsequently, he clearly supported a petition to award Busse with the Order of St. Vladimir, fourth degree, for writing the textbooks: “this award would be quite gratifying and precious for him as a token of imperial benevolence and favorable approval on the part of the higher authorities” as it was expressed by Busse himself (Zhurnaly, 1828-1835, p. 409).

Other textbooks were also prepared under Chizhov’s supervision and on his instructions, including, for example, Francoeur’s course (1831), with Chizhov overseeing the disbursement of payment to the translator (about one thousand rubles), and the varnishing of the diagrams, and the changes made in the book (“almost all of the projections have been altered, since the method common among the French is new to [our] teachers”), and determining the book’s intended audience – the book was suited both for uyezd schools and for the lower grades of gymnasia, and the first section can be taught using the monitorial method, while the last section must be taught in the normal fashion “and only to the best and most capable students (whose number undoubtedly will never be very great)” (Zhurnaly, 1828-1835, pp. 236-237).

A new edition of Chizhov’s beloved Bellavène was also prepared, now titled *A Course Composed by A. Ya. Kushakevich and A. S. Kinderev* (for example, 1846). Presenting it, Chizhov wrote on December 27, 1834:

I have the honor of presenting to the Committee the elementary foundations of pure mathematics for use in gymnasia. They contain Arithmetic, elementary Algebra (ending here with second-degree equations and Logarithms). Geometry, Linear Trigonometry, with applications to practical problems and the application of Algebra to Geometry (including conic sections). These foundations have been composed under my direction and supervision by Messrs. Kinderev and Kushakevich, in accordance with the curriculum on this subject approved and confirmed by the Committee for the Organization of Educational Institutions. To this end, the Course in Pure Mathematics by Bellavène, translated by them previously, has now been completely revised and adapted for the needs of gymnasia, with the exception only of Geometry, in which only a very few changes and corrections had to be made.

The elementary foundations of pure mathematics presented here in my view entirely meet the intended purpose (*Zhurnaly*, 1828-1835, p. 782).

The book, however, was met with comments by Professor Perevoschikov of Moscow University, which were forwarded to Chizhov for review. On June 15, 1835, Chizhov reiterated his former opinion, noting, however, that certain alterations would be made in the book in keeping with Perevoschikov's wishes ("elementary continued fractions in the section on arithmetic are not altogether sufficiently demonstrated" – *Zhurnaly*, 1828-1835, p. 802). At a meeting on July 20, 1835, Chizhov's opinion was read out loud; he noted, not without venom, that

Mr. Perevoshchikov, evidently wishing merely to indicate which articles in general may be used to expand the various parts of the aforementioned course, in most of his remarks about it had only the completeness of the discipline in view, while paying no attention whatever to the very purpose of such a course. (*Zhurnaly*, 1828-1835, p. 811).

Based on this review,

The committee has resolved to present for further consideration to the Minister of Public Education both the aforementioned manuscript by Messrs. Kushakevich and Kinderev with Mr. Perevoshchikov's comments, and the translators' annotations on the latter, while reporting at the same time that this book has been composed in conformity with the required program and with the arrangement of academic subjects that exists in gymnasia, for which reason the Committee adheres to its previously stated opinion regarding it (*Zhurnaly*, 1828-1835, p. 811).

Certain general remarks by Chizhov

Although the present article is devoted specifically to textbooks, it must be pointed out that, while working on the Committee, Chizhov also presented his views on other aspects of teaching mathematics. Among these, we should note the negative opinion submitted by him concerning the extremely inadequate number of hours that were to be allocated for the teaching of mathematics in gymnasia with the Greek language (*Zhurnaly*, 1828-1835, pp. 117-119). Chizhov writes:

Mathematics as a practical and empirical Logic, ever since it has existed, has always and everywhere been recognized as one of the main subjects constituting a sound general education, since by stimulating the power of reasoning and teaching the strictest precision and accuracy in judgments, it thereby facilitates the development of many capabilities in the students. For this reason, it must be taught in a manner worthy of it, and in particular in the upper grades of Gymnasia, that is, thoroughly.

Chizhov then writes that the suggested number of hours for grades 4-7 is absolutely insufficient; that with such a method of teaching in gymnasia, special preparatory classes would need to be introduced for education at the University; and finally, that

there will be even greater difficulty in finding (or teaching) such skillful teachers as would be capable of teaching children in three years that which requires seven, and more hours than what has been allocated, and in such a way that in the remaining 4 years they would not forget anything. These teachers must not teach like professors. They must pay attention and make sure that each pupil has understood the proof offered by him or the problem solved by him, and force pupils to repeat this over and over again.

The decree issued in 1828, however, implemented a significant reduction in the course in mathematics.

The Committee's journals also contain *Model Instructions for Teachers in Schools for the Children of Government Office Workers* (Zhurnaly, 1828-1835, pp. 149-157), one section of which is addressed to teachers of mathematics. Although this document contains no information about its authorship, it may be supposed that Chizhov was among those who participated in its composition or at least in its approval, especially since the views developed in the Instructions are clearly similar to those expressed by him elsewhere:

35. The main efforts of the mathematics teacher must be directed at developing in the pupils, without relying on their memory alone, such power of reasoning as might help them to comprehend on their own the sequential concatenation and origination of mathematical verities. To this end, it is most useful to follow the rule of not passing from one proposition to the next without first convincing the pupils of the truth of the former with a clear and precise proof.

36. Such a method of teaching not only facilitates a fundamental comprehension of the mathematical sciences, but at the same time can be greatly beneficial for the correct formation of the reasoning ability in general, imperceptibly imparting the skill of strictly ordering ideas in speeches and compositions, drawing natural conclusion, and not making false inferences. In short, a skillful Teacher of mathematics can in a certain way replace the teacher of Logic and be an important aid even to the teacher of literature.

Discussion and conclusion

In December 1834, Chizhov wrote: "I have the honor to report that, as of today, all instructions given to me by the Committee to oversee and direct the adaptation of translations and the composition of textbooks on technical drawing and elementary mathematics for uyezd schools and for gymnasia have been carried out" (Zhurnaly,

1828-1835, p. 782). Naturally, this did not mean that work on ratifying new textbooks stopped: the Committee for the Organization of Educational Institutions was officially terminated only in 1850; but in reality its activities ceased earlier, while the approbation, authorization, and support of new textbooks was carried out both before and after this by various other agencies, until it finally became the purview of the so-called Scientific Committee of the Ministry of Education (Georgievsky, 1902).

It must be emphasized that never prior to the revolution of 1917, let alone during the 1820s-1830s, was such a uniformity achieved as may be remembered by people who were schooled in Soviet times. Kolyagin and Savvina (2013), who disapproved of those deviations from this uniformity that did occur during the 1990s, sarcastically write that the “much-vaunted variety of alternative approaches existed in the nineteenth century as well” (p. 59). In fact, it is likely that the possibility of a lack of alternative approaches simply did not enter anyone’s mind at the time, and what people were troubled by was not an excessive quantity of textbooks, but on the contrary a shortage of good textbooks (or even not such good ones – we should remember that textbooks were quite expensive).

At the same time, one feature that developed at that time and even earlier endured for quite a long time. Chizhov might have been described as the person in charge of mathematics in the country (even though mathematics was taught to only a very small part of its population). His power was not unlimited – those who were in charge of education as a whole were free to ignore, as we have seen, the opinions of lesser authorities. Moreover, other professors (the same Perevoschikov) could disagree with him on certain matters, and he would have to overcome their opinions somehow. However, Chizhov’s position was extremely powerful, and to repeat, such a position – that of the main expert on the teaching of mathematics – emerged again and again over the next hundred and fifty years.

Russian mathematics education developed indissociably from mathematics education abroad; distinctions between them were recognized, but Russian mathematics education was built on the basis of foreign (German and French) mathematics education, and no *a priori* preference was given to Russian sources.

School mathematics had not yet acquired its clear boundaries at this time; it included a number of topics that later came to be studied in other courses or stopped being studied in school altogether. The teaching of mathematics was viewed as achieving practical goals, on the one hand, but on the other hand, general developmental goals as well--teaching students logic and even serving as an aid to the study of language and literature. In this respect, Chizhov and Russian education adhered to common European norms.

During the 1840s-1860s, more detailed curricula and new textbooks appeared, and later – during the 1880s-1890s, when the number of educational institutions increased still more – a new generation of textbooks and problem books appeared, which became the foundation of Soviet schools as well, and which, while making use of German and French methodological insights, nonetheless were far more oriented toward their own enormous market. This development, which took mathematics education far from what had been accomplished during the 1820s-1830s, was nonetheless based on the crucial steps that had been taken at that time toward the recognition of what school mathematics education needed and the preparation of well-conceived sets of textbooks.

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Differential calculus in a journal for Dutch school teachers (1754-1764)

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Abstract

From 1754-1769, a monthly mathematics journal for teachers was published in the Netherlands. The journal was a combination of mathematics and news; the news sheet contained items on vacancies, comparative exams for new posts and other items which were of relevance to teachers in Dutch schools. The mathematics section contained some theory, many mathematical problems and their solutions, sent by readers and published a few months after the problems and discussions on some of the solutions. The readers were encouraged to send mathematical problems, in the form of questions. The content provides insight in the mathematical knowledge of teachers in the 18th century. Most of the questions were on arithmetic, algebra and geometry combined with algebra, however, there was also a small number of problems solved with fluxions, differential calculus, using Newton's dot notation. In this paper the use of differential calculus by the teachers in the 18th century is analysed and discussed.

Keywords: history of mathematics education; mathematics journal for teachers; knowledge of differentiation; Dutch teachers; 18th century

Introduction

In recent years several authors have published on mathematical journals for teachers, all journals from the 19th century. Preveraud (2015) wrote about American mathematical journals in the 19th century and their connection with the French textbooks, written for the Ecole Polytechnique. Furinghetti (2017) wrote about mathematical journals for teachers in Italy and the development of mathematics teachers' professional identity, also in the 19th century. Pizzarelli (2017) researched mathematics in educational journals in Turin in the second half of the 19th century. Oller (2017) published an article on the mathematics section in *El Progreso Matematico*, a journal for teachers in primary education in Spain at the end of the 19th century. As both Oller and Furinghetti remark, during the 19th century the number of mathematics journals grew continuously, including mathematics journals for teachers. That was no doubt related to the development of the profession of mathematics teachers, through teacher education and teacher associations (Furinghetti, 2017; Schubring, 2015). However in the Netherlands already in the mid-18th century, during about 16 years a mathematics journal for teachers flourished, based on private initiative and more than half a century before the start of a modest system of teacher education.

In 1754 in Purmerend, a small town in the north west of the Netherlands, a new type of journal was published, different from the many periodicals which were available during that period in the Republic. This new journal was meant for a specific group of people. *Mathematische Liefhebberye* [Mathematical Pastimes] was a journal on mathematics, for teachers in primary schools, and for teachers in private schools who offered lessons in mathematical subjects (Krüger, 2017). A large part of the content consisted of problems, entered by the editor or by readers, and the solutions of those problems, sent by the readers. The published solutions mostly showed a sound knowledge of and competency in mathematical methods that spread among users of mathematics during the 17th century, such as solving systems of linear equations and of higher order polynomial equations, use of trigonometry and of spherical trigonometry, etc. However, a small number of solutions, often new solutions of older problems from well-known authors, showed the use of a new technique: differential calculus, mostly in the dot notation of Newton.

The development of calculus and its use in mathematical applications in the 18th century is discussed by many authors, e.g. Bos (1993), van Maanen (2006) and Struik (1995). Not much attention has yet been paid to the circulation of knowledge about calculus amongst the ‘minor’ users of mathematics, the practitioners (van Maanen, 2006). One such group was formed by teachers: primary school teachers who wished to improve their basic skills in mathematics, teachers who worked in and perhaps owned a private school (called a ‘French’ school) and teachers who wished to attract private students for instruction in more advanced mathematics.

This unexpected use of differentiation to present a new method of solving old problems gives rise to some questions.

- How common was the knowledge and use of differential calculus among those teachers?
- Which problems in the journal were solved with help of differentiation?
- What was the position of calculus in the ‘body of knowledge’ of this group of teachers?

Mathematics in Dutch education in the 18th century

As described before (Krüger, 2014; Krüger, 2017) at the start of the 18th century most villages had a primary school; in towns there were usually more schools. Children could learn basic arithmetic, usually after learning to read and to write. In many towns there were privately owned primary schools, the French schools, which offered comprehensive primary education, with more subjects and often more mathematical topics. Thus more advanced arithmetic and other mathematical subjects, e.g. accounting, geometry, algebra, trigonometry and navigation, could be

learned through private tuition, in specialized mathematics institutes and through self-instruction.

At universities, for example in Leiden, Utrecht and Franeker, mathematics was part of the undergraduate courses. Sometimes also lectures on architecture, hydraulics and other applied mathematical topics were offered. Some mathematical subjects were offered in Dutch language, such as practical geometry (surveying techniques) and fortification at the universities of Leiden (Duytsche Mathematique or Dutch Mathematics) (Krüger, 2010) and Franeker (Dijkstra, 2012), navigation and fortification at some of the ‘illustere scholen’¹.

Professor Willem Jacob ‘s Gravesande (1688-1742), from 1717 professor in astronomy, mathematics and philosophy at Leiden University, was an admirer of Newton and did much to propagate ideas of Newton on the Continent. He taught experimental physics, combined explicitly mathematics and physics and emphasized the practical usefulness of mathematics (van Dijk, 2011). It is likely that ‘s Gravesande as well as his colleagues and successors, Petrus van Musschenbroek (1692-1761) and Johannes Lulofs (1711-1768) gave lectures in differential and integral calculus (van Dijk, 2011). However, not many teachers in primary education would have had the opportunity to follow lectures at a university.

There was a demand for mathematics teaching, especially mathematics that could be applied in practical situations. Throughout the 17th and the 18th century knowledge of practical mathematics gained more and more relevance, for example for navigation, civilian and military architecture, surveying, water management, military engineering and the fine arts (Krüger, 2012; van Maanen, 2006). The demand for teaching of mathematical subjects naturally led to a demand for teachers of mathematics. As in 16th century England (Rogers, 2012), mathematics education was very much a matter of ‘grass roots’ activities, with an emphasis on useful mathematics.

However, there was no national system of secondary education, nor were there institutes for teacher education. Whoever wished to become a teacher had to find a post as help-teacher in a school. After some years a help-teacher could apply for a teaching position of his own in a primary school. In the 18th century it was expected in certain regions to sit a comparative examination for such a teaching position (Krüger, 2017). These examinations included more and more often mathematics, sometimes consisting only of a few arithmetic problems; but it happened also that the mathematics exam consisted of four to five subjects and took many hours. If one wished to start a private (French) school or to teach in a French school a broad knowledge of mathematics definitely was an asset. So for teachers, knowledge of

¹ These institutes offered university type courses, without right of promotion. They were situated in the larger towns, such as Amsterdam, Rotterdam, Middelburg and Deventer.

mathematics was important for their career options, however, they had to acquire the knowledge by themselves.

Mathematische Liefhebberye, a mathematics journal for teachers

Mathematische Liefhebberye was a monthly, published from April 1754 until December 1769 by the librarian Pieter Jordaan² (Krüger, 2018). It consisted of two parts: the *Nieuws* [News], for teachers of Dutch and French schools and *Mathematische Liefhebberye* [Mathematical Pastimes]. Important items in the section with news were vacancies and the mathematics questions of the comparative examinations for a teaching position, especially in the west of the country. The mathematics section focused initially on arithmetic and algebra, also solutions of the questions of examinations were published regularly. It was the first journal of this kind in the Netherlands. Mathematical topics were sometimes discussed in more general journals, such as *Boekzaal* and the *Journal littéraire de la Haye*, of which 's Gravesande was one of the editors (Jorink & Zuidervaart, 2012). However, these were general journals, for an erudite audience, while *Mathematische Liefhebberye* was aimed at teachers in general and teachers of mathematics.

We will discuss a few characteristics of the journal which are relevant to the topic of this paper. For a more extensive description of content and contributors, see (Krüger, 2017; Krüger, 2018). Though the name suggests a recreational journal, the aims stated by Jordaan and by the longest serving editor, teacher and mathematician Jacob Oostwoud (1714-1784), indicate that it was a journal meant to improve the mathematical knowledge of teachers and to enable teachers skilled in mathematics to share their knowledge with a larger group of colleagues. Readers were encouraged to send problems and solutions; there were also occasionally discussions about published solutions. Several teachers contributed regularly by sending problems; a much larger group contributed by sending solutions. Oostwoud and his successor, Louis Schut, published the names of the contributors of problems, the names of the readers who solved the problems and also which problems each one solved and for many problems the source, often from Dutch mathematics books, but also from German and English books and journals.

The first editors, the teachers P. Karman and P. Molenaar, treated series extensively. They also published some other problems on algebra, arithmetic and probability. Oostwoud, mathematics editor from November 1754 until July 1765, started with the treatment of arithmetic and with exercises which consisted of both simple and more advanced problems. His aim was to give opportunity to improve mathematical skills for different levels; for those teachers who were starting to learn

² The digitalized version of all year volumes is available at the site of the library of the University of Amsterdam (<http://uba.uva.nl/home>)

mathematics and for those who already had mathematical skills in varying degree. A method to improve the skills of those who were not advanced in mathematics was explanation of a topic, followed by some exemplary problems and problems for the readers to solve. Thus Oostwoud explained several rules in arithmetic, the procedures in modern algebra, equations, cossic algebra, geometric, arithmetic and harmonic series, fractions, simple calculations in probability, special numbers, logarithms, some topics in spherical trigonometry, etc.

Besides these clarifications there were always many problems to solve, varying from rather simple to more complex. Oostwoud selected from existing publications and published problems sent by readers; these also often came from other publications.

Examples of the type of problems are

- simple questions on arithmetic and algebra;
- modelling of a situation, resulting in (systems of) linear and quadratic equations;
- problems to do with trade;
- calculations with series;
- simple questions on probabilities;
- problems solved with proportionality, also in mechanics;
- questions about mixtures of substances;
- questions about distances travelled;
- problems in plane and solid geometry;
- calculation of a maximum or minimum.

In November 1758 the first thousand problems had been published, and on that occasion Oostwoud gave a categorization of these problems, which differs very much from the topics mentioned above. He gave again a categorization after the second thousand of problems was published, that number was reached in March 1763. The third thousand questions were complete in July 1769. See table 1.

Table 1 Categorization of three times thousand problems

Category by Oostwoud	1-1000	-2000	-3000	Total
	'54-'58	'59-'63	'63-'69	
Interesting problems	97	247	139	483
Problems on history or other important matters	11	13	9	33
Proofs	5	7	6	18
Probability	10	11	--	21

Calculation of most and least (maximum and minimum)	58	15	2	75
Mechanics	5	--	--	5
Finding (a) word(s) as solution	5	14	22	41

There is some overlap between the first category and the others. In the solutions of problems on ‘most and least’ quite often differential calculus was used. Oostwoud did not mention problems on algebra, arithmetic, series, geometry or other common mathematical subjects. He seemed to mention mainly the topics which were a bit out of the ordinary and those problems which were a bit more demanding.

Use of differential calculus

Already in the solution of one of the first problems, April 1754, when Molenaar and Karman were editors, differential calculus was used. The last time this method was used occurred in a problem of February 1764. In the next paragraph some examples are shown. All together 70 of the 75 problems on maximum/minimum were solved with help of differential calculus; in some cases solutions with and without differential calculus were published. The use of differentiation reached a peak in 1756 and 1757 and then diminished. See also Figure 1. During the time Schut was editor, no differential calculus was shown.

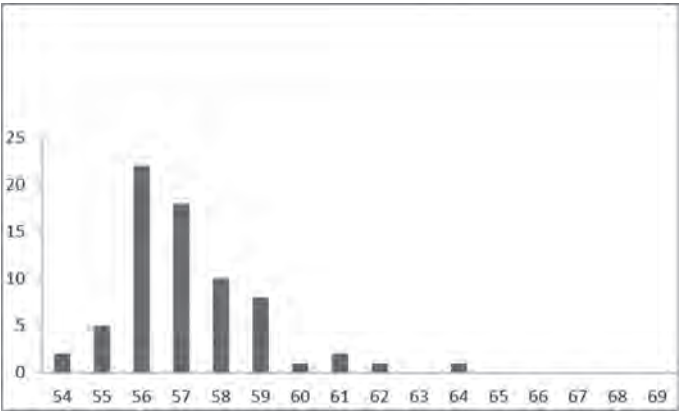


Fig. 1. The frequency of solutions with differential calculus, per year.

The large increase in solutions by differentiation in 1756 and 1757 was entirely due to Oostwoud himself. In 1756 he published 19 (out of 22) of those problems, originating from *Mathematisches Sinnen-Confect* [Mathematical Delights] by Paul Halcke (1719). In 1757 also 13 of the 19 maximum/minimum problems were added by Oostwoud, again from Halcke. In 1767 he published a Dutch edition of Halcke’s

book, with many additions, referrals to Dutch mathematics authors and also application of differential calculus for maximum/minimum problems. Halcke himself, a 'Rechenmeister' and member of the Mathematical Society in Hamburg did not use calculus in his book, which treated arithmetic and algebra. All together 43 problems solved with differential calculus originated from Halcke's book.

Some examples³

1754: 33 (by Molenaar & Karman, no source mentioned)

A measurer of seeds needs a new measuring vessel; he wishes to have it made of the least amount of wood possible. Which will be the ratio of the height to the width?

Solution by D.T.V.W and Jan van Leeuwen, translation of the first part of the solution, see also Figure 2⁴:

Let the volume be a . The proportion of diameter and circumference equals 1 to p [$p=\pi$]. Let the radius be x and the height y . Then $1:p :: 2x:2px$ is the circumference. The area of the bottom = pxx . The volume = $pyxx=a$.

The lateral surface equals $2pxy$. The total area = $2pxy+pxx$ which should be a minimum.

Continued as follows: Which Fluxion equals (Welker Fluxie is, Figure 2)

Welker Fluxie is $\frac{2py\dot{x} + 2p\dot{y}x + 2p\dot{x}x}{2p} = 0$
 Of $\dot{y}x + x\dot{y} + \dot{x}x = 0$
 En $pyxx=a$ Welker Fluxie is $= \frac{p\dot{x}xy + 2py\dot{x}x}{px} = 0$
 Of $\dot{x}y + 2y\dot{x} = 0$
 Dies $\dot{y}x + x\dot{y} + \dot{x}x = 0$ } Sub.
 En $2y\dot{x} + x\dot{y} = 0$ }

 $\dot{x}x - y\dot{x} = 0$
 $\dot{x}x - y\dot{x} = 0$
 Ergo $x - y = 0$ En dus $x = y$
 Dat is, de hoogte gelyk aan de halve wyte.
D. T. V. W.
 NB. Is mede Beantwoord door Jan van Leeuwen
 te Rotterdam.
 34 VOOR-

Fig. 2. The last part of the solution of problem 33 (1754), by two readers.

³ All transcriptions and translations of the exercises are by the author.

⁴ The letter symbols were all written in italics.

A typical problem from Halcke (1719), provided by Oostwoud, is number 337 in 1756 (Halcke nr. 371)

Divide a given number q in 4 parts, of which $a:b:c=r:s:t$. The sum of the four products abc , bcd , cda and dab is the maximum. What are the parts?

Solutions were sent by 5 readers, out of 22 who sent solutions that month.

Steld dedeelen $= rx, sx, tx, \text{ en } q-r-s-t, x$,
 zoo is $qst+qrt+qrs, xx - sst-2rst-stt-rrt-rtt-rrs-rss, x^3$, of $(st+rt+rs, q=m, \text{ en } stt+2rst+stt+rrt+rtt+rrs+rss=n, \text{ nemende}) mxx - nx^3 = \text{de Maxim. wiens differentie is}$

$$\frac{2mxx - 3nx^2}{3nx} = 0$$

 $x = \frac{2m}{3n}$, $r, s, \text{ en } t$, als mede q nemende
 als boven, komt $m=1560$, en $n=210$, dien-
 volgens $x = \frac{16}{21}$, &c.
 S. H.

Fig. 3. The solution by S.H. to problem 337 (1756).

Translation:

Let the parts = rx, sx, tx and $q-(r+s+t)x$, then $(qst+qrt+qrs)xx - (sst+2rst+stt+rrt+rtt+rrs+rss)x^3$ or

(let $(st+rt+rs)q=m$ and $sst+2rst+stt+rrt+rtt+rrs+rss=n$)

$mxx - nx^3$ = the maxim(um), of which the differential is⁵

$2mxdx - 3nx^2dx=0$ Divide by $3nxdx$

Etc.

Without the use of calculus an expression for x would be found by means of lengthy calculations of products of polynomials, summing those products, finding a suitable factor for division and so diminishing the degree of the polynomial. See for an example Halcke (1719, p. 300) or the translation by Oostwoud (1767, p. 422), both available on-line⁶.

5 For technical reasons the dot-notation is replaced by dx, dy , etc. in transcription

6 Halcke (1719), *Deliciae Mathematicae* in Google books en zie ook <http://echo.mpiwg-berlin.mpg.de/MPIWG:6XTQN17A>. Oostwoud (1767), *Mathemaisch Zinnenconfect* zie Google books.

A different problem, contributed by a reader, G. de Gooyer, in 1757 is 555. No source was provided.

Two straight roads, both East-West, at a distance of 135 rods. At DE Daphne is running West, at a velocity of $\frac{1}{2}$ rod/second. At AO runs het pursuer Apollo, at a velocity of 36 rod/minute. However, on the rough land between the roads his velocity is only 27 rods/minute. Which course should Apollo choose to reach his mistress in the shortest time and how much time will he need?

Solutions were sent by 3 people, out of 15 who sent solutions that month. In Figure 4 is shown part of the solution sent by J. Kok and J. Kooyman, both teachers at Texel.

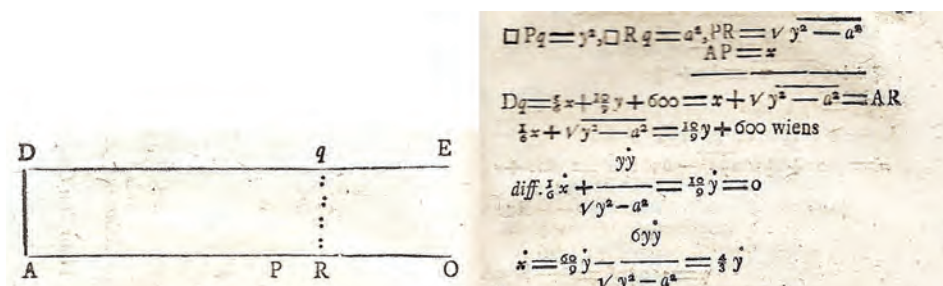


Fig. 4. Part of the solution to problem 555 (1757), by Kok and Kooyman.

The position of differential calculus in the journal

The problems sent by the readers were published if they met the criteria of the editor(s). Between 1754 and December 1758 about 90 persons sent problems which were published; and between January 1759 and December 1769 about 100 persons saw their questions published. Of these contributors about ten persons sent problems which were solved with help of differential calculus. All those had to do with maximum or minimum; the sources mentioned were Paul Halcke (1719), Laurens Praalder (1753), Gerard Kinckhuysen (1723), Frans van Schooten jr. (1659, 1660) or problems previously published by Molenaar and Karman, but without a published solution. For all those problems more traditional solutions were known.

All together a few hundred people participated in sending solutions to all type of problems. About 27 of them used differential calculus in solutions, sometimes only once or twice, others such as G. de Gooyer, J. Kok and J. Kooyman, very regularly. So it was a relatively small group that used this method. However, it looks as if gradually more readers started using differentiation, perhaps because some became more aware of its practical use. Besides the relative low number of contributions two matters are remarkable.

Firstly: there was no explanation of the procedure of differentiation or the mathematical background. Oostwoud and others regularly gave an explanation, of virtually every topic in the journal; however, there was no explanation at all on differential calculus. Not how to use it, nor when to use it.

Secondly: problems on maximum/minimum were the only problems for which differentiation was used in the solution. The notation was nearly always Newtonian, the words used were both fluxion ('fluxie') and differentiate ('differentie').

So only a minority of the contributors used calculus, they only used it for one type of problem and there was no information on the why and how of the use. It looks as if a small group of people knew how to use it and that some were interested to learn this method to solve a well-defined type of problem. But somehow it seemed not to be part of 'official' mathematics. Since calculus was not asked in the comparative examinations for teachers, a large group would not be bothered, perhaps only those who were more curious. Also the topic might be taught to private students by skilled teachers in mathematics, such as Oostwoud at Oost-Zaandam and R. van Vreeden at Arnhem.

From where did the knowledge about use of differential calculus come? Mathematical knowledge could be dispersed through books, private lessons or lectures, written manuscripts such as dictations from private lessons and journals. Most teachers had a simple background; in the 18th century most of them would not have the means to attend university lectures or to take private lessons from a professor, such as 's Gravesande or Lulofs. Manuscripts of simple people such as teachers or artisans are extremely rare, they usually aren't preserved. So that leaves books and articles in journals, available to the Dutch teachers of mathematics.

Books and journals as sources of knowledge about calculus

Oostwoud mentioned more than 120 authors as source for published problems and for different types of solutions. Though he mentioned no publications in relation to the use of differential calculus, some of the books and journals mentioned in relation to other topics might contain something about calculus as well. We will first look at some books, published before 1750, which had possibly relevant information and were available to Dutch teachers.

Books

Let us start with a well-known and respected mathematics teacher and prolific author of mathematics books: Abraham de Graaf (1635->1717). He published on arithmetic, accounting, geometry, algebra, trigonometry, navigation, etc. and was often referred to by Oostwoud and others. It is very likely that books by De Graaf

and also the publications by Struyck and by Hellingwerf (see below) were a source of knowledge for Oostwoud and others who were interested in the new technique of differentiation. In *Analysis of stelkunstige ontknoping in de meetkunstige werkstukken* [Analysis or algebraic solutions in geometrical problems] (1706) De Graaf dealt with maximum and minimum, tangents to curves, etc. It is a theoretical treatise building on his previous work and using Cartesian geometry. De Graaf mentioned Descartes, Huygens, Slusius, Hudde, Frans van Schooten and also Von Tschirnhaus⁷, but not Leibniz or Newton. De Graaf based his treatment of tangent lines, maximum and minimum etc. on the work of these 17th century mathematicians from the northern and southern Netherlands. Their work and also *Analysis* by De Graaf may be considered a forerunner of the calculus of Leibniz and Newton (see also Jahnke, 2009). For example Proposition XVIII states:

If in a curve one has two points infinitely close together then the extended chord is tangent to both points.

Consequences

1- If a line CK touches the curve in C and on the curve is another point D, infinitely close to C, then CK also touches the curve in D.

De Graaf used specific symbols; next to x, y, z for variables, he used a, b, c for parameters, f, g, h for infinitely small and m, n, o for infinitely large. In the chapter on tangents (how to find tangents on curves) one finds the following example (Figure 5).

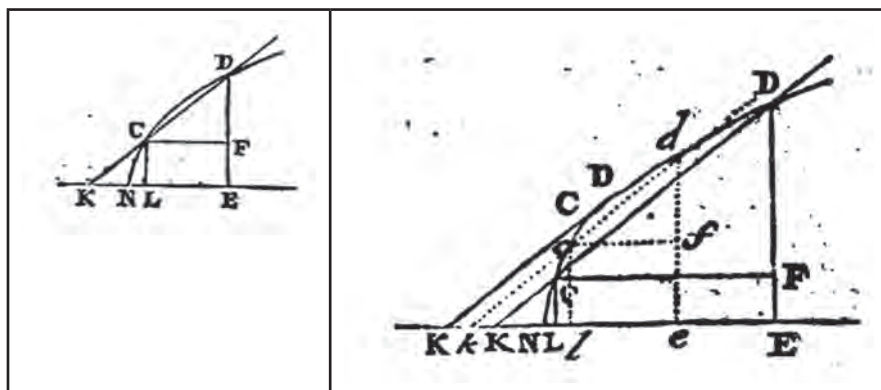


Fig. 5. A. de Graaf, to find the tangent to a point of a curve LCD with known equation. (*Analysis*, pp 14-18)

Compilation of text by De Graaf (p.15)

⁷ Christiaan Huygens (1629-1695), Jan Hudde (1628-1704), Frans van Schooten Jr. (1615-1660), René François de Sluse (also Sluze, Slusius) (1622-1685) were mathematicians from the Northern and Southern (Sluse) Netherlands, Ehrenfried Walther von Tschirnhaus (1651-1708) was a Saxon mathematician and philosopher.

NL= x , LC= y , CF= f , DF= g , KL= z and $z:y=f:g$, thus $f=(gz)/y$

Let the equation of the curve be $y^3=rrx$

NE= $(x+f)$, ED= $(y+g)$

$y^3+3yyg+3ygg+g^3=rrx+rrf$; subtraction of $y^3=rrx$

$3yyg+3ygg+g^3=rrf=(rrgz)/y$; divide by g and multiply by y

$3y^3+3yyg+ygg=rrz$

If $f=0$ then also $g=0$

Thus $3y^3=3rrx=rrz$ or $3x=z=KL$

One could say that De Graaf dispersed through his *Analysis* the concepts of infinitely small distances, laying the ground for calculus.

In 1718 Pieter Hellingwerf, a mathematician from Hoorn, published *Wiskonstige Oeffening* [Mathematical Practice]. This book contains chapters on mechanics, music, sundials, specific gravity and mathematical miscellaneous, more or less an overview of existing knowledge, with mathematics applied to physics and reports of experiments by the author. In the last chapter, on p.421, Hellingwerf presented in a short text on “The flow of magnitudes” a definition of differentiation and integration. He mentioned differentiating (‘differentiëren’), flow (‘vloeyen’), reduction on the infinitely small (‘reductie op het oneyndig kleyn’) etc. He mentioned De Graaf and his symbols, but gave an example using dot notation, similar to the notation in use in *Mathematische Liefhebberye* (Figure 6).

This Hellingwerf does not figure among the authors mentioned by Oostwoud, however the style of this text in combination with the overall character of the publication (overview of existing knowledge) suggest that the procedure of ‘flow’ (differentiation) was a relatively new, but known procedure.

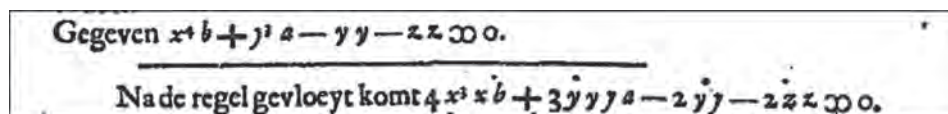


Fig. 6. Hellingwerf (1717), p.421. De Graaf used the reversed symbol instead of ∞ .

This Hellingwerf does not figure among the authors mentioned by Oostwoud, however the style of this text in combination with the overall character of the publication (overview of existing knowledge) suggest that the procedure of ‘flow’ (differentiation) was a relatively new, but not unusual procedure.

In 1738 Joan Christoffel van Sprögel published the first volume of *Grond-beginzelen van alle de mathematische wetenschappen* [Principles of all mathematical sciences], a translation of *Anfangsgründe aller mathematischen Wissenschaften* by Christian Wolff (1710). This work contained clear definitions and descriptions of differential calculus, with all rules for differentiation of algebraic expressions explained in a lucid way, using the notation of Leibniz. Applications mentioned are tangents to curves, but not problems with maximum/minimum. Wolff was quite popular in the Netherlands, Oostwoud also mentioned him a few times.

In 1740 Nicholaas Struyck (1686-1769), a mathematician, who wrote about probability and several other mathematical topics, published *Inleyding tot de algemeene geographie met eenige andere natuur- en sterrekundige verhandelingen* [Introduction to general geography with some treatises on Physics and Astronomy]. Struyck was an admirer of Newton and mentioned him several times. In the second part (Physics and astronomy) he used differentiation of algebraic expressions a few times, always mentioning 'flux' and always with dot-notation, see for example p.168. Struyck was a member of the Royal Society in London and correspondent of the Académie des Sciences in Paris. Oostwoud referred to Struyck regularly, for example in 1757, in the solution of exercise 275.

In 1749 Symon Panzer, a teacher of mathematics, astronomy and navigation in Embden⁸ published *Mathematische Rariteit-kamer* [Mathematical Repository]. The book was mainly about algebra; it contained in four volumes many exercises and their solutions on basic algebra, polynomial equations up to degree 6, arithmetic, geometric and harmonic series and applications, rules to find the largest and the smallest (maximum and minimum), logarithms, etc. As origin of many exercises Panzer mentioned Meissner, Halcke, De Graaf and Kinckhuysen. On p. 441 treatment of maximum and minimum starts. Panzer mentioned first the use of arithmetic progression, referring to De Graaf and Kinckhuysen, followed by differential calculus and the names of Leibniz and Wolff. There are definitions, rules for differentiating algebraic expressions and applications in finding maximum and minimum, using problems from Halcke's *Mathematisches Sinnen-confect* as source. He used the notation of Leibniz. Panzer was very well known to Oostwoud and his readers and more than 30 times one of his exercises or solutions were used in *Mathematische Liefhebberye*. Besides the notation used, this book is very similar to what Oostwoud wrote. Both Oostwoud and Panzer used Paul Halcke as a major source for exercises and both were a member of the Mathematical Society in Hamburg.

In 1750 Johan Lulofs, professor in mathematics, astronomy and philosophy at Leiden University, published *Inleiding tot eene natuur- en wiskundige beschouwing des aardkloots* [Introduction to a physical and mathematical view of the Earth], in which

⁸ Embden is situated in Saxony (Germany), very close to Groningen (Netherlands).

he combined geography with physics and mathematics. Lulofs used occasionally differential calculus, he mentioned amongst others Newton and Keill, a follower of Newton. However, he used the Leibniz notation. Lulofs' work was known to Oostwoud, he mentioned him occasionally.

The lists of authors, which were provided by Oostwoud and later Schut, also contained many authors from France, Germany and England; sometimes but not always translated into Dutch. Regarding these authors there seems to be no link with calculus.

Journals

Sometimes Oostwoud mentioned a journal as a source for published problems. Examples were *De maandelykse Berlynsche schatkamer der wijsheid* [The Monthly Treasure Trove of Wisdom from Berlin], het *Europeis Magazyn der byzondere zaken* [European Magazine of Extraordinary Matters], the *Ladies Diary* (also *Woman's Almanach*), the *Gentleman's Diary* and the *Imperial Magazine*. The English journals were mentioned from 1759 onwards, and provided some examples of differentiation (fluxion). The first two journals (Dutch) were also mentioned before 1758, but appeared to offer no relevant information about differentiation.

Discussion

In this mathematics journal, solutions with differential calculus were limited to one type of problem and a relatively small group of readers. There are indications that the group of users increased, possibly through the examples in the journal. Though by the mid-eighteenth century even in Leiden, the centre of promotion of Newton's idea, the Leibniz-notation had become more common in mathematical publications; Newton's dot-notation prevailed in *Mathematische Liefhebberye*.

The body of knowledge of teachers in Dutch and French primary schools was defined by the practical use of mathematics. In the first place arithmetic, a quite extensive topic. Other important topics were algebra, with polynomial equations, and of course geometry. Algebra in geometry, linked with the name of Descartes, had become respectable and quite common. Potential private students were those who might want to learn surveying (lessons in geometry, trigonometry, logarithms and algebra), navigation (geometry, spherical trigonometry, astronomy, algebra and logarithms) or commerce (arithmetic, algebra and accounting). Differential calculus was finding its applications, for instance in geography, astronomy and for mathematics, in the determination of tangents to curves and in finding maxima and minima of polynomials.

Teachers of mathematics did not develop new mathematical knowledge; they were instrumental in dispersing the existing knowledge; most often the knowledge about the practical use of mathematics. Finding a maximum or minimum (volume, area, costs, etc.) evidently belonged to their domain, tangents to curves far less so.

This explains to some extent why the use of differentiation was limited to this one type of problems and also why only a limited group of these teachers could use or was willing to learn differentiation. On the other hand, the use of this new method by a growing group of teachers shows that there was interest in new developments and new methods and that among those teachers there were quite a few with considerable knowledge of mathematics. For one thing mathematical competency enabled one to improve one's income, by acquiring better paid positions and through attracting private students. There was not yet a structure with secondary education to attract mathematics teachers, so in primary education one would find a large variety of mathematical competency among teachers.

Vermij (2003) discusses the introduction and dispersal of Newton's ideas in a network of science and mathematics amateurs in Amsterdam in the 1690's by David Gregory. This network, in which Adriaan Verwer, a merchant from Rotterdam and Joannes Makreel, a broker and amateur mathematician, were central figures, probably formed the first group of admirers of Newton in the Netherlands. Makreel also had been part of a network with Ewald von Tschirnhausen and Abraham de Graaf, in the 1680's and he had close connections with Bernard Nieuwentijt (1658-1719), a physician and regent in Purmerend. Van Nieuwentijt published during 1694-1695 two tracts on calculus, in which he showed himself a defender of Newton and criticized Leibniz. He was a very popular and influential author in the Netherlands and elsewhere through his physico-theologian work from 1715, *Het regt gebruik der wereldbeschouwing* [The Religious Philosopher: Or, the Right Use of Contemplating the Works of the Creator] (Jorink & Zuidervaart, 2012). So there was an early group of non-academic mathematical amateurs and authors of textbooks whose members knew about and were admirers of Newton's mathematical work, in Amsterdam and vicinity. Through influential authors such as De Graaf, Nieuwentijt and Struyck, it is likely that other teachers of mathematics in the area learned about fluxions and differentiation.

The lack of discussion of the procedure and of the background of differential calculus indicate that as far as these teachers were concerned calculus was not yet part of the established body of knowledge. It had the character of a method which worked well in practice, but the mathematical foundation was not too clear. An instructive book in Dutch language on calculus was only published in 1775 by Johannes Arent Fas, assistant professor at Leiden University. He wrote the book for his students at the course for Duytsche Mathematique at Leiden University.

It is intriguing that the Newton dot-notation persistently was used, with few exceptions, in *Mathematische Liefhebberye*. By 1754 the Leibniz notation or at least a notation dx , dy was at least as usual, if not more common. One reason may be typographical. If the fraction notation is used, one needs more than one line unless the fraction is printed very small. In *Mathematische Liefhebberye* the fractions were printed very small, which made reading more difficult. A dot-notation was relatively quick to write and not too complicated in printing.

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Teaching mathematics in Moroccan high schools in the past fifty years

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Abstract

This paper will trace the educational development in the teaching of mathematics in Moroccan high schools over the past fifty years. The discussion will cover the main reasons behind three major reforms of mathematical education, and present innovations in the concepts and pedagogical approaches of their corresponding curricula. We will also proceed to the analysis of the educational material associated with each reform (academic studies, committee reports, official documents, textbooks... etc.), with the aim of highlighting the evolution of mathematical concepts taught in Moroccan high schools over the last half of the past century.

Keywords: arabization, high schools, National Charter, reform, teaching

Introduction

History shows that any curriculum reform within a country reflects its internal evolution as well as the external factors, such as international reform movements, inciting the re-evaluation of the established norms. The reforms of mathematical education in Morocco are no exception to this rule. In fact, Morocco has experienced three major reforms in the teaching of mathematics since its independence in 1956. These reforms are: the Reform of modern mathematics (1960's), the Reform introducing Arabization (1980's), and the Reform advocated for by The National Charter for Education and Training (turn of the 20th century). While the first reform concerned mostly high school curricula, the two other reforms affected the middle and primary school programs as well¹.

We present in this paper the motivation behind each one of these reforms, the general objectives they aimed to achieve, and the pedagogical orientations defining

¹ The Moroccan school system consists of three cycles: primary school, middle school and high school. Each cycle has its own characteristics (a different local, a specific organizational structure, particular profiles for its teachers ...etc). The number of years in each cycle and the names given to it have been modified over the years as follows:

- Number of years: 1956-1964: (6,3,3), 1965-1988: (5,4,3), and 1989-2017: (6,3,3).

- Cycles' names: 1956-1989: (primary, first cycle of secondary, second cycle of secondary), 1989-2002: (first cycle of fundamental education, second cycle of fundamental education, secondary cycle), and 2003-2017: (primary, intermediate secondary, qualifying secondary).

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them. Emphasis will be placed mainly on high school programs, while some aspects of the lower levels may be mentioned occasionally.

The reform of modern mathematics

The teaching of mathematics underwent a major reform in several countries since the late fifties of the 20th century. This renovation revolved mainly around the introduction of “modern mathematics” in pre-university education. Several studies have highlighted the reasons behind this major pedagogical shift (Carsalade, Goichot & Mermier, 2013; Bkouche, 2013), while the premises of this change can be noticed as early as the 30’s of the 20th century (Barbin, 2013). The new perception of mathematics was explained for instance by the official Moroccan document presenting the programs, named Official Programs and Instructions (1969)², in the following words³:

It is important first to have a precise idea of what has become of mathematics. A few years ago, school curricula provided our pupils with a conception of mathematics from the ancient Greeks and the mid-nineteenth century. After about a hundred years of profound evolution, the researches and discoveries of mathematicians, and their reflections on the reality and the object of mathematics, have led to new ideas expressed in a new language. Current mathematics is no longer exclusively, nor is it essentially, what Augustus called “the science of the indirect measurement of magnitudes”.(p.6)

The document also stated that the new mathematics gives more importance to the relationship between objects than to the objects themselves, making of set theory and algebraic structures central topics of the curricula.

Any “calculation”, in the most general sense, has two main components: the objects on which one operates and the operative rules. Of these two constituents, only the latter is essential. Mathematics appears as the science of relations and systems of relations, the nature of the objects to which these relations apply becoming irrelevant. (p.7)

(...) Finally, the study of a structure, based solely on a small number of axioms unrelated to the objects studied, is an exercise of pure reasoning. It teaches to order and enchain thoughts according to a rigorous method, admirably developing clarity of mind and rigor of judgment. (p.9)

Another element that characterizes this new vision of mathematics is the abandonment of traditional geometry and its replacement by vector and affine geometries. Arguments of famous mathematicians were cited to justify this choice. One of these

2 In the following we use MEN(1969) to refer to this document.

3 All translations are by the author of this paper.

arguments - as reported by the MEN(1969) - is that of the French mathematician Jean Dieudonné, who stated:

It seems to me that the goal is to overcome two definite psychological difficulties:

- (i) The student must be made aware of the necessity of an axiomatic treatment of mathematics.
- (ii) The student must be familiarized as soon as possible with the constant handling of certain abstract notions, the most difficult to assimilate being undoubtedly that of application (or “transformation”) and even more perhaps that of calculation on applications.

As it can be said without exaggeration that either difficulties are truly cornerstones of the entire modern mathematical edifice, all the other aspects of teaching in the first years should be consciously subordinated to the assimilation of these ideas. (...) it is thus desirable to free the pupil as soon as possible from the straitjacket of traditional “figures” (point, right, plan being an exception of course) by mentioning them as little as possible, in favor of the idea of a geometric transformation of the whole plan and of space, which must be constantly emphasized and illustrated by numerous examples. (p. 21)

In Morocco, modern mathematics appeared in school programs from 1962 on. It is worth noting that the introduction of modern mathematics in Morocco faced less obstacles compared to other countries⁴. The former director of the Ministry of National Education in Morocco during the 1970's, Mohammed Akkar (2002), explained in this regard:

The needs of higher education and the concern of the public opinion for the growing gap between the evolution of sciences and that of education led to this reform in France. In Morocco, it was a decision of the head of programs and mathematics education, J.P.Nuss, who was strongly influenced by the ideas and movements circulating in France and Belgium at the time. The “modernization” of the programs was carried out in Morocco in two main stages: one in 1962 by Nuss and the other in 1968 by Peureux. (p.180)

Another excerpt coming from the MEN(1969), added the following:

4 For example, in France despite the fact that we had started preparing for this reform since the 1950s as it is well explained by Bernard Charlot (1986), and despite the creation of IREM to support it, voices had been raised against this reform since the early 1970s. Carsalade et al. (2013) reports that:

By 1972 with the introduction of the program of the fourth year, the very spirit of modern mathematics began to be attacked from all sides; the criticism goes beyond the group of refractory teachers and spreads to scientists, the public, the press. On all sides, we condemn the excess of abstraction, the heaviness of the new programs and their dogmatism. (p.241)

Many countries indulged in the renovation of mathematical education. The reform began in Morocco in the aftermath of independence. Our methods and programs follow a normal evolution adapted to the realities of our teaching. (p.8).

The instructions of this reform encouraged teachers to use the language of modern instead of traditional mathematics in their teaching. These instructions stated:

The language of this theory is the best fit for mathematics. It allows for a clear, precise, and simple expression, better and easier to grasp than ordinary language. The teacher, having become aware of the vagueness of the traditional mathematical language, will make a very wide use of the “new” vocabulary thus placed at his disposal. Certainly, he will retain, with regret, certain words consecrated by custom. (p. 10)

These instructions also stipulated that this new conception of mathematics would be an effective means to train students intellectually, consequently achieving one of the fundamental aims of mathematical education.

This conception leads to an exceptional economy of thought, by standardizing the mathematical tool. It created an instrument so general and powerful that it now applies to all activities of the mind, for example, to the social sciences (...). Considering the essence of the mathematical method as the main aim of mathematical education, this novel approach helps achieve intellectual training more efficiently than any other method. (p. 9)

The main reasons for supporting this reform seem commendable. On one hand, symbolism and structures provide a safe, simple and rigorous language. On the other hand, the study of algebraic structures develops intellectual training, clarity of mind and thought, and rigor of judgment.

Despite the general support of this reform in Morocco, some Moroccan authors called out on the absence of a national need justifying it. Those authors believed the reform to be an experiment in which France, former colonial power, tested the introduction of modern mathematics in Morocco before including them in French curricula. For instance, for al Jabri (1985) this reform is an illustration of the dependence of the Moroccan educational system on its French counterpart. In this regard, al Jabri (1985) said:

These “reviews and amendments” were not, one day, born of a desire to create a purely national program that responds to the requirements of the desired “Moroccan National School”. They were, in most cases, unclear, answering other motives far from these requirements. If we have counted the number of times our programs have undergone operations of “adjustment and change”, and we search for the hidden motives dictating those, we will find they come down mostly to one of three things.

<One of them is>⁵ The almost complete subordination to the educational system in France and its developments. This dependence manifests itself in two basic aspects. The first is the imitation of the developments happening in France blindly, without taking into account our reality, our circumstances and our specific needs. The second is the desire of French circles to conduct a particular experiment in Morocco, and study its results, in order to apply it in France if found valid. This is what happened when Morocco - before France - started teaching modern mathematics in all high schools' sections ... (pp.80-81)

This argument has its origin in the difficulties faced by Morocco at this time in managing the legacy left by the protectorate period in relation to education. These difficulties suggest that the reform of mathematics education of this period did not meet the needs of the Moroccan reality (see Appendix for more details).

The reform introducing Arabization

This reform brought about two major changes. The first concerned the language of teaching, which was changed from French to Arabic (hence the term Arabization)⁶, while the second related to the abandonment of modern mathematics from school programs.

While many disciplines such as philosophy, history and geography, transitioned to Arabic without much difficulty, the Arabization of scientific disciplines turned out to be harder than expected⁷. As a result, the Arabization of the latter subjects only began in the late 1970's. Mathematics itself was taught in Arabic as early as the 1960's in primary schools. However, as it seemed challenging at the time to propagate translation to subsequent cycles, it was decided to go back to French from the third year of primary school (French as a foreign language was first taught to students that same year at the time). In 1978, the plan of Arabization for scientific disciplines was resumed and preparations for its concretization were launched. Finally, the reform was carried out for primary school curricula in 1980, middle school in 1983, and high school in 1987.

5 The two other motives are: the random developments of our educational system and the balance of power among the categories of "ruling elite".

6 Arabization of education was one of the four principles expected to be at the basis of any reform of the educational system after the independence of Morocco in the late 1950's. The other three principles are Generalization, Unification and Moroccanization (see Appendix for more details).

7 The implementation of Arabization encountered several difficulties not only due to technical reasons (like the lack of trained teachers, the preparation of manuals...etc), but also to political and ideological reasons. (See Appendix for more details)

The remarks made by one of the official texts testify to the interest given by educational leaders to Arabization:

If the language of instruction is an obstacle to the communication, understanding and assimilation of mathematical concepts, the decision to put Arabic in the place of a foreign language should be one of the most commended changes of this reform. Arabization in our opinion is an effective means and a strong and solid base for any reform aiming at raising the level of mathematical education in our country. (MEN, 1983, p.12)

Regarding modern mathematics, the process of its abandonment began in the mid-1970's, when several 'users' of mathematics (namely teachers of other disciplines and professionals in scientific fields) criticized both the content and the teaching approach taken in the new high school programs. Critics of the curriculum focused on its elitist character, which they also described as solely catering for the needs of heavily scientific professions such as engineering and mathematical research.

Subsequently, a meeting was held in Rabat in 1975 organized by the ministry of National Education, bringing together spokespersons from different levels of education (from primary school to university) and different disciplines (math, physics, economics ...). Among the arguments brought up in that symposium was the fact that teaching math, particularly in high school, failed to achieve the objectives set for it. It could not target a large enough audience of learners, failed to prepare students for higher studies in disciplines other than mathematics, and did not accommodate the needs for professional careers on demand in the labor market. Critics, however, turned a blind eye to the role of geometry in the development of intelligence and learning aptitudes stated among the objectives of the old program. Other points brought up in the discussion included:

- The excess of formalism and the abundance of vocabulary emptied mathematics of its meaning.
- Solving problems involving computations or geometric configurations created enormous difficulties for students.
- The exercises and problems on algebraic structures are often superficial and purposeless⁸.
- The complete abandonment of the classical geometries of the plane and space and their replacement by the affine geometry turned out to be more harmful than beneficial.

The culmination of these two changes led the Ministry of National Education to create a commission called The National Committee for Reflection and Reform of

⁸ Often at the level of high school, the exercises given to students were superficial and concerned properties that the majority of students already knew in other forms, which made them unable to see the interest of such exercises.

Mathematics Education in 1978. This commission included representatives from all sectors interested in the teaching of mathematics. The points to be discussed by the commission, as set by the minister, were:

- The Arabization of mathematical programs.
- Coordination between the mathematical curricula from different educational levels.
- Coordination between the teaching of mathematics and that of other disciplines.

The commission made general recommendations concerning the aims of mathematical education, in addition to specific remarks to improve the programs taught. Recommendations insisted that the goals of high school mathematics and the minimum level required to integrate into the job market must be clearly stated. Concerning the programs, the committee recommended reducing the number of general concepts and logical symbols, enriching the teaching of geometry, striving to enhance numerical computational skills, devoting the necessary time to practice exercises, and introducing examples inspired by other disciplines.

The programs generated from this second reform were considerably less 'formal' than those of the 60-70's. They notably contained more geometry and incentives to observe, to experiment and to solve problems leading to the construction of knowledge. High school programs were also relieved of logical rigidity and conceptual abstraction⁹. Additional changes included the restoration of traditional geometry, bridging the gap between middle school and high school programs, and openness to other disciplines through textbook exercises and mathematical activities.

It may be noted that, despite the Arabization of mathematical education, there has been an introduction of activity sessions in the form of mathematical exercises in French since the early 1990's. This is due to the fact that the teaching of mathematics was never Arabized in higher education. These activity sessions would allow students to surmount language barriers when pursuing higher degrees in French¹⁰.

9 The official programs explained that it is possible to use the terminology of sets theory without it being the object of study.

The vocabulary of sets and their related symbols can be used in various paragraphs of the program without being the subject of a study per se. Whenever necessary and without being studied in themselves, the vocabulary and symbols associated with the sets presented in the first year are progressively complemented by the following elements: application of a set to a set, bijection, commutativity, associativity, distributivity, neutral element. (MEN (1987), p.23)

10 Generally this was achieved by the introduction of one hour of translation (Arabic to French), once a week for scientific subjects (mathematics, physics and chemistry, and natural sciences). These hours were provided by translation professors (who were trained for this purpose), and in their absence, by professors of the concerned disciplines.

During this same period, a change was introduced to the baccalaureate examination system. The new system divided the baccalaureate exam in several stages, taking into account the accumulation of school results during all three years of high school. This encouraged average students to succeed through regular work while depriving bright pupils from developing a deep interest in mathematics¹¹. In order to remedy this situation, a number of activities, such as math Olympiads and similar competitions, have been set up to enable talented students to enhance their mathematical vocation.

Once again, the stability of the program did not last long. The rapid progress in new technologies during the 1990's called for yet another reform to keep up with the emerging trends of the modern world. The impact of computing and the advent of automatic calculation methods created new social and technical needs in almost all fields. It became necessary for programs to propose activities and working methods that account for the era of large amount of information, networks and computing. All of these rapid changes made of reforming mathematical programs once more an urgent necessity.

The reform advocated by The National Charter for Education and Training

The reform advocated by The National Charter for Education and Training was part of a global review aiming at restructuring the entire Moroccan educational system. This review strived to offer students a school focusing on useful and functional knowledge, in which basic education is a solid and guaranteed foundation for all fields. It also aimed to achieve the Education for All goals in 2010, to adapt the curriculum to the needs of individuals and society and to improve the quality of teaching and learning. (Belfkih 2000, p.80, Mili, 2017, p.10).

King Hassan II appointed a commission in the second half of the 1990's to implement these goals¹². The work of this commission was gathered in a basic

11 Since the grades obtained in the class tests were accumulated in the final grade of the baccalaureate exam, and since the teachers were expected to make several exams per trimester, students had to give priority to their exams' preparation at the expense of their own learning. This also did not leave enough time for teachers to dive into the depths of the curricula, nor did it allow students to go beyond the class lessons by their own means.

12 In addition to the President, the Special Education-Training Commission has 33 members, including representatives of political parties (14) and trade unions (8) in Parliament. The 11 other members were chosen individually, among the oulamas, the economic operators and the heads of non-governmental organizations and associations of parents of pupils (Belfkih, 2000, p.80)

document called the National Charter for Education and Training¹³. The purpose of this charter is detailed by Belfkih (2000) as follows:

Despite the undeniable development of the educational system since independence, a widening gap has been appearing between the expectations of the nation and the solutions put forward by schools. This has led to the creation of a National Charter for Education and Training which will lay the foundations of a new Moroccan school system at the start of the 21st century while taking into account the new economic and technological climate. (p.77)

The charter, which summed up the basis of trending reflections on education, set out the aspects to be renovated in the curricula and teaching methods. Another main feature of this reform was the explicit adoption of skills' pedagogy -which aimed globally to develop five types of skills: communication skills, methodological skills, strategic skills, cultural competences, technological skills- for all disciplines. In the case of mathematics, the said reform has brought about a new vision emphasizing the aspects illustrated below through excerpts from official texts:

Objectives of teaching mathematics:

In view of the above, and in accordance with the provisions of the National Charter of Education and Training, the general objectives of teaching mathematics in secondary education should reflect the importance of mathematical culture and its contribution to the integration of citizens in a society that is constantly evolving. (MEN(2006),p.10)

Objectives in terms of competencies:

1. Promote positive values and attitudes towards mathematics among students to enhance their confidence in their mathematical skills and enable them to appreciate the role of mathematics in the development of the individual and society.
2. Develop students' ability to solve problems, communicate mathematically, use logical reasoning, and establish connections between ideas
3. Provide students with solid foundation skills to prepare them for higher studies and other professional endeavors.

13 This charter appears to represent a national consensus on the teaching issue. Belfkih (2000) reports in this regard:

While there is unanimous agreement on the need for reform, the debate on the foundations and objectives of the reform has exacerbated passions and hardened positions. The ideological and political dimensions will prevail, leading the various protagonists to hide behind positions of principle and the validity of their convictions. Therefore, the search for a national consensus, or at least the widest possible agreement around the issues of the reform of the educational system, will slowly become the only way of reconciliation between the nation and its school. (p.79)

According to this reform, mathematics is attributed a new role and function in the training of students. We can read at MEN (2006):

Mathematical training is not limited to the formal knowledge of definitions, theorems, results and techniques, but should be made so that these acquisitions are significant by the ability to use them and synthesize them in the face of challenges and problems. The teaching of mathematics should contribute to the development of the student's aptitude for personal work and self-development. It should aim at strengthening their willingness to seek, to communicate and to explain their position. (p. 6)

The teaching function of mathematics must be built in accordance with the student's mental and emotional composition. This education, which should be adapted to the pupil's reality, in accordance with the cultural, socio-economic data of their country, must also remain open to the technological and scientific developments of their world. (p. 7)

Among the main features of this reform, we can mention the use of problems and numerical methods to combine experimentation with reasoning, the highlighting of the algorithmic aspect of mathematics, and the introduction of elements from the history of mathematics.

Another novelty brought by the reform was the matching of teaching material with corresponding competences in the program description. Thus, the content of each lesson was aligned with the skills and abilities it aims to develop in students. In addition, the adoption of several textbooks for each grade level allowed for multiple interpretations of the same program.

Conclusion

The reforms presented in this text allow us to draw several conclusions concerning the aims of mathematics' education in Morocco, the approaches used for its teaching and its role in society.

For the first reform, the main goal of mathematics' education was to provide high school students with tools to quickly access mathematics used in research and/or in high level scientific and technical trainings. Thus, the teaching was focused on an abstract and general presentation of the studied concepts favoring their theoretical aspects. In doing so, the mathematics taught was catering for specific higher studies and neglecting other disciplines which use mathematics in its more classical form (human sciences, short and medium term courses and vocational training ... etc). This reduces the different roles mathematics can play to the role of a course preparing senior technical staff and researchers in the field itself.

It is in this perspective that, during the preparation of the second reform, the criticism voiced by the various pedagogical actors of the time, and especially by the official commissions, against modern mathematics was fundamentally intended to show the usefulness of classical mathematics. The recommendations then advocated that secondary education should emphasize mathematics' role as a "tool for other disciplines". In addition, this vision must ensure that the mathematics taught is more accessible to the majority of students and not only to those who will be devoted to scientific careers and discipline-centered research.

Moreover, this reform led to a change of language of instruction (from French to Arabic), which ended up not being pursued for higher education. This led to the introduction of certain pedagogical provisions, notably in the form of translation sessions, aimed at facilitating the pursuit of higher education in French. Also, the change in the system of baccalaureate examination led to the introduction of extra-curricular activities to help the emergence of the mathematical vocation of students who showed interest in the discipline.

The third reform, which is not specifically concerned with mathematics, has consisted of a revision of the mathematical teaching system at the level of objectives, content and the teaching approach. Mathematics' education was not only aimed at providing content but also at developing skills and competences to use acquired knowledge, or to associate the acquisition of knowledge with the skills to be used for. This was done in accordance with one of the general objectives advocated for by this reform which is to prioritize practical knowledge and functional know-how.

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Appendix: Historical overview on the educational system in Morocco¹⁴

It is well known that Morocco was under French protectorate from 1912 to 1956. During this period, the French were de facto ruling over Morocco. They had introduced several types of education that Morocco inherited following its independence in 1956¹⁵. This appendix gives an overview of each of these types.

Traditional teaching¹⁶: The primary and secondary levels were taught in traditional schools, known as *msid* or *madrasa*, while its higher level was taught at the University of Fez, Qarawiyyine. The disciplines generally taught there were the Arabic language, history, geography, utilitarian mathematics, in addition to the disciplines related to the juridico-religious corpus (learning by heart of the Koran, basic principles of Islam, the Muslim jurisdiction, theology ...). The language of instruction was Arabic and French was introduced as a foreign language at the secondary level.

Modern Muslim-Teaching¹⁷: This type of education, intended for Muslim Moroccans, was divided into several types according to the social class of the pupils:

- Schools of notables which were reserved for children of notables, big traders, high officials, agents of local authorities.
- Urban schools for the children of workers, craftsmen, petty traders and small officials (for the inhabitants of urban centers).
- Rural schools devoted to the children of workers in the agricultural field (for the inhabitants of the countryside).

In general, the majority of pupils in these schools enter the fields of work after completion of primary education. Some laureates, especially those from schools of

14 Based on the books: Baina (1981), Al Jabri (1985) & Al-Mu'tassim (1991) and on the papers: Belfkih (2000) & (Mili, 2017).

15 For a very brief overview of the history of Morocco, the reader can see: <http://www.localhistories.org/morocco.html>

16 This education is inherited from the time before the protectorate, but was renovated, at the level of structures and content, during the period of the protectorate in order to control it and to ensure that Moroccan students who wanted to continue their higher education were not obliged to go to the middle-East (Egypt). This trip would have exposed them to the reforming ideas that were circulating at that time in eastern Arabic countries.

17 The term 'modern-Muslim' assigned to this type of education aims to distinguish it from other types of education which were also in force in Morocco at the time. The adjective 'modern' sets it apart from the traditional teaching mentioned above, whereas the term 'Muslim' aims to separate it from both the 'European education' which was intended for children of settlers and Europeans who resided in Morocco, and the 'Hebrew teaching' intended for the Moroccan Jewish community (which existed before the protectorate and was modernized during the protectorate).

notables, could access secondary education. It was taught in institutions that were called Muslim colleges.

This categorization of schools for Moroccans was the result of the educational policy adopted by the protectorate. One of the principles¹⁸ of this policy was that the education given to Moroccan children must preserve and respect the existing social categories within the Moroccan society. The point of view defended by Marshal Lyautey, with respect to this question, is explained by al-Muʿtassim (1991) in these words:

He<Lyautey> objected to the policy of integration and assimilation, and went as far as respecting the local institutions he found in Morocco and keeping them as they are (...) Desiring to preserve local institutions, Lyautey subjected his own conception of social justice to his idea of the beautiful hierarchy. As such, the large number of “national” schools that he created, of all kinds, was in reality designed to match the hierarchical structure of the Moroccan society” (p.23-24)

In parallel with these types of education totally supervised by the protectorate, there was an Arabized education taking place in ‘private’ schools founded by individuals following the incitement of the nationalist movement. These schools appeared for the first time in Morocco in the 20’s of the 20th century as a response to colonial policy in education. This pedagogy was also an attempt to renovate and modernize traditional teaching. In addition to religious subjects, the subjects taught in modern education were taught in Arabic¹⁹.

Another aspect worth mentioning was the compartmentalization of different types of education. A student’s transition from one type of education to another was rather an exception. Let us quote again, al Muʿtassim (1991) on this subject:

Some students from urban and rural schools joined the Qarawiyyine. A minority of the sons of the rural notables (Caid) were transferred to the schools of the notables. In exceptional cases, some Moroccans from the schools of the upper classes moved from the secondary school to the secondary school of the French system (lycée). In any case, Moroccans were generally not accepted in schools dedicated to the children of French settlers.(p.33, note 31)

18 Another principle is that Moroccans should receive a limited education, so as not to create a class of intellectuals as in Europe; it is a danger that must be avoided. Education for notables should give them the opportunity to work in administration and commerce. As for the education intended for the other classes, it had to allow a professional qualification according to the economic environment of the child. In the cities, teaching was preparing for manual work for handicrafts or building, while in the countryside, education was oriented towards agriculture and livestock.(Baina, 1981, pp.110-111)

19 For more details about these schools see al Muʿtassim (1991).

At the time of independence, the Moroccan authorities were aware of the role that education plays in the country's development and tried to set up a unified education in the early years of independence. In fact, this desire to build a unified education was one of the concerns of the Moroccan authorities several years before independence. The following quote by al-Mu'tassim (1991) illustrates this well:

The French isolated Moroccans from each other by introducing different types of education. All public schools will then have to be unified in a national educational system. (...). Muhammad V <the king of Morocco at the time> referred in 1946 to the need for a unified, free and compulsory public educational system designed to make the school "the savior of the nation."(p.110)

Unification was one of the four principles established by the Royal Commission on Educational Reform, created in 1957, which was to form the basis of all educational reform. The other three principles were: generalization, Moroccanization and Arabization. Generalization means allowing all Moroccan children at school age to access education. Unification involves the merging of the various types of education that existed during the French protectorate in a single type of public education. Moroccanization implies that all staff working in education should be Moroccan. Finally, Arabization means Arabic language should become the main language of instruction.

In reality, the implementation of the four principles above would remain, particularly problematic at the time of the independence of Morocco and even several decades later. The following quote highlights some difficulties related to the principle of unification:

If we consider all these various teachings, we find that we are faced with an "amalgam" that would make unification very difficult: First, the very fact of this diversity was already complicating the situation; but the support given by each social class - who shared power - to its favorite teaching made matters worse. Everyone therefore advocated "his" type of teaching as the "model" of unification. (Baina, 1981, p.171)

With regards to the principle of generalization, its realization proved to be much more complicated than expected as well. The problem of schooling has even become progressively more complicated by the loss of many students who leave school before acquiring a qualification allowing them to enter the workforce well prepared. Belfkih (2000) considers this aspect as one of the indicators of the crisis of the Moroccan educational system at the turn of the 21st century:

In terms of performance, many indicators show that the Moroccan school has performed poorly given the efforts made by the community. Certainly, the legacy out of the colonial period was particularly heavy. As the school had no other purpose at that time than to ensure the reproduction of a very small

indigenous elite, the overwhelming majority of Moroccans could not access it. However, while a large part of this deficit has been absorbed in the post-independence phase, the goal of widespread enrollment has proved difficult to achieve. In addition, the modes of operation that have prevailed within the school have generated relatively low returns and efficiency, which will translate into even greater losses, to the point that today most beneficiaries leave the educational system without any real qualification and, therefore, without being equipped to begin the path of integration into the working life. (p.83)

As for the principle of Moroccanization, there was no significant progress for secondary education until the mid-1980's. Indeed, still in the 70's, teachers from Eastern European countries (Bulgaria, Romania, Poland) were still called to teach science subjects and teachers from Arab and Muslim countries (Jordan, Egypt ...) were brought to teach history, geography, philosophy ... etc in order to meet the growing demand of secondary school teachers. To overcome this lack, in addition to the laureates of universities, whose number did not cover the need of teachers, specific institutions (Ecole Normale Supérieure) were created for teacher training in the late 1970's. These institutions have helped fill the teacher gap within less than a decade.

The introduction of the principle of Arabization has not been easy either and has experienced many hesitations in the early years of independence. Its implementation has been hampered by difficulties of technical as well as political nature. Some aspects of the technical difficulties are illustrated in the following quote:

When popular pressure at the beginning of independence imposed a sort of "generalization" of education, this led to an automatic Arabization from the bottom up. But as the Arabization of primary education was completed - after several attempts - we found ourselves faced with a great dilemma: to continue the Arabization of science and mathematics in secondary education without having enough teachers (...) or return to bilingualism in primary education. After a year or two of hesitation, where students translated their knowledge of mathematics and science from Arabic into French in the <first class of middle school> observation class, it was finally decided (October 1970) to return to French to teach mathematics at the primary level. (Al Jabri 1985, p.90)

When faced with political difficulties, they were linked to the visions and convictions of the various actors involved in teaching. The following quote highlights a political dimension related to Arabization:

The question of Arabization was, in fact, an essentially political affair. At the level of the people, those who advocated a complete and rapid Arabization were those of the government and the ministry which had a traditional and Arabized training. (...) While people with a bilingual education insisted on bilingualism <in teaching>. At the level of political parties, the economic and social composition of party members dictated, to a large extent, a certain approach to the question of Arabization.

<For example> the Istiqlal party, which had a traditional and Arabized base, had always advocated for a complete and rapid Arabization. While the National Union of Popular Forces party based on industrial workers in urban centers supported an advanced bilingual approach. (Al Muʿtassim, 1991, p.114)

In spite of these divergences, a new process of Arabization of the scientific disciplines was started during the year 1979-1980 starting from the third grade. It progressed year after year until its arrival to the first year of high school during the year 1987-1988. However, technical education and higher education kept the French language as the language of instruction. In fact, the problem of the language of teaching scientific subjects was not yet settled and the debate is still open to date. Even the national charter has only accentuated the ambiguity with regard to this issue, as evidenced by the following quote from the chairman of the commission that drew up the charter. Belfkih (2000) stated:

In order to provide high-level scientific and technological options in the Arabic language and in the most promising languages in the various fields of knowledge at the higher education stage, a vigorous effort must be made. In keeping with this orientation, high school education in the most specialized disciplines will be taught in the languages used in the corresponding courses at the university. Correlatively, it is planned to promote the teaching of foreign languages, introducing the first of these languages from the age of seven years and the second language from the age of ten years. (Belfkih, 2000, p.83)

Nowadays, the process of reverting to the teaching of scientific subjects in foreign language(s) in secondary education is under process, after being triggered in 2015. The reversion is implemented through the creation of international baccalaureate options. In this type of baccalaureate, the scientific subjects (mathematics, physics-chemistry, and natural sciences) are taught in French, English or Spanish. Of course, these options, which are in the minority, coexist with the ordinary baccalaureate where these subjects are taught in Arabic.

Anton Dakitsch collection – the scope of mathematics teaching in Brazilian industrial education in the 1950s

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Abstract

*As early as in the 1930s, the Brazilian government expressed their interest in renewing and expanding Vocational Education. A representative factor in this movement was the hiring of foreign professionals and the production of didactic material. Anton Dakitsch was one of those professionals. He came to Brazil and worked in Industrial Education as a teacher, author of textbooks and left us a collection of books and technical journals. In this work at hand, one of Dakitsch's textbooks is analysed concerning the content of mathematics. I aim to assess the status assigned to mathematics within the theme of standardization of paper size, proposed by Anton Dakitsch in his 1950 textbook entitled *Standardization of paper size in general* [free translation], aimed at Industrial Education. To this purpose I consider the way the mathematical instructional content is presented and dealt with will be determinant of its status as a 'tool' or as an 'object'.*

Keywords: History mathematics education; Brazil; vocational teaching; mathematics textbooks

Introduction

In the early 1930s, a demand for public policies aimed at renewing and expanding professional education in Brazil arose. The hiring of foreign professionals and the production of didactic material were representative factors in this movement. Anton Dakitsch, one of those professionals, came to Brazil and worked in Industrial Education as a teacher and an author of textbooks. He left us a collection of books and technical journals, today allocated to IFF Campos, Rio de Janeiro State. But to explore the Dakitsch's collection with a focus on the books that were oriented towards vocational education, a representative approach was the documentary research, to investigate teaching careers and to consider the cultural diversity that

1 This paper is an integral component of the research project entitled *Anton Dakitsch – a Swiss Instructor in the Brazilian Vocational Education System* (2011-2016), developed by the author and Wania Manso (coord.) through a partnership with: UFF- Fluminense Federal University; IFF- Fluminense Federal Institute; PUC-Rio – The Pontifical Catholic University of Rio de Janeiro. This project has been funded by MEC/CAPES (The Brazilian Department for Education and Culture /Department of Staff Development) and by FAPERJ (Research Support Foundation of Rio de Janeiro State). FAPERJ's funding has enabled the listing and renovation of Dakitsch's personal collection.

interweaves Dakitsch's instructional production. I have used that collection as the primary source.

In this work at hand, one of Dakitsch's textbooks is analyzed concerning the content of mathematics. My aim is to assess the status assigned to mathematics within the theme of standardization of paper sizes proposed by Anton Dakitsch in his 1950 textbook *Padronização de papéis em geral* [Standardization of paper size in general] (Dakitsch, 1950), aimed at Industrial Education. And I have explored the content of the discipline of mathematics – as a *tool* or as an *object* – according to Douady (1991), in order to explore the role played by mathematics within the paper size standardisation approach as proposed by Dakitsch.

Anton Dakitsch – a Swiss teacher in the Brazilian vocational education

Dakitsch arrived in Brazil in 1942 and worked in Industrial Education as author of several textbooks, and as a teacher for four decades. The arrival in Brazil of 42 experienced Swiss teachers of diverse backgrounds, made headline news in January 22, 1942 – ‘*Forty two Swiss teachers for the official technical magisterium*’.

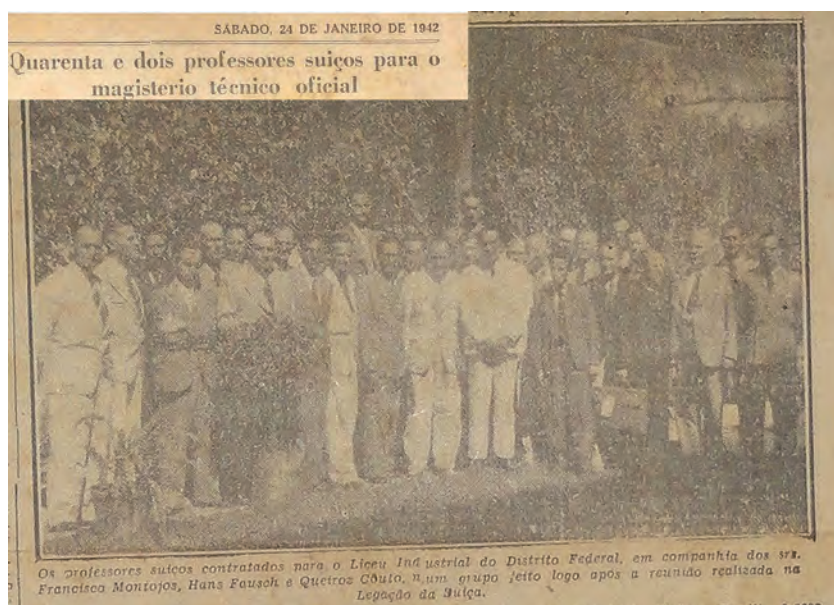


Fig. 1. The Swiss teachers hired by Liceu Industrial, Federal District, together with Mr. F. Montojos, Mr. H. Fausch and Mr. Q. Couto. Dakitsch's collection. Newspaper clipping: *Correio da Manhã*, Jan 24, 1942.

Amongst them was Anton Dakitsch (1909-1993), who taught vocational school between 1942 and 1980. A timeline shows Dakitsch working in Industrial Education as a teacher and as well as author of textbooks, and shows the Brazilian context of vocational educational reforms (Almeida & Almeida, 2012).

Aug 30, 1909 – Born in Brugg, in the Canton of Aargau, Switzerland. Parents: Anton Dakitsch and Anna Elise Dakitsch. Marital Status - married.

1936 – Entered a Master`s Degree in Binding from the School of Industrial Arts in Bern (Switzerland, Kunstgewerbeschule).

1941 – Signed contract for Vocational Technical Instruction with Brazil`s Department of Education and Health.

1942 – Came to Brazil.

1942 – Brazilian Industrial Education Reform.

1942-1957 – Hired as teacher by Industrial Secondary School and National School, Brasilia Federal District.

1948-1954 – Expert Technician working in Teacher Development Courses supplied by the Covenant Agreement between MES/CBAI (Brazil`s Department of Education and Health/Brazilian-American Commission for Industrial Education).

1949-1956 – Author of 13 textbooks on various themes aimed for Industrial Education: a) Books edited by MES/CBAI: Book binding course, 1949, 2v.; Book binding course, Teacher`s guide, 1949; Book gilding, 1950; Technologies, 1950, v.1; Technologies – operations, 1950, v. 2; Technologies – tasks, 1950, v. 3; Technologies – Teacher`s guide, 1950, v.4; Paper size standardization, 1950; Planning and Organization of Typography Workshops, 1953; Suggestions for Planning and Organization of Typography Workshops, 1953. b) Books edited by MEC/DNE (National Education Department): Cardboard crafting, 1956, 2v.; Painted Paper Making, 1956.

1951– Entered the Brazilian Registration for Professor, MEC/DEI (Brazil`s Department of Education and Culture/Section of Industrial Education), Typography Course, Disciplines: Technological Culture, Technical Design, Technology.

1954– Naturalized Brazilian.

1972-1979 – Hired as teacher by the Campos Federal Technical School.

1980 – Retired.

1993 – Passed away

See below some photos of Anton Dakitsch. The first is his passport photo, and according to his wife the following photos show Dakitsch around 40 and 70 years old.

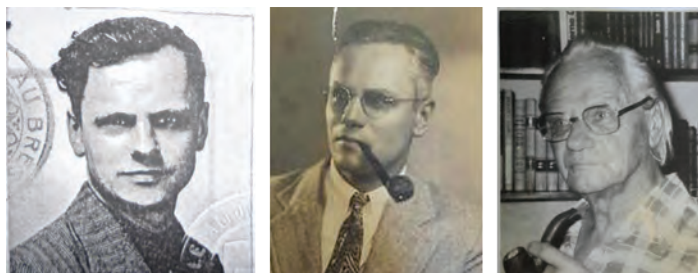


Fig. 2. Dakitsch's pictures. Dakitsch's collection.

Brazilian policies for industrial instruction and for hiring foreign instructors eiducation

In 1940, the São Paulo University Polytechnic School Professor Roberto Mange, Swiss in origin, was nominated to go to Switzerland and select properly-fit foreign teachers. Mange selected 42 experienced teachers, including Dakitsch, who arrived in Brazil in January 22, 1942.²



Fig. 3. Covers of Dakitsch's textbooks: Standardization of papers size in general; Typography; Book Binding.

² Also in 1942 the Industrial Education Reform was put in place. There is a legislation that provides for the organizational base of the Industrial Education: 1942 - Decree-law no. 4.073, of Jan 30, 1942, the Organic Law of Industrial Instruction; 1942 - Decree-law no. 4.127 of Feb 25, 1942 provides for the organizational base of the federal network of industrial education centers; 1943 - Motion nº 28 (embraced by the 1st Conference of Education Ministers and Principals of the American Republics, held in Panama) established the Covenant Agreement between MES/CBAI (Brazil's Department of Education and Health/ Brazilian-American Commission for Industrial Education) and the US for the exchange of educators, ideas and instructional methods.

Industrial Education was basically organized into 8 Sections. The Typography and Binding Course that encompassed the discipline of Standardization belonged to Section VIII - Graphic Arts. Dakitsch's book *Standardization of paper size in general* was published aimed at Graphic Arts.

In his 13 textbooks for Industrial Education, major themes were paper size standardization, book binding, book gilding and cardboard crafting. These books show that various disciplines of Industrial Education share a mathematics basis and thus are a research source for the history of mathematics education.

Mathematics teaching in Brazilian industrial education

In order to explore the role played by mathematics, I considered the paper size standardization approach as proposed by Dakitsch in his textbook *Standardization of paper size in general* (idem, 1950). For this purpose I considered the way the mathematical instructional content is presented and dealt with, which will be determinant of its status as a tool or as an object (Douady, 1991).

Dakitsch designed and published this book by MES/CBAI in 1950, volume 6 of the Series C – Technical Education, aimed to Industrial Education, more specifically Graphic Arts, a Printing and Binding Course. The author's works show that paper size standardization is a major line of study to qualify graphic designers, given its application in typography and book binding, amongst other disciplines.

As a graphic arts teacher Dakitsch emphasized in several of his books that it was important to study the standardization of paper for vocational training. So in order to grasp some aspects of the scope of mathematics teaching in Brazilian industrial education in the 1950s, I have used his textbooks collection as the primary source

According to Dakitsch (1950) his book aims to “spread the knowledge about the norms that guide the rational use of paper in compliance with DIN standards” (p. 1). His book is divided into three sections:

1. Standardization of paper size;
2. Types of paper: specifications, endurance testing and quality assessments;
3. Instructions for typographers.

The table of contents of Section 1 ‘Standardization of paper size’ lists theme-related specifications and market-oriented standardization of paper sizes – the envelopes and market-oriented types of paper.

1. Standardization of paper size:

- Paper sizes (notes), international standard;
- Use of the main series.
- Standard sizes for all market-oriented types of paper;
- Underlying principles;
- Envelopes.
- How to fold drawings;
- Size of elongated paper sheets; business letters.

The presentation of the theme in Dakitsch's textbook starts with the statement "The standardization of paper sizes consists of a harmonious set of models aiming not only at saving paper but also at having sustainable labor throughout the process" (idem, Presentation).

Dakitsch explains that standardization, although not a new trend, has not yet become universal due to the need of adapting the existing equipment to new standards. He then builds the case for standardization using the DIN paper size system. The DIN – Deutsches Institut für Normung [German Institute for Standardization] has set an international standard for paper size.

According to Dakitsch, in the DIN System, the base A_0 size of paper is defined as having a rectangular area of 1 m^2 which when folded in half results in another sheet with the same ratio of length of sides.

The author explains that this condition is met only when the width (x) to length (y) ratio of the sheet is $1 : \sqrt{2}$, that is $1 : 1.4142$. He adds a figure showing the geometric construction of the square root of 2 and points out that the square root is the diagonal of the square with the side measuring one unit.

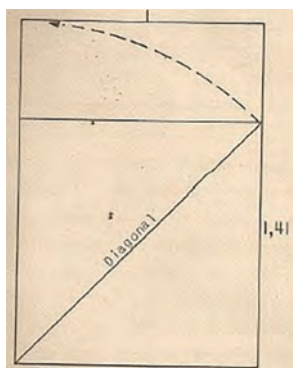


Fig. 4. Geometric construction of the r sizes ratio $1 : \sqrt{2}$ or $1 : 1.41$. (idem, p. 4)

Figure 5 shows the development of the author's explanation: an equation system allows us to get values $xy = 1 \text{ m}^2$.

Pelo sistema de equações;

$$\frac{x}{y} = \frac{1}{\sqrt{2}}$$

$$xy = 1 \text{ m}^2$$

obtêm-se os valores

$$x = 841 \text{ mm}$$

$$y = 1189 \text{ mm}$$

Fig. 5. Excerpt. (idem, p. 5)

Translation of the excerpt above: The equation system allows us to get values. It results in $x = 841 \text{ mm}$ and $y = 1189 \text{ mm}$.

Dakitsch notes that “By the successive division of the A_0 sheet is obtained the formats of the A Series, denominated A_0, A_2, A_3, A_4 , etc.” (Idem, p. 3). The figure below illustrates this sequence.

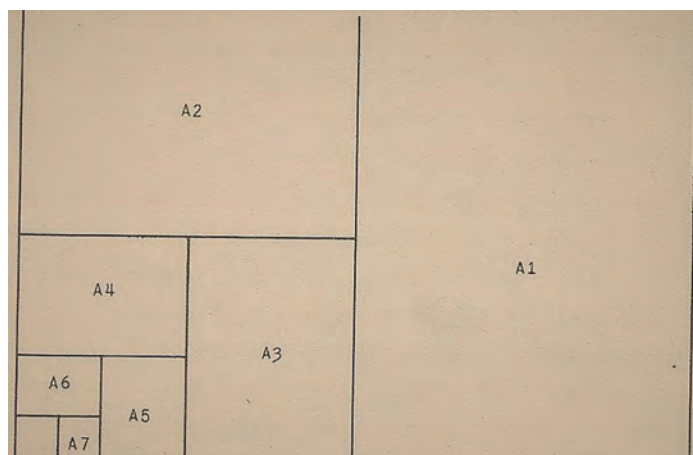


Fig. 6. International Format A series. (Idem, p. 4)

Then Dakitsch (idem) summarized the principles of paper sizes.

- 1) Each size is an $\sqrt{2}$ geometric mean (half the size) of its predecessor and twice as big as the following size. Thus, A_5 is half the size of A_4 and twice the size of A_6 .

- 2) The formats are geometrically similar and the ratio between the sizes is $1 : \sqrt{2}$.
- 3) The metric system is used, so the standard size A_0 corresponds to the area of 1 m^2 . (p. 6)

In order to to make the spread of standardized sizes easier and to ensure its universal usage, four series of standards were designed. So, in addition to the major Series A, there are also Series B, C and D. Figure 7 shows the table of sizes of the series A, B, C and D (Idem, p. 2.)

1.2	A	B	C	D
Formato internacional				
	DIMENSÕES EM MILÍMETROS			
0	841 x 1189	1000 x 1414	917 x 1297	771 x 1090
1	594 x 841	707 x 1000	648 x 917	545 x 771
2	420 x 594	500 x 707	458 x 648	385 x 545
3	297 x 420	353 x 500	324 x 458	272 x 385
4	210 x 297	250 x 353	229 x 324	192 x 272
5	148 x 210	176 x 250	162 x 229	136 x 192
6	105 x 148	125 x 176	114 x 162	96 x 136
7	74 x 105	88 x 125	81 x 114	68 x 96
8	52 x 74	62 x 88	57 x 81	48 x 68
9	37 x 52	44 x 62	---	---
10	26 x 37	31 x 44	---	---

Fig. 7. The four Series A, B, C and D of standards. (Idem, p.2)

Translation of the text in figure 7: International format A, B, C, D. Sizes in millimeters.

The following example of the practical application of Series C shows that the paper standarization can be useful in the production of books and envelopes.

In the case of the book industry, the A_1 format is 610 mm x 860 mm. In series B, the size B_1 is 707 mm x 1000 mm and if a margin of 20 mm is allowed on each side, then the total paper size is 727 mm x 1020 mm. As the B_2 format is half of the B_1 format, the sheet will have a total size of 727 mm x 510 mm and this allows paper to be used.

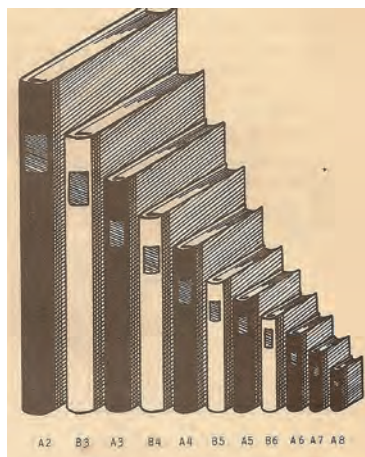


Fig. 8. Books industry. (Idem, p. 3)

Another example of the practical application of Series C presents the envelope with window C6 DIN 680 that is, with the 114 mm x 162 mm according to the previous table. The C format series belongs to auxiliary sizes, amongst which are series B and C, used for envelopes and a relative common choice for small-sized paper applications: a) visiting cards, congratulations cards, forms, postcards, letters, brochures, flyers; b) format folded longitudinally; format with a jacket, envelopes, larger remittances; sheets inside binders.

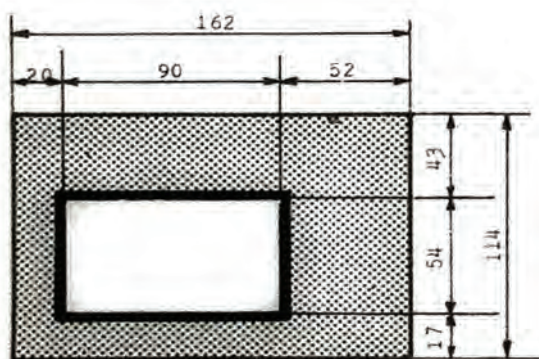


Fig. 9. Envelope with window, C6 DIN 680, that is with the 114 mm to 162 mm ration. (Idem, p. 12)

Dakitsch's approach shows that the study of standardization of paper sizes involves a wide range of mathematical contents. All these are contents of the primary school, with specific application to the industrial education in the 1950's. To grasp some aspect of the scope of the mathematics in this case, I follow Douady's assertion that

the way the mathematical instructional content is presented and dealt with will be determinant of its function as a *tool* or as an *object* (Douady, 1991).

Douady presents the meaning of a *tool*:

We say that a concept is a *tool* when the interest is focused on its use for solving a problem. A tool is involved in a specific context, by somebody, at a given time. A given tool may be *adapted* to several problems, several tools may be adapted to a given problem. (Douady, 1991, p. 115)

And she explains when a concept is an object:

A concept is an *object* when it is considered in a cultural dimension, as a piece of knowledge independent of any context, of any person, which has a place in the body of the socially recognized scientific knowledge. An object is mathematically defined. This can be by various means: properties, an effective construction, or by an existence theorem. The status as object allows the capitalization and the structuration of knowings and the extension of the body of knowledge. (Douady, 1991, p. 116)

The mathematical principles that underlie the DIN paper size system, according to Dakitsch rely upon the ratio between segments and upon proportions. These mathematical contents acquire, thus, the status of *object* of the mathematical knowledge to be discussed by the author of the textbook and to be learned by students. Other mathematical concepts as measurement systems, operations with decimals, areas of geometrical figures are used to develop the baseline content. As prerequisites, their status is that of *tools*, that is, they are used for solving a problem.

Wrap-up

Dakitsch's approach depicts the role played by mathematics in the instruction of paper standardization. But the instructional content may be viewed from two indissociable perspectives: its mathematical specificities and its practical usages. Thus, the work of Dakitsch shows that the practical application of the mathematical content justifies and should motivate the teaching of paper standardization in Industrial Education in Brazil, thus aggregating instructional value to this discipline, given its application in typography, book binding and cardboard crafting, amongst other possible uses in the field of graphic arts.

When applied to the solution of practical problems in the technical, professional and economic spheres, the status of *tool* of the mathematical content that underlies paper standardization shows that the usage of the DIN paper size system leads to saving paper and labor. The practical outcomes of adopting the DIN System point to the need of setting an International Technical Norm that allows for a larger scope

of the system application, since, as remarked by Dakitsch, the usage of this system was not universal in the 1950s. And still in relation to technology, in 1950, the gap between the new paper size format and the existing equipment hindered the move towards the universal usage of the DIN paper size System.

Dakitsch's textbook published in 1950, which was the object of my analysis, as well as other works of his collection that span over 1910 through 1950 and which were published in several languages and countries, are all oriented towards the instruction of graphic arts. They constitute the primary source for studying Dakitsch and as such, they reveal a close connection between the technological base available at that time and the design of instructional materials. The current Technical Education in Brazil does not provide the teaching of disciplines such as paper standardization and book binding. Anton Dakitsch's history can contribute to the records of the history of Brazilian Industrial Education - a history we know very little about.

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Didier Henrion, an enigmatic introducer of Dutch mathematics in Paris

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Abstract

The renewal of scientific thinking in the first half of the 17th century, along with the use of new methods of computing by engineers on the battlefield had to impact mathematics which was taught to future officers. The Kingdom of France needed the officers to be trained in the new methods, but no college, not even Universities such as the Sorbonne, offered courses on these matters. Paris had no military Academies comparable to the Academia Real Mathematica in Madrid, except Pluvinel's Academy, where young noblemen were trained in horse riding and fencing, but probably no or few mathematical topics.

From 1613 to the middle of the 1630's, the only available mathematical textbooks in French were written by an unknown mathematician named D. Henrion, who introduced himself as a private teacher in the prefaces and the dedications of his numerous books. In fact, many new subjects could be discovered in his treatises, sometimes for the first time, like logarithms, trigonometry, fortification and the use of the proportional compass.

The starting point of our research was a basic question: what does D stand for in "D. Henrion"? This question may be of no importance to historians of mathematical thought, but it's so simple that it had to be solved first. The search for Henrion's first name led us to investigate his family affairs, and allowed us to better understand how he became the most important mathematical popularizer of his time.

Keywords: biographic research; practical geometry; circulation of knowledge; mathematics textbooks

Introduction

In 1610, Henry IV, King of France, was killed by a religious fanatic at the time he was about to send his armies to support the Protestant side in the War of the Jülich Succession. The alliance between France and the young Dutch Republic would survive the death of King Henry, as the two countries had Spain as a common enemy. France had overcome the religious wars thanks to the promulgation of the Edict of Nantes in 1598, but Louis XIII, and then Louis XIV, carried the war beyond the borders. The Kingdom of France needed a disciplined army, and therefore well-trained officers. Many young French noblemen gained experience with military matters in the Low Countries during the Eighty Years War, but being a nobleman, even adept at fencing, couldn't be enough for modern war (Parker, 1988). You had to learn mathematics to become a rational leader, capable of understanding new theories of

fortification and ballistics. The new Art of War came from the United Provinces, along with new practices in mathematics and physics. There was a need in France for this new knowledge.

Till the end of the 16th century, very little mathematics and physics were taught in French colleges, and it appears that mathematics teaching would develop in Paris through private lessons (Le Dividich, 1998). One of the most prolific writers on that subject in the beginning of the 17th century signed under the pen name 'DHPEM', which stands for "D. Henrion professeur és mathématiques". Some of his treatises were amongst the first ever published in France on a variety of subjects already known in the Low Countries, such as logarithms, trigonometry, Dutch fortification and even the slide rule.

But we hardly know Henrion's life, as a teacher as well as an engineer or even as a person. His relationship with the Low Countries, especially the Leiden Engineering School, is still to determine. His personal scientific skills remain particularly obscure, numerous parts of his books having been identified as translations of major works by Clavius and others. Nevertheless, he certainly played an important part in the restoration of mathematics studies for the French nobility in the first half of the 17th century. His books may have been the only available sources in vernacular to young noblemen preparing for an officer career in the French Army, as Jean Itard stresses in his biographical note for the *Dictionary of Scientific Biography* (Itard, 2008). So, Henrion is one of the first (self-proclaimed) teachers of mathematics, and if his contribution to mathematics is insignificant, his role as a pedagogue must not be underestimated. However we know almost nothing about him, his training in mathematics, his lessons, etc. To the contrary, we know much more about the life and works of mathematical giants such as Descartes, Huygens or Newton, who did not have to teach to earn a living. Yet without teaching, it is likely that mathematics wouldn't have developed so well. Research about mathematics teaching can't be done without taking into account teachers, and this should be undertaken from an external point of view, considering the living conditions of the educational actors in their time and places.

When these teachers are also mathematical practitioners, we must expand our view to other circles, such as engineers, instrument makers or printers. Given the rare information that is preserved about people from these circles, the task is quite difficult. The existing literature about Parisian mathematicians and private teachers in the first half of the 17th century is very poor so far (Le Dividich, 1998). Even if we may not be able to trace down the real conditions of the circulation of knowledge in non-academic circles, our goal is to contribute to a better knowledge of this topic. To dig deeper where we stand, this paper focuses on our current research about Henrion, his personal life and his role as a transmitter of the new mathematical techniques of his time in Paris.

The common knowledge about Henrion

As far as biography is concerned, the major source about Henrion's life is the classical *Biographie de Michaud* (Michaud, 1857), which conveniently gives essential information in a minimum of lines:

HENRION (DENIS), mathematician, born in France towards the end of the 16th century. He entered the service of the United Provinces as an engineer very early. In 1607, he came to Paris where he taught mathematics, and had for students many young men of noble families. He died around 1640 after having published a lot of works and translations as follows [...] (pp. 212–213)¹

Itard (2008) corrects several details, including the date of death, brought back to 1632. This is attested by a mention of Henrion's widow on the title page of the 6th edition of his translation of *Euclid's Elements* (Henrion, 1632). But Itard admits that “information about Henrion is very scarce and imprecise”. The lack of identified personal documents led the former biographers to take as a truth all information they found in Henrion's own works. As a matter of fact, he only gave details about his life when he had to justify priority or legitimacy, but we can infer at least two major points from his prefaces.

First of all, he surely was a teacher, as we can understand from his own prefaces, which contain many references to his “schoolboys” or his “Gentlemen disciples” (Henrion 1613, fol. ã iij). We can read that he started his teaching in Paris twenty years before the publication of the second volume of his *Memoires* (Henrion, 1627, fol. ã iij), namely around 1607. It is notable that he calls himself a mathematician at the frontispiece of his first treatises, but from 1621 on, he chooses to introduce himself as a mathematics teacher. We can't explain this change yet, but it could be linked to a growing notoriety as a Parisian teacher.

Second, he may have followed Jacques Alleaume, who had been recruited by the States General of the United Provinces to work along with Stevin and Marolois (De Waard, 1912, col. 17-19). Henrion himself mentions several trips to Holland (Henrion, 1627, p. 339; 1630, pp. 25 & 38) and reports various aspects of Dutch practices in the field (Henrion, 1627, fol. ã vj). We eventually find the nature of his job in his own words in 1623, justifying the low quality of his *Euclid* by his absence: “while I was in Holland for the exercise of my duties as an engineer at the service of Mr. Prince of Orange and for the States General of the United Provinces of the Low Countries” [pendant que j'estois allé en Hollande pour l'exercice de ma charge

1 « HENRION (DENIS), mathématicien, né en France vers la fin du 16^e siècle, entra fort jeune comme ingénieur au service des Provinces-Unies. En 1607, il vint à Paris où il professa les mathématiques, et eut pour élèves beaucoup de jeunes gens de familles nobles. Il mourut vers 1640, après avoir publié un grand nombre d'ouvrages et de traductions dont voici les titres [...] » (*All translation are by the present author*)

d'ingenieur de Mons^r le Prince d'Orange & de Messieurs les Estats generaux des Provinces unies du pays bas] (Henrion, 1623, p. 4).

We should add that Henrion was certainly not poor. Almost all of his books were published at the author's expense and sold at home "A Paris, en l'Isle du Palais, a l'image St Michel". If we consider for instance the *Memoires mathematiques* (Henrion, 1613), with their 365 pages full of tables, figures and formulas, the expense might have been quite high for an unknown mathematician without patrons. In fact, the *Memoires* are dedicated to Sully, former French Prime Minister, whom the author confesses not to know personally. This could be considered as miscalculation, because Sully was no longer as powerful as during the reign of Henry IV, but he remained a prominent figure of the Protestant party. Henrion's dedication was a first (miss)step in a sponsorship search strategy, which would be improved over time.

Henrion's patrons and sources

Henrion's dedicatory strategy has been analyzed in depth by Aurélien Ruellet in his study *La Maison de Salomon* (Ruellet, 2016, pp. 106-112). Ruellet shows that Henrion used three different kinds of targets and language registers:

- As we already mentioned for the *Memoires mathematiques*, the compilations intended for the education of the nobility were offered to important men in the Kingdom, but not randomly at all: Sully was a major figure of the Huguenot party, and he was still the Grand Master of Artillery. The dedication emphasizes their common acquaintances, especially Jacques Alleaume, whose great skills are praised. In the same way, the *Collection mathematique* (Henrion, 1621) is dedicated to 'Monsieur', the young brother of King Louis XIII, to draw his attention at the very moment when the question of allocating positions in his royal household would arise. In both cases, the text of the dedication refers both to a project of education for French noblemen and to the service of the country.
- More often, the books are dedicated to Henrion's pupils or their parents. Every time he mentions the mathematical and personal qualities of his students, intimately linking mathematical knowledge with military profession: the skills used in wartime are compared with the ones needed in times of peace, namely scientific knowledge. These dedications can be seen as acknowledgment of good wages or even of funds for publication. Anyway, they contribute to the promotion of the author and his pedagogical abilities amongst the Parisian society.
- Finally, minor works such as translations or reissues are dedicated to personal acquaintances. In this case, Henrion doesn't bother with writing; he just

copies excerpts from other dedications, borrowing more than once expressions from Jean Errard's works (see below).

There are no mathematicians in the group of patrons we just examined, but we noticed similarities of expressions in the dedications with another famous mathematical practitioner, Jean Errard, who is better known as the principal engineer and military architect of King Henry IV of France. As we have shown in our PhD thesis², we can find in Henrion's early works strong similitude with Errard's own works: for instance, his *Geometry* follows exactly the same pattern as Errard's one, whose last three books are even fully copied into Henrion's second edition in 1619; his *Fortification* is intended to complete Errard's one, giving rigorous demonstrations that were missing.

This is already visible on the frontispiece of the *Memoires*, "collected and prepared in favor of the French nobility" (fig. 1, left), clearly a sequel of Errard's *Fortification* "Preface to the French nobility" (Errard, 1600). Moreover, the *Memoires* last chapter is intended to "serve as an explanation and addition to the second and third books of Fortifications by M. Errard". This intention is actually implemented through extensive explanations of Errard's concealed computations, and new propositions of construction. Henrion rewrites the original text, keeping only the essential propositions and completing them by the precise vocabulary he could have found in Marolois' *Fortification* (Marolois, 1615). In fact, Henrion's initial project was a new edition of Errard's *Fortification*, but he had given up after hearing that it was also a project of Errard's nephew (Henrion, 1621, *Briefve instruction* p. 1). Henrion didn't give another plausible reason for giving up: the Errardian system had fallen into oblivion and the new trend had come from the Low Countries, namely the Dutch manner of fortifying initiated by Stevin, fully described by Marolois and many others after him. The main differences between the two systems lie in the initial choices. Errard recommends the use of right angles at the tips of the bastions, and the adjustment of all the lengths on a unit of 16 fathoms. For Marolois, the angles vary according to the number of sides of the fortress, and the lengths do not matter by themselves: they must only meet certain proportions.

In his first treatise on fortification (Henrion, 1613), Henrion tried to improve Errard's propositions by providing proofs and detailed calculations, plus new case studies. In the second one (Henrion, 1623), he leaves Errard's construction principles to concentrate on the new Dutch manner of fortification.

A similar change in Henrion's teaching subjects can be seen in practical geometry. His collected works in 1613 and 1621 contain numerous construction problems with

2 *La fortification géométrique de Jean Errard et l'école française de fortification, 1550-1650* [Jean Errard's geometric fortification and the French School of fortification, 1550-1650] (defended on December 6, 2016 under supervision of Dr. Evelyne Barbin at Nantes University).

his comments on them, intended to improve Errard's *Geometrie et pratique generale d'icelle* (Errard, 1594). In 1621, when Henrion decides to seize the counterfeit copies of his *Collection* (Henrion 1621), he provides the new printer with his commentaries on Errard's *Geometrie*; which will constitute books 2 to 4 of his practical geometry treatise. If we study the organization of this treatise, especially the order of the description and the considered objects, we can recognize Marolois' and Errard's works as the two major sources.

But even if Errard is the most important amongst Henrion's acknowledged masters, the priority should be given to Stevin, at least at the beginning of Henrion's career as a mathematics writer. Observing the frontispieces in figure 1 highlights the unspoken filiation. Several indicators confirm the importance of Stevin's *Memoires* (Stevin, 1605-1608) for Henrion. First, the mention of Stevin as reference author for the study of triangles and more generally for practical geometry. Henrion certainly had perused Stevin's volume, despite its size that prevented it to be used as a hand-book in the field (Henrion, 1613, p. 1). He even informs the reader that Stevin's text about the measure of the parabola must be corrected:

Now we will advise the reader, who possesses the *Memoires Mathematiques* of the learned Stevin (in the French version) to correct the impression here, which is the 16. p. of the second book of his practical Geometry practice. Instead of adding to rectilinear triangle ABC a third of it as said above, here is added its half: so that triangle ABC being 20, the content of the parabola is made 30, instead of being only 26 two-thirds (p. 328).

In fact, proposition 16 of Stevin's second book (Stevin, 1605-1608, p. 81) contradicts Archimedes' quadrature of the parabola it refers to. Surprisingly, Stevin uses a wrong method to calculate the area of the parabola passing through points A, B and C, by taking three halves of the area of triangle ABC, instead of its four thirds. In the 1634 edition, Girard will correct the 16th proposition, making 30 the area of triangle ABC in order to add 10 which is the third of 30, as if it had only been a typo in the former edition.

Even if the name of Stevin is quoted only twice in Henrion's *Memoires*, plus the correction we mentioned above, he remains along with Errard his main inspiration as far as contents are concerned. A large part of Henrion's *Geometry* (Henrion, 1613, pp. 159-343) is modeled on Stevin's one, after a list of definitions entirely taken from Errard. Similarly, the chapter on trigonometry (Ibid., pp. 33-158) is a paraphrase of Stevin's work on triangles. Nevertheless Henrion acknowledges his sources, even when he stresses the originality of 40 geometrical construction problems he created, and the use of the proportional compass to solve them. Let's recall that he pays his debt to Jacques Alleaume for this instrument too. But opinion is divided about his borrowings, and Henrion's contemporaries can be as fierce as him.



Fig. 1. Comparing the frontispieces of Henrion's and Stevin's *Memoires*

Henrion judged by his contemporaries

He surely was a controversial personality, as we will show through several excerpts from books published in his time. First of all, Honorat de Meynier, an independent Provençal mathematical practitioner, a Stevin's opponent who condemns Henrion's plagiarism (Meynier, 1614):

Monsieur Henrion, of whom I have heard the accusation by the best mathematicians in Paris of having attributed to himself and having uncovered in his said *Memoires* the most beautiful and useful secrets of Monsieur Alleaume, who has the honor of walking in the forefront of the French mathematicians (p. 80).

But we already know that Henrion recognized Alleaume as his master. Meynier's quarreling remarks should be taken more generally as a part of his anti-Protestant crusade. We find a defense of our "Professeur és mathématiques" written by another practitioner, René Le Normant, who had participated in the Dutch War of Independence (Le Normant, 1632):

I recognize M. Henrion as greatly skilled in all sciences of Mathematics, not by hearsay, but by having contact with him and noticing his very learned writings, which show that he is greatly versed in the fortifications according to the custom of Holland & France. I learned from him some sciences, a part of which was not used in Holland where I did my war studies. In Paris, there are some very learned Mathematicians, but on the opposite side, there are a lot of

ignorant, who call themselves Mathematicians, who interfere with this science, and teach their lessons without demonstration (p. 242, mispaginated 142)

But Le Normant was a soldier more than a mathematician. Would he be able to tell the difference between original works and translations? What we could take as the fatal blow was dealt by Claude Mydorge (Mydorge, 1630), another mathematical amateur, in his commentaries on a quadrature of the circle previously explained by Henrion, under his pen name DHPEM:

We are ashamed of the impudence of this presumptuous Censor, who asserts in his comment, that nobody so far would have taught this ratio between this curvilinear excess to the square [...] This piece, reported by him on this question is not his invention, but has the same quality as the rest of his remarks, that is to say furtive and stolen elsewhere. If you ask the good Longomontanus for news, he will show that he has already published it in Denmark [...] No offense to this new Cyclometer [i.e. Henrion] or his new Treatise on curves, it is a curvilinear understanding that leads you to admit such absurdities. If this counterfeit money is used in Denmark, France, or at least Paris, will never spot it, or it will be valued only amongst ignorant (pp. 60–62)³.

Caught red-handed? We could play Henrion's advocate, noticing that he never claimed to be the inventor of this quadrature, but only the first one to explain it in France.

Apparently Mydorge did not know the identity of his opponent, hidden behind his DHPEM acronym. It is true that this pseudonym allowed Henrion to advertise for his own books when editing or translating other mathematicians' works. According to his contemporary Claude Hardy, he even paid scholars for translations, because his knowledge of Latin was too weak to allow him to read major mathematical authors as Clavius and Ptolemy. We'll probably never know exactly who were the translators (business confidentiality), but we can notice troubling coincidences. The name of Pierre Herigone (ca 1580-1643) has been often mistaken for Henrion's pen name, and we can argue that Herigone was one of the aforesaid translators. In fact, Herigone's *Fortification* (Herigone, 1634, t. 3, pp. 179-231) or his edition of Euclid's *Elements* (Ibid., t. 1, not paginated) contain many texts and figures similar or even identical to Henrion's ones. But Herigone acknowledges Clavius as the source of his own Euclid's edition, so we can reasonably assume that Henrion's edition of Euclid's *Elements* consists of a commented translation of Clavius's one.

So, who was Henrion actually? What was his relationship with the Parisian mathematical circle? How did he get money to publish so many books without needing

3 By this figure of style, Mydorge ridicules his opponent's crooked mind with reference to his project about a treatise on curvilinear objects.

the support of booksellers? In order to gain deeper knowledge of his personal life, we had to turn to the notarial archives.

Studying notarial archives

Fortunately, Henrion lived in Paris, so that we could expect getting information at the French Archives Nationales (AN). The Parisian AN center owns the complete records of the 122 historical notarial firms of the city from the end of the 15th century. Its website provides several constantly evolving research instruments. Even if it is still far from complete due to millions of documents held, the catalogue gave us a first success with keyword ‘Henrion’.

We know that the revelation is not terrific, but as it was our first simple question we are pleased to confirm that *D.* stands for *Didier*. In February 1612, Henrion signed a lease agreement (fig. 2) about two cellars in a house situated “en l’Isle du Palais, à l’image Saint Michel”, that is to say the same address at the frontispiece of Henrion’s books.

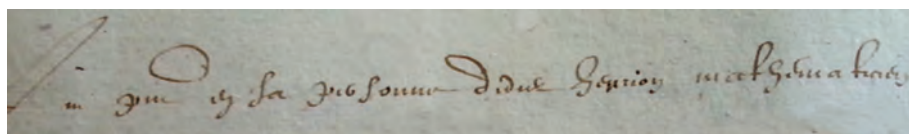


Fig. 2. First line from the 1612 lease agreement

This first line reads: “Fut p[re]sent en sa personne didier henrion mathematicien” [Was p[re]sent in his person didier henrion mathematician]: no doubts about his first name. We learn from this contract that Henrion was not the owner of the place but had a lease agreement himself. Twice again in 1613 and in 1618, the same kind of lease agreement was established between Henrion and new tenants. The cellars were to be used as warehouses and shops by a wine dealer, a carpenter and finally a merchant, all of them living in the neighborhood. Thus, different parts of the basement were subleased at least till 1622, but we’ve lost track of the house after that year (search still in progress).

Keeping on researching in the catalogue, we only found further mention of Henrion as the deceased husband of Jehanne Le Villain, from 1633 on. We had to follow this family track to learn more about our mathematics teacher. Surprisingly, a notarial act dated April 1633 relates to repair estimates for two houses and a farm we had never heard about before. In 1635, Le Villain benefits from an income annuity, constituted for her and her daughters. So it is now possible to describe Didier’s nuclear family (fig. 3):

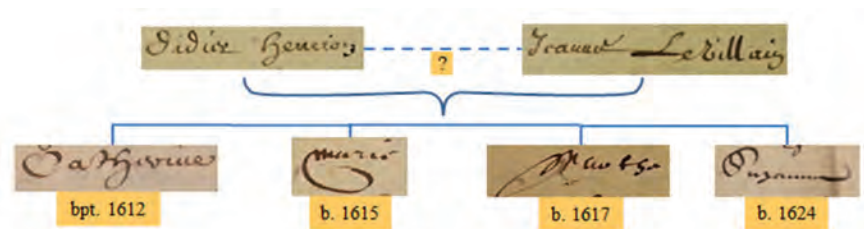


Fig. 3. The Henrion couple and their daughters Catherine, Marie, Marthe and Suzanne

The fact that the eldest daughter was baptized in 1612 in Charenton is precious because Charenton hosted the unique Protestant Church of Paris. Thus Didier Henrion and his wife worshipped there and consequently they were Protestants. Did they marry there too? Unfortunately, the registers of Charenton Protestant Church disappeared in 1871 in the great fire of Paris City Hall, so we may not know more about Le Villain this way. The last deed for 1633 is dated December and reveals more important matters. We can read that all previous deeds concerning Le Villain are consequences of the due execution of the Will of Jean Henrion. But who was he? And what was his relationship with Didier?

This appears to be the essential point of our story. Jean Henrion, Didier's uncle⁴, was an important lawyer at the Châtelet of Paris and the provost of several towns in the Parisian region, including the Savis farm in Belleville. This farm was a huge food supplier for Paris, so as an administrator, Jean Henrion was certainly wealthy. In fact, we found his Will, dating back to September 1602. The list of his properties is considerable: he definitely was very rich. At the very end of this Will, Didier Henrion is described as "absent", and consequently unable to hold the position of universal legatee of half the properties of his uncle. But in case he would "come back", Didier would recover the legacy. Absent? Coming back? We have at least two interpretations: the physical absence of an engineer at the service of the United Provinces or the spiritual absence of a heretic. The second one seems the most plausible, considered the phrasing of the Will, but we still have to discover whether Didier had to abjure his faith, and when. Nevertheless, another good surprise was given by Jean Henrion's wife, Jacqueline du Lis (or du Lys) (fig. 4):

Jacqueline du Lis survived her husband by ten years. She was the executor of his Will, but we have not managed yet to untie the thread of her relationship with the heirs (her two nieces and Didier).

The important fact is that Jacqueline had two brothers, Luc and Charles, who were apparently good friends of Jean. Charles du Lis is well-known as private secretary of François Viète, who committed him to deciphering the correspondence

4 This relationship is evidenced by Jean Henrion's will, which mentions Didier as his nephew.

of the King's foreign enemies. Before becoming an advocate in Paris Parliament, Charles had edited "with the author's agreement" a French translation of Viète's mathematical works, for which the King gave his royal privilege in June 1606. Unfortunately, the book has never been put to press.

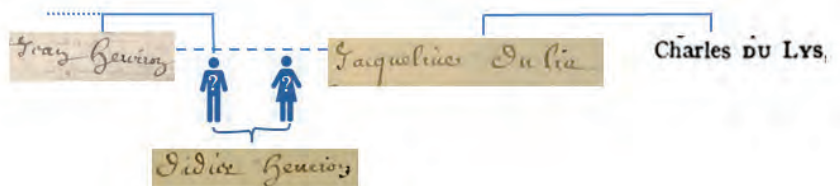


Fig. 4. The previous generation track

Charles du Lis was not alone to undertake this huge work. The privilege mentions another famous Viète's secretary, Pierre Alleaume, whose son Jacques Alleaume was a close friend of Didier Henrion's. The world of Protestant mathematical practitioners was a small world.

What next?

This archival research was quite new for us, but it allowed us to investigate the world of 17th century lawyers, as well as secret family affairs. It is sometimes strange to read documents related to life and death of real persons without regards to their mathematical achievements, but in our opinion it should be done more often to understand the real conditions of mathematical thinking by human beings.

We still have to find many things before having a good view on Henrion's life, for instance his Will, the date and place of his death, and how he managed to "come back" in the execution of his uncle's Will. We need more information about his visits to the United Provinces, to clarify the matter of his possible training at Leyden School of Engineers. We would also like to know more about his relation to Pierre Hérigone, whose *Cursus mathematicus* (Hérigone, 1634) borrows many elements from Henrion. So many things we do not know about a simple Parisian mathematics teacher of the beginning of the 17th century!

Acknowledgment. We would like to thank Jenneke Krüger for her kindness and support.

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Arithmetic in Joan Benejam's *La Enseñanza Racional* (1888)

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Abstract

*Joan Benejam was born on the island of Menorca in 1846. He became a teacher in 1866 and in 1874 he became elementary school teacher back in his home town, where he spent the rest of his life until his retirement in 1912. His life was completely devoted to teaching and education. He wrote papers, he gave lectures, he wrote children's plays, etc. He also founded and directed several journals addressed to elementary teachers. Here, we will focus on one of such journals published in 1888: *La Enseñanza Racional* [The Rational Teaching]. This journal contained sections devoted to geography, grammar, history, arithmetics, etc. In this paper, we will describe and analyse the arithmetic section of this journal. This will give us an idea of the pedagogical ideas of its author as well as of the situation of arithmetic teaching at the time. Even if this is a very particular case, it constitutes an interesting example of how the ideas of the so-called Regenerationism (which tried to reform and modernize the country) were received and, in some sense, put into practice in the field of education.*

Keywords: arithmetic, journal, 19th century, Spain, Joan Benejam

Introduction

The 19th century in Spain was a period of great instability. The country was involved in several wars and many different governments followed on one another. By the end of the century, most of the overseas colonies had been lost and the formerly great Spanish Empire faced a clear decline (Esdaile, 2000).

This situation gave rise to an intellectual movement, the Regenerationism, which tried to understand the underlying reasons of the Spanish decadence in order to overcome them (Carr, 2001). The name of this movement came from the word 'regeneration' as opposed to 'decay' and one of its main characters was Joaquín Costa. Some of his most famous slogans were "school and pantry" or "close the tomb of El Cid with two locks", pointing out that it was time to stop looking back at a lost glorious past and to start looking at a future where the main goals should be to reduce the poverty and increase the education of the people.

Regenerationism was a very active movement which gave rise to many journals devoted to disseminate its ideas in different areas. As for education, it deeply influenced the so-called *Institución Libre de Enseñanza* [Free Educational Institution]. This institution, which existed between 1876 and 1936 tried to impose progressive ideas on the educational practices of that time. For example it proposed the

coeducation, the low importance of textbooks, the involvement of the family in the teaching process, the interest of non-formal education by means of excursions, poetry and theater sessions, etc. (Molero Pintado, 2000).

In this work, we are going to focus on a rather unknown and not very influential character, Joan Benejam Vives. He was an elementary school teacher in a very small town in the island of Menorca. He did not write important books and his ideas were not particularly groundbreaking, but his life and his work illustrate the importance of these somewhat anonymous people in the process of innovating and improving education. In fact, the role played by teachers like Benejam in order to improve education by disseminating and sharing innovative ideas was fundamental because they acted as 'loudspeakers' bringing these ideas from the main cultural centers to more isolated regions. The case of Benejam is an interesting example which is worth studying, in addition, due to his intense activity (as we will see in the next section) writing books and editing journals.

More particularly we are going to analyze the arithmetic section of one of the many journals that Benejam edited during his life. This section presents some interesting features that can be related with more general ideas to exemplify how the teaching of mathematics can also be used as a mean to educate in the broad sense of the term.

Joan Benejam Vives. Life and works

Joan Benejam Vives (Figure 1) was born in a small town of Menorca, called Ciutadella, on May 27, 1846. At the age of 16, he entered Barcelona's Normal School in order to become an elementary teacher. In 1866, he graduated and started his teaching career in Blanes, near Barcelona. Soon, in 1869 and back in his native town, he founded a private school called Colegio Ciudadelano [School from Ciutadella]. In 1874, he became a public servant gaining a position as elementary teacher at Ciutadella's public school. He kept this position until his retirement in 1912. For a short period of time, between 1912 and 1914, he lived in La Habana, where he was involved in adult education activities. Finally, from 1917 until his death in 1922, he was devoted to local politics at Ciutadella's town hall. For a more detailed account of Benejam's life we refer the reader to the works by Adrover, Vallespir and Villalonga (1986) or, more recently, by Vilafranca i Manguan (2002).



Fig. 1. Portrait of Joan Benejam.

As we have just seen, Benejam's life took place in a quite narrow geographical area. The years he spent as a student in Barcelona and as a teacher in Blanes (about 60 km to the north of Barcelona on the Mediterranean coastline of Spain) from 1862 to 1869 were the longest period he lived far from Ciudadela. After that, he only left the island for his two years stay in La Habana and for some short occasional journeys¹.

On the other hand, Benejam was a very prolific author. He published around 70 articles and more than 40 books. Most of them were devoted to pedagogy and teaching, but he also wrote on local culture (a Menorquin-Spanish dictionary, for instance) and he was the author of some poetry and plays.

If we focus on Benejam's strictly pedagogical books, he wrote about many different topics. We can mention, for example, books about curricular contents such as *Vulgarizaciones Científicas* [Scientific Vulgarizations] (Benejam, 1898) devoted to natural sciences or *La Aurora de la Lectura* [The Dawn of Reading] (Benejam, 1890) devoted to the teaching of reading. He also wrote books like *La Alegría de la Escuela* [The Joy of School] (Benejam, 1899), addressed to inexperienced teachers and even a sketch of a renewed educational system (Benejam, 1908).

Benejam founded and edited 5 journals of pedagogical content addressed to teachers and families. Some of the names of these journals clearly show their scope: *La Enseñanza Práctica* [The Practical Teaching] (1894-1895), *La Escuela y el Hogar* [School and home] (1906-1910), *Alma de Maestro* [Soul of Teacher] (1915-1916), etc. As we can see, most of these periodicals did not have a very long life, possibly because they were mainly personal projects that had to be abandoned due to the lack of the time or money required to continue them.

Finally, it is interesting to point out that Benejam even invented a device designed to teach geography, the *Didascosmos* (Benejam, 1893). Roughly speaking, it

¹ In a local journal from 1899, for instance, we find a short note informing about his departure to Palma and Barcelona due to the summer season (*El Demócrata* [The Democrat], year L. Ciudadela, July 29, 1899. No. 13, p. 2).

was a model that included several geographical features such as mountains, islands, rivers, railways, tunnels, bridges, etc. The idea of the author is better expressed in his own words (Benejam, 1893):

Maps, drawings [...] are not enough to show the students the world we live in. It is necessary to get closer to reality: see and touch those things the books talk about [...]. Excursions put the children in the presence of nature but only a few landscapes [...] can be reached. But excursions hardly ever take place and only with great difficulties. Let us bring a small world to the schools... (p. iii)²

However, to date, we have not been able to find any surviving *Didascosmos*.

La Enseñanza Racional. An overview

The journal *La Enseñanza Racional* [The Rational Teaching] (Figure 2) was the first periodical edited by Benejam (Benejam, 1888). Only 24 issues, each of them containing 12 pages, were published during 1888 (and maybe 1889). The page numbering was continuous from 1 to 288. Its periodicity is unknown, but it is likely that it was irregular. In fact, the last issue of the journal ended with the expression “end of the first part”. Its publication was discontinued and the second part never came out.

By that time, Benejam had already been teaching for more than twenty years, so he was an experienced teacher who had probably had the time to reflect about curricular contents as well as about his own teaching practice. In addition, by 1888, Benejam had already published books about the teaching of different topics like arithmetic, grammar or sciences. Thus, this journal might be worth analyzing in order to have a closer look at Benejam’s pedagogical ideas.



Fig. 2. Top of the cover page of the first issue of *La Enseñanza Racional*.

The title of the journal literally means ‘the rational teaching’. This name resembles that of an institution initiated by the anarchist pedagogue from Barcelona Francisco

2 Mapas, grabados, esferas, no alcanzan a llevar al ánimo y al convencimiento del niño la idea del mundo que habitamos. Es necesario acercarnos más a la realidad: ver, palpar las imágenes de aquellas cosas que nos hablan los libros [...]. Las excursiones escolares colocan al niño en presencia de la naturaleza; pero solo es dable ver por su medio algunos paisajes [...]. Pero las excursiones escolares no pueden practicarse sin grandes dificultades, por cuya razón apenas se practican. Llevemos un pequeño mundo corpóreo a los centros de enseñanza... (Translation by the author).

Ferrer i Guardia (Avilés, 2006), the Escuela Racional [Rational School]. This movement led to the so-called rationalist or modern school that spread through Europe and the United States (Avrich, 1980) and which promoted freethought and collaborative learning among other progressive ideas. Nevertheless, as Vilafranca i Manguan (2002, p. 320) points out, it is not possible to consider this journal as part of the rationalist movement since Ferrer's school was not created until 1901. In any case, many of the ideas of Ferrer's school were similar to those pursued by the Institución Libre de Enseñanza [Free Educational Institution] that we have presented in the introduction.

In the first issue of the journal, we can find a very short introduction where the author provided some information about his motivations, objectives or scope. The journal is described as "one of a kind" and it is addressed not only to teachers, but also to parents and to "friends of childhood". Benejam's main objective is to "introduce easy lessons and useful exercises in the school and at home". Finally, the author is aware of the novelty and difficulty of his proposals: "This teacher who is writing, understands how difficult it is to implement the proposed exercises and lessons due to the current organization of our schools".

The journal contained several different sections. Now, we give their names and a brief description of them.

- A joyful lesson. This section presented a hypothetical dialogue between a teacher and one or more students trying to show an ideal lesson in the classroom. The topics were diverse but always related to geography, from 'The mountains' to 'A journey through Germany and Holland' passing by 'The movements of the Earth'.
- Grammar through examples. This section introduced the most important grammatical concepts using them in many examples.
- Writing exercises. This section usually presented some topics the students had to write about. It also provided some examples of compositions about the given topic and, in some issues, dictation exercises.
- Nature in the presence of the children. This section included the description of a lesson about natural sciences together with didactical suggestions for the teachers to carry them out.
- Religious, social and aesthetical education. In this section, with a structure quite similar to others, religious and ethical aspects of life were treated with the students.

3 In 1886 the freedom of teaching was abolished in Spain. This was, of course, an obstacle to innovation.

- Commented poetry. In this section, we can find short poems intended to be read by the students and some explanations about the meaning of the poem, the figures of speech that contains, etc.
- Arithmetic. This section dealt with elementary arithmetic, presenting both didactical suggestions for the teachers and problems to be proposed to the students. We will get back to this section later.

The order in which the sections appeared in the journal was the same in which we have presented them above. The first five sections were included in all the issues of the journal. The arithmetic section was not present in the last issues (21-24). The poetry section was not regular, appearing in the issues 4-6, 9-12, 14, 15, 18 and 20-23.

The arithmetic section

As we have already pointed out, this section appeared only in issues 1 to 20 and, if present, it was always the closing section of the journal. Benejam did not state the ages of the students to whom the contents of the journal were addressed. However, he was an elementary school teacher so, according to the Spanish system of that time, his actual pupils were between 6 and 9 years old. We may assume that the author had his own students in mind when writing the journal.

The contents of the arithmetic section were organized into three degrees, somehow according to their difficulty and depth. This organization into degrees, which we will further describe in a forthcoming section, was also used in the grammar section. In Table 1 we show the sequence of contents corresponding to each degree throughout the 20 issues.

Table 1. Sequence of contents of the arithmetic section.

	First degree	Second degree	Third degree
Issue 1	Decimal number system	Decimal number system	Decimal number system
Issue 2	Addition of natural numbers	Addition of natural numbers	Addition of natural numbers
Issue 3	Subtraction of natural numbers	Subtraction of natural numbers	Subtraction of natural numbers
Issue 4	Multiplication of natural numbers	Multiplication of natural numbers	Multiplication of natural numbers
Issue 5	Division of natural numbers	Division of natural numbers	Division of natural numbers
Issue 6	Fractions	Common fractions	Common fractions

Issue 7	The meter	Decimal fractions	Operations with fractions
Issue 8	The liter	Operations with decimal numbers	Operations with fractions
Issue 9	The gram	Decimal metric system	Decimal fractions
Issue 10	The peseta ⁴	The square meter	The square meter
Issue 11	Syllabus	The cubic meter	Other units of surface
Issue 12		The liter	Volume
Issue 13		The gram	The liter
Issue 14		Monetary units	The gram
Issue 15		Review of operations	Miscellaneous arithmetic problems
Issue 16		The rule of three	
Issue 17		The rule of interest	
Issue 18		Interest and discount	
Issue 19		Discount	
Issue 20		Proportionality problems	

As we can see, the main topics covered in the section were natural numbers, positive rational numbers, monetary units and the decimal metric system and proportionality with its most common applications. Roughly speaking, these contents correspond to the arithmetic contents in the official curriculum of elementary primary education at that time.

In the case of the first degree, issues 11 to 20 were devoted to present a detailed syllabus. The contents corresponding to the second degree (third column of Table 1) show the usual sequencing: beginning with the number system, then natural numbers and their operations, fractions and decimal numbers, measurement and ending with the rule of three as the culmination of elementary arithmetic knowledge.

In addition to the degree structure, the arithmetic section was also divided into some kind of subsections or parts according to the type of material that they presented. Namely, we can find expositions, exercises and calculation exercises.

⁴ The peseta was the Spanish currency from 1868 (when it unified the 21 different currencies that were simultaneously used throughout the country at that time) until 2002 (when it was replaced by the euro).

The expositions were usually presented at the beginning of the section. They consisted in more or less detailed discourses proposed to the potential teacher to be used in the classroom. For example, Figure 3 shows the explanation of how to reduce a set of fractions to a common denominator.

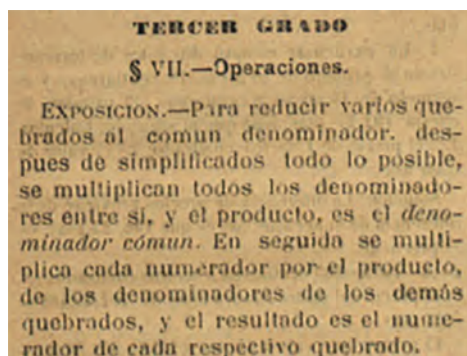


Fig. 3. Reducing fractions to common denominator (Issue 7, p. 84)⁵.

The exercises appeared just after the expositions and they were conceived as a practice to be proposed to the students as a revision or, sometimes, as an extension of the contents introduced during the exposition. The statements of these exercises were short and they consisted either of topics to be explained by the student or of short questions to be answered orally. In Figure 4, for instance, the students are asked, among other things, how many cents there are in a dime.

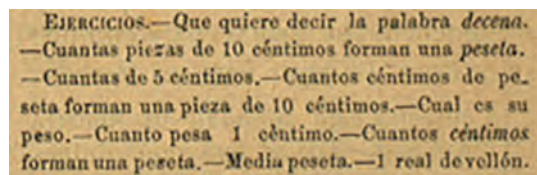


Fig. 4. Exercises about Spanish currency (Issue 10, p. 118)⁶.

Calculation exercises were introduced, for each degree, at the end of the section after the exposition and the exercises. They were of a more applied nature and they mostly (but not always) consisted of arithmetic problems. More than 200 problems were proposed as calculation exercises and, in the case of verbal problems, they

5 To reduce several fractions to their common denominator, after simplifying them as much as possible, all the denominators are multiplied and the result is the common denominator. Then, every numerator must be multiplied by the denominators of the other fractions, the result being the numerator of each of the fractions. (Translation by the author).

6 What does the word 'ten' mean. — How many coins of ten cents are there in a peseta — How many of 5 cents. — How many cents are there in a dime. — How much does it weigh. — How much does a cent weigh. — How many cents are there in a peseta. — Half a peseta. — 1 royal vellon. (Translation by the author).

were often presented in commercial, agricultural or industrial contexts. In Figure 5, we see an example where the payment for a job must be distributed among the two workers that did it. It is interesting to point out that nowhere the author states that the distribution must be proportional and, moreover, at that point proportionality had not been introduced yet. In fact, as we see in the figure, the problem appears in a list under the title “problems involving the four operation”.

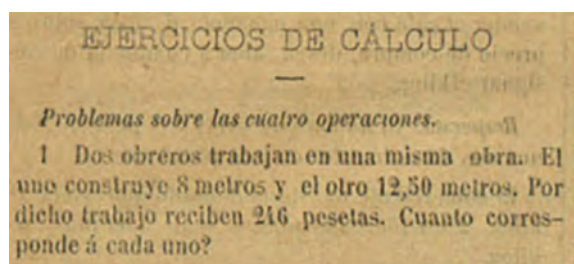


Fig. 5. Problem about a proportional distribution (Issue 6, p. 71)⁷.

It is noteworthy that there are no figures or illustrations along with the problems. In fact, neither of the sections contained any kind of illustration. This was probably due to the printing costs, that Benejam was likely to pay himself.

A closer look at the calculation exercises

Karp (2015) points out the interest of studying problem sets. Thus, we are going to have a closer look at the calculation exercises that are part of the arithmetic section of Benejam's journal. The problems that can be found in it, have different degrees of difficulty and also seem to pursue several diverse goals.

Some of them were intended to be solved mentally. As we can see in Figure 6, when some problems had to be solved mentally, the author explicitly stated it. These problems could range from performing simple arithmetic operations (like the addition of 15 and 3 we see on the top of the left hand side of Figure 6, for instance) to solving simple verbal problems (on the right hand side of Figure 6 we read: “a window had 8 glasses and 3 broke up, how many of them were left?”). In some cases the exercises are proposed with the objective that the students develop strategies to carry out certain computations (like the ones involving fractions at the bottom of the left hand side of Figure 6, for instance).

⁷ Two workers are doing the same work. One of them build 8 meters and the other 12,50 meters. For this work they earn 246 pesetas. How much does it correspond to each of them? (Translation by the author).

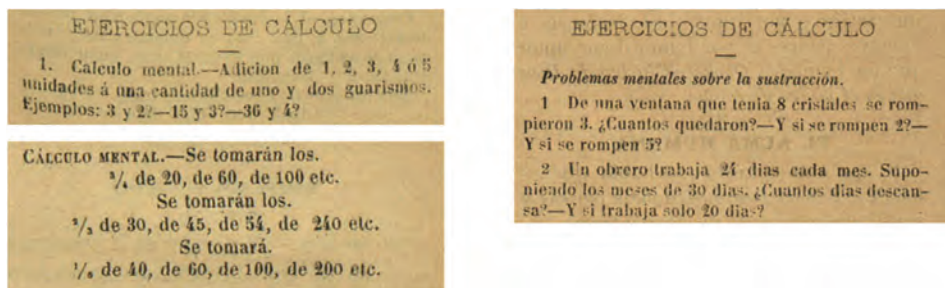


Fig. 6. Some examples of exercises to be solved mentally. Top left (Issue 2, p. 23)⁸, bottom left (Issue 7, p. 82)⁹ and right (Issue 14, p.167)¹⁰.

Some easy exercises, particularly in those involving just the reading or writing of numbers in the first degree, the statements are used to introduce interesting or useful facts.

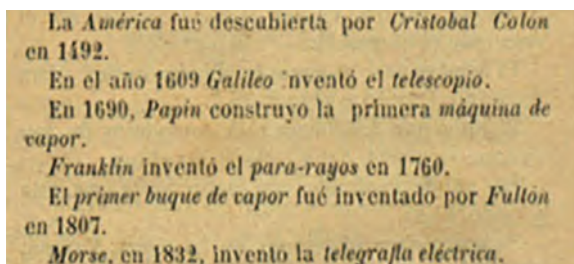


Fig. 7. Learning from the statement of an exercise (Issue 1, p. 12)¹¹.

8 Mental calculations. – Addition of 1, 2, 3, 4 or 5 units to numbers of one and two digits. Examples: 3 plus 2? – 15 plus 3? – 36 plus 4? (Translation by the author).

9 Mental calculations. – Compute $\frac{3}{4}$ of 20, of 60, of 100 etc. Compute $\frac{2}{3}$ of 30, of 45, of 54, of 240 etc. Compute $\frac{1}{5}$ of 40, of 60 of 100, of 200 etc. (Translation by the author).

10 Mental problems about subtraction.

1 On a window containing 8 glasses 3 were broken. How many were left? – And if 2 were broken? – And if 5 were broken?

2 A worker works 24 days each month. Assuming months have 30 days, how many days does he rest? – And if he works 20 days? (Translation by the author).

11 America was discovered by Cristobal Colón in 1492.

In the year 1609, Galileo invented the telescope.

In 1690, Papin constructed the first steam engine.

Franklin invented the lightning rod in 1760.

The first steam boat was invented by Fulton in 1807.

Morse, in 1832, invented electric telegraphy.

(Translation by the author).

In Figure 7, for example, we see some sentences that the students were supposed to read out loud. In doing so, they could learn that Galileo invented the telescope in 1609 or that the first steam boat was invented by Fulton in 1807. This practice was not uncommon in Spain at that time (Meavilla & Oller, 2014).

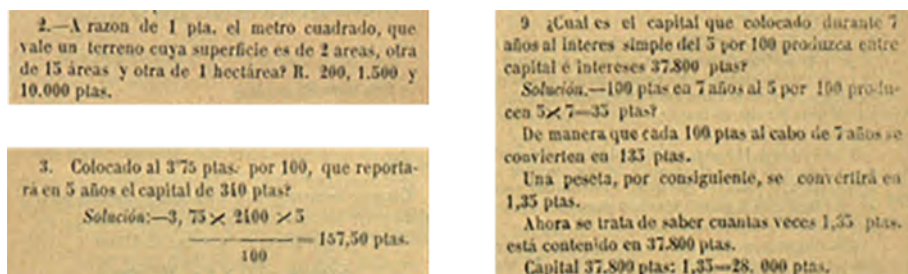


Fig. 8. Different degree of detail in the solutions. Top left (Issue 16, p. 191)¹², bottom left (Issue 17, p. 204)¹³ and right (Issue 11, p.132)¹⁴.

In the case of verbal problems, some of them were given without its solution, while in some other cases a solution was provided. When the solution was provided, the degree of detail was variable. In some cases (top left of Figure 8), only the numerical solution can be found. Sometimes, we can also find the arithmetical operations that led to the numerical solution (bottom left of Figure 8). Finally, in some cases, the author provided a completely worked-out solution that included explanations (right side of Figure 8).

Finally, a few problems included useless data; i.e., data not required to obtain the solution. This feature is still rather uncommon, even though it is generally accepted that presenting such kind of problems to our students can be very formative. In

12 At a price of 1 peseta per square meter, what is the value of a field with a surface of 2 areas, other of 15 areas and other of 1 hectare? R 200, 1500 and 10000 pesetas. (Translation by the author).

13 At an interest of 3,75 pesetas per 100, what will be the benefit after 5 years of an amount of 340 pesetas?

Solution: - $(3,75 \times 2400 \times 5)/100 = 157,50$ pesetas.

(Translation by the author). Note that the 2400 in the solution should be a 340. In addition, the final result is incorrect.

14 What is the capital that after 7 years at a simple interest of 5 per 100 produces 37800 pesetas between capital and interests?

Solution. - 100 pesetas in 7 years at an interest of 5 per 100 produce $5 \times 7 = 35$ pesetas.

So every 100 pesetas become, after 7 years, 135 pesetas.

One peseta, consequently, will become 1,35 pesetas.

Now, we have to know how many times 1,35 pesetas is contained in 37800 pesetas.

Capital 37800 pesetas : 1,35 = 28000 pesetas.

(Translation by the author).

Figure 9 we see an example and how in the solution of the problem the author explicitly mentioned that “the number of packages and the weight of each of them are useless data in this problem”.

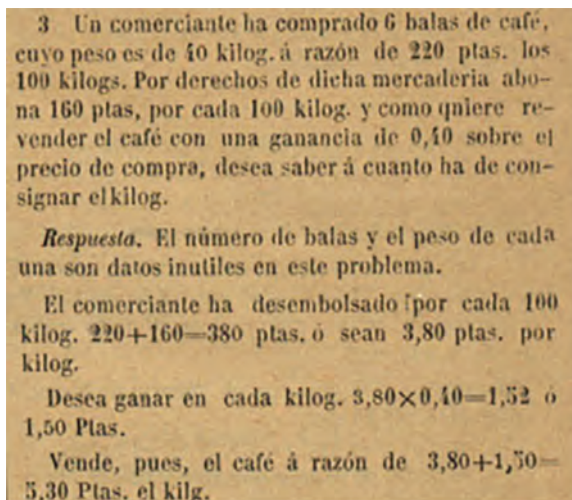


Fig. 9. Problem including useless data (Issue 6, p. 72)¹⁵.

Some words about the organization of the contents into degrees

As mentioned previously, the contents from the arithmetic section were organized according to three degrees. Only this section and the grammar section were organized in this way, even if other subjects like religious and moral education or natural sciences were very important at that time and received much attention in the journal. The reason for only organizing arithmetic and grammar in degrees may have been that it was easier to do so than in other subjects, due to the nature of their contents.

Benejam did not provide many explanations about his possible inspirations or motivations to follow such organization. In fact, neither the official curriculum nor the textbooks made that explicit distinction in degrees. The only information that we

¹⁵ A merchant has bought 6 packages of coffee, each weighting 40 kilograms, for 220 pesetas every 100 kilograms. He pays as taxes 160 pesetas every 100 kilograms and since he wishes to sell the coffee with a benefit of 0,40 with respect to the buying price, he wants to know at what price he must sell each kilogram.

Answer. The number of packages and their weight are useless data in this problem.

The merchant has paid for every 100 kilograms $220 + 160 = 380$ pesetas, that is, 3,80 pesetas per kilogram.

He wishes to win in each kilogram $3,80 \times 0,40 = 1,52$ or 1,50 pesetas.

So he sells the coffee at a price of $3,80 + 1,50 = 5,30$ pesetas per kilogram.

(Translation by the author).

find appears in a short footnote on the first issue of the journal (p. 11), where the author described them as “three concentric circumferences”. This description and the underlying idea make the degrees somewhat reminiscent of the modern concept of spiral curriculum (Bruner, 1960) since, in fact, it shares some of its features such as the revisiting of topics or the increasing level of difficulty.

Since the idea of progression is essential in designing a curriculum, it is relevant to analyze how Benejam constructed the different levels of difficulty. As an example, we will focus on the case of the addition of natural numbers as presented on the second issue of the journal (pp. 22-24). In Table 2 we see the main ideas regarding this topic, the sequence in which they are introduced and the degree each of them belongs to.

Table 2. Addition of natural numbers (Issue 2).

First degree	1. Example of how to perform the algorithm with only two summands
Second degree	2. Nature (abstract or particular) of the summands and the sum 3. Example of how to perform the algorithm with four summands
Third degree	4. Definition of problem, data and unknown 5. Consequence over the result of increasing one summand 6. Checking the correctness of a sum by rearranging the summands

The first degree dealt only with algorithmic issues, which were revisited in the second degree together with a first step into more conceptual aspects. Finally, the third degree mainly dealt with conceptual issues. This organization of contents let us infer that Benejam's idea of difficulty was mostly related to conceptual aspects rather than to reckoning abilities (Prytz, 2015). Contents evolve from performing simple procedures to the understanding of them and the third degree deals mainly with cognitive abilities. This stress on cognitive abilities can be seen as a particularity of Benejam's work, at least when compared with the Spanish official curriculum at that time and with the curricula from other countries. In the Swedish case, for example, Prytz (2015, p. 318) identifies only one term referring to cognitive abilities during the period 1850-1950.

As we mentioned before, there was no explicit indication in the journal about the age of the students. If we assume ages 6 to 9, it would be possible to identify the three degrees with ages 6-7, 7-8 and 8-9, respectively. This, of course, would coincide with the usual organization of students into years and the distinction into degrees would be less interesting. However, we do not think that this is what Benejam had in mind. In fact, we think that his idea was that the students could proceed through these degrees regardless their age. A gifted student of age 6 could rapidly advance to the third degree, while a student age 9 could still be struggling in the second degree.

This would be consistent with the author not mentioning ages and would oppose to the rigid curricular organization into courses according to age.

An example of innovation

As we have already mentioned, the arithmetic contents that we can find in the corresponding section Benejam’ journal were essentially the same that could be found in the official curriculum. Nevertheless, the treatment of the contents was, in some cases, different from the usual.

Here, we are going to briefly present and analyze de case of the introduction of the number system. As seen in Table 1, this topic was covered in the first issue of the journal (pp. 10-12).

In order to see how this topic was usually introduced at that time, we can have a look at some textbooks. Here, we focus on two contemporary books published in 1899 (Olivares, 1899) and (Guerola, 1899). Both these books were written by elementary teachers (like Benejam himself was) and they were addressed to elementary primary school pupils.

In Table 3, we present the main concepts related to the topic of number system as well as the order in which they were presented by the three compared authors.

Table 3. Sequence of contents related to number system (Issue 1).

	Benejam	Olivares	Guerola
Digits	1	4	4
Names of compound units (from tens to millions)	2	3	3
Idea of decimal and positional system	3	5	6
Concept of unit	4	1	1
Numbers as sets of units. Abstract and particular numbers	5	2	2
Absolute and relative value of digits	6	6	5
Roman numerals	7	-	7

We can see that both textbooks give a very similar treatment to the topic. They begin presenting the concept of unit and define abstract numbers as set of units. Then, they give names to the compound units, introduce the idea of digit and finish with the notion of positional system. Roman numerals are introduced at the end of the chapter only by Guerola.

On the other hand, Benejam did not begin with the abstract concept of unit. Rather, he first introduced the digits as mere symbols as well as the naming and writing conventions for numbers on the first degree. After that, on the second degree, he introduced the concepts of unit and number, defining both abstract and particular numbers. Finally, on the third degree, Benejam focused on the relative value of digits and on the decomposition of a number into units, tens, etc. He concluded with a description of the roman numerals and the roman number system.

As for the exercises, in the first degree Benejam only proposed the writing and reading of small numbers, with at most two digits. In addition to some 'open' proposal like writing down the number of books that a student owns, we find some exercises whose learning goal is not only mathematical. Thus, Benejam proposed that the teacher exemplified the number 5 mentioning the five continents with their names, or the number 4 by the mention of the four cardinal points, etc. Exercises for the second and third degree proceeded in the same way but increasing the number of digits and also proposing exercises of a more theoretical flavor. For instance, on the third degree, Benejam proposed to discuss the values of 0 according to its position as an extension exercise.

Final comments

We think that Benejam's work in this journal provides a good example of the efforts of anonymous teachers to innovate and to disseminate their innovations in a period when education in Spain was trying, sometimes successfully, to modernize.

In 1921, Benejam published a short note of 2 pages in a journal called *El Menorquín* [The Menorquin] (Year III, February 1921. No. III, pp. 1-2), which was published in Buenos Aires addressed to expatriates. The eloquent title of that note was *Contra la Corriente* [Against the Grain] and, since it was published only one year before his death, we may consider it as some kind of pedagogical testament.

As such, this short text contains many of Benejam's main ideas regarding teaching. For example, he says: "the education that comes from a book turns out to be a dead education and we need an alive education", and related to this: "in the school we deal with dead ideas [...] Boredom is nearly unavoidable" and also: "facts, real facts and not the imaginary ones are those to be preferred".

These ideas, quite similar to those of the Institución Libre de Enseñanza [Free Educational Institution], were rather general, but we have seen them exemplified in his journal and, particularly, in the arithmetic section. Thus, the use of particular situations as a starting point can be related to the search of an alive education, while exercises like the mental calculations may have been conceived to avoid boredom

and to engage the students. Finally, most of the problems included in the section were stated in a real context.

Benejam's 1921 note includes also a very short and simple statement that, nevertheless, encloses a very strong idea: "it is easy to teach, but it is difficult to educate". The main goal of Benejam was always to educate his pupils and not only to teach them and this could also be achieved while doing arithmetic. The following excerpt comes from Issue 10 where the peseta (the Spanish currency at the time) was being introduced in the first degree

There are gold and silver coins which are more worthy than a peseta: you don't have them, do you? You only have a few copper coins. Nevertheless, do not spend them in vain because some day you may need them [...]. Well, each time you have a coin, think over for a while before spending it (p. 118)¹⁶

Even though it belongs to the arithmetic section, there is no mathematics in the previous fragment. However, it contains an important piece of advice for the students and it shows that we can contribute to the education of our students at the same time that we teach them mathematics.

Acknowledgment. I wish to thank Professor Luis Puig for pointing out the possible relation between the name of the journal and Ferrer i Guardia's ideas. This work was partially funded by the Government of Aragón and by the European Social Fund (Research group S36_17D "Investigación en Educación Matemática").

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16 Monedas hay de plata y oro que valen más que una peseta: vosotros no las poseéis, ¿verdad? Sólo disponéis de algunas piezas de cobre. Mas no por esto las gastéis inútilmente, porque alguna vez tendréis necesidad de ellas [...] Pues bien, cada vez que poseáis alguna moneda, reflexionad un poco antes de gastarla. (Translation by the author).

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Geometry for women in teacher training schools in the late 19th century in Spain¹

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Abstract

At the beginning of the second half of the 19th century the teaching of mathematics for boys and girls was different in Spain. In the same way, teacher education for men and for women was different. In particular, in a new law adopted in 1857, geometry was excluded both for girls in primary education and in teacher training schools for women. Instead, the law established that tasks appropriate for women had to be taught.

In this text I present how geometry is reintroduced in teacher education for women throughout the second half of the 19th century precisely linked to those tasks appropriate for women, and I briefly analyse two textbooks in which geometry is presented in this way, one written by a man and one by a woman and both published in Valencia, Spain.

Keywords: geometry teaching; teacher training; 19th century; gender difference

Introduction

The need to organise teacher training schools for women began to be considered in Spain in the second half of the nineteenth century in the *Ley de Instrucción Pública* (Law of Public Instruction), adopted in 1857², and known as ‘Ley Moyano’ [Moyano’s Law], by the name of its promoter.

However, in spite of the progressive character of this law in its intention to improve women’s education, the law established important differences in the syllabus of mathematics, both for boys and girls in primary education and for men and women in the teacher training schools.

Before Moyano’s Law, teacher training schools for primary teachers had begun to be established in Spain in the 1830s with the name of “Escuelas Normales” (Normal Schools). The first one was created in 1839 in the capital of Spain, Madrid,

1 This paper is a slightly different version of parts of a chapter written in Spanish for the catalogue of the exhibition *Escoles i Mestres. Dos segles de història i memòria*, held at the Cultural Centre of the University of Valencia from November 28, 2017 to March 18, 2018 (Puig, 2017). All quoted texts written in Spanish have been translated by the author.

2 The law was signed by Her Majesty the Queen on September 9, 1857, and was published in *Gaceta de Madrid*, September 10, 1857, num. 1710, pp. 1-3.

with the name of ‘Escuela Normal Central’ [Central Normal School]. The character of ‘central’ was defined in the main aim of the school and was twofold: first, to serve as a model for the other Normal Schools, and second to train not only prospective teachers of primary education, but also prospective teachers of Normal Schools. After the creation of this Central Normal School in Madrid, there followed other Normal Schools in provinces, the one in my hometown, Valencia, in 1845. All these teacher training schools were exclusively for men.

The first teacher training school for women was created in 1847 in Pamplona with the name of ‘Escuela Normal de Maestras’³, and by the time of Moyano’s Law only three others had been created. The teacher training school for women in Valencia was created in 1867, ten years after Moyano’s Law.

Thirty years later, in 1897, two books were published in Valencia, one by Carmen Cervera, a book of 110 pages on fractions entitled *Lijero* (sic) *estudio de las fracciones comunes* [Slight study of common fractions], and one by Francisca Ferrer, *Elementos de Geometría plana y descriptiva y nociones de dibujo, con aplicación a las labores de la maestra* [Elements of plane and descriptive geometry and notions of drawing, with application to the tasks befitting the woman teacher].



Fig. 1. Title page of Francisca Ferrer's book

3 In Spanish, nouns are usually marked according to gender: “maestra” is feminine for teacher, the masculine being “maestro”.

To my knowledge, these are the only two books of mathematics, intended for the teaching of mathematics to the pupils of teacher training schools for women and written by women who taught at the teacher training school for women of Valencia⁴, from its opening in 1897 until its integration in the Normal School of Teachers of Valencia in 1967, when teacher training schools in Spain were no longer segregated by sex.

I present in this text a brief account of the differences in the teaching of mathematics in teacher training schools for men and for women in the second part of the nineteenth century in Spain, and I will use Francisca Ferrer's book to illustrate how these differences evolved in the case of geometry.

Teacher education and mathematics in Moyano's Law

First of all let me say that there was a difference in the way that Moyano's Law considered teacher education for men and teacher education for women. Moyano's Law established that there would be a teacher training school for men in each of the fifty provinces of Spain, but, as far as teacher training schools for women were concerned, the law only stated that the Government would try to promote them "where convenient", without establishing their sources of funding.

With regard to the differences between what had to be taught, the content established in Moyano's Law for teacher education in the schools for men and the schools for women was not only different, but also established in a different way.

Indeed, Articles 68, 69 and 70 of the law listed what had to be studied to obtain the degrees of primary school teacher (in the case of men), in two levels (elementary and higher). In the case of mathematics, the articles specified that "Arithmetic" and "Notions of geometry, technical drawing and surveying" would be studied for the elementary degree, and, for the higher degree, in addition to the above, "Notions of algebra". However, for the degree of primary school teacher in the case of women what the law established in its article 71 is that

[...] it is required:

First. To have studied with the extension due in a Teacher Training School the subjects that the primary education of girls embraces, elementary or higher, according to the title to which they aspire.

4 Actually, Francisca Ferrer was not a teacher of the teacher training school for women of Valencia when she wrote her book. And when she was hired at that school as a specialist teacher, she did not teach geometry, but drawing and calligraphy. Carmen Cervera was an assistant teacher when she wrote her book. She got a permanent position some years later, but she did not then teach arithmetic.

Second. To have been instructed in the principles of Education and Teaching Methods⁵.

It is therefore necessary to examine the syllabus for the primary education of girls established by Moyano's Law, in order to know what had to be studied by women aspiring to become teachers. However, what the law established for the primary education of girls is not explicitly stated, rather the law referred to what was established for boys with the indication that some parts had to be omitted and replaced by others. Indeed, Article 5 reads as follows:

In the levels elementary and higher of the primary education of girls, the studies dealt with in the sixth paragraph of article 2 and the first and third paragraphs of article 4 will be omitted and they will be replaced with:

First. Tasks befitting their sex⁶.

Second. Elements of Drawing applied to the same tasks.

Third. Slight notions of domestic hygiene.

And, in the paragraphs indicated, the subjects that appear for the boys, but that must be omitted for the girls are:

Sixth. Brief notions of Agriculture, Industry and Commerce, according to localities.

[...]

First. Principles of Geometry, Technical Drawing and Surveying.

[...]

Third. General notions of physics and natural history adapted to the most common needs of life.

So, as far as mathematics is concerned, girls will not study geometry to give room to the "tasks befitting their sex", "elements of Drawing applied to the same tasks ("labores")" and "slight notions of domestic hygiene".

5 The instruction "in the principles of Education and Teaching Methods" was also included in the syllabus of the teacher training schools for men with the same wording.

6 I translate literally the expression in Spanish "Labores propias del sexo", that refers to tasks such as sewing, embroidering, weaving and housework, which were seen as the ones befitting women. The expression "Labores propias del sexo", meaning this kind of tasks, was shortened to "labores" (tasks) in the school context to refer to the subject taught in women's schools having this content. A similar shortened version was used to indicate in any kind of official forms or identification documents the profession of a housewife: "sus labores", her tasks. In what follows, when I translate a Spanish text with this use of "labores", I will add between brackets the Spanish word "labores" to the English translation "tasks".

Of the paragraphs contained in the syllabus for boys that are not be omitted in the case of girls, the only thing left of mathematics is the fifth paragraph of Article 2, which says:

Fifth. Principles of Arithmetic, with the legal system of measures, weights and currencies.

Since Moyano's Law established that in the teacher training schools for women they have to study "the subjects that the primary education of girls embraces", the only subject they have to study are those "Principles of Arithmetic, with the legal system of measures, weights and currencies". They don't have to study geometry, nor algebra.

What geometry for primary education

Geometry was excluded from teacher education of women, but in the case of men it appears linked with Technical Drawing and Surveying: "Principles of Geometry, Technical Drawing and Surveying". This is a consequence of a conception of the kind of geometry suitable to primary education and to the education of primary teachers that is clearly stated in the book that was the official textbook of Pedagogy for teacher training schools from 1850 to 1905. This book of 390 pages dedicates less than a page to the teaching of geometry, containing this assertion:

Geometry must, in our opinion, remain rigorously enclosed in the study of its usual applications. In schools, the difficult theories of this science must be left aside, for they are at least useless when they cannot be well understood. The most useful application of geometry, especially in Spain and generally in all rural schools, is surveying, which should be reduced frequently to practical exercises. The teaching of Geometry in schools should be limited to establishing easy principles and to demonstrating its application immediately. (Avendaño & Cerderera, 1850, pp. 254-255)

This option to limit the study of geometry to its applications defined the subject of geometry in the general syllabuses, and the content of textbooks for teacher education. Geometry was presented in the teacher training schools for men applied to surveying. When geometry found its way to the syllabuses for women, the general idea of its study in application to a field was imported, but the field of application was changed. There was no way to present the use of geometry in surveying, since surveying was a subject that also had to be omitted for girls in Moyano's Law, and furthermore it was not seen as a field usual in women's lives. Instead, the tasks befitting their sex was the field chosen for the application of geometry.

Mathematics in teacher education from Moyano's Law till the end of the 19th century

Following Moyano's Law, a general regulation with a more detailed syllabus for teacher training schools for men was published in 1858⁷, but this was not the case for teacher training schools for women. Table 1 shows the parts on mathematics in this syllabus.

Table 1. Mathematics for teacher education from 1857 till 1881

Men	Women
Elementary level: <ul style="list-style-type: none"> • Arithmetic • Notions of geometry, technical drawing and surveying 	Lack of a general regulation. Regulations for each newly created teacher training school for women.
Higher level: <ul style="list-style-type: none"> • Complements of Arithmetic and notions of algebra • Elements of geometry, technical drawing and surveying 	

In the case of teacher training school for women, due to the absence of a general regulation, regulations appeared independently for each teacher training school as they were created, or new regulations were adopted for teacher training schools already established. These regulations were proposed by the faculty of the school or the promoters of the new schools, and were approved by the Ministry of Development, which was in charge of the educational system. The situation regarding the inclusion of geometry was very varied: a good number of schools did not include it, a few included it as an independent subject, and other schools linked it to the tasks ("labores").

It was not until 1881 that a Royal Decree⁸ established the syllabuses and regulations both for teacher training schools for men and for teacher training schools for women. In this Royal Decree, the subject "Drawing applied to tasks ("labores")⁹ with slight notions of geometry" is included, generalizing the initiative of several teacher training schools for women that had included in their syllabuses geometry

7 Royal Decree September 20, *Gaceta de Madrid*, September 23, 1858, num. 266, pp. 1-2.

8 *Gaceta de Madrid*, August 31, 1881, num. 243, p. 614.

9 "Tasks", "labores" in Spanish, means in this Royal Decree "the tasks befitting her sex", shortened to "tasks", "labores". See footnote 6.

applied to tasks (“labores”)¹⁰. This had been the case of the teacher training school for women of Valencia, where it was included in its general regulations of 1867¹¹ for the syllabus of teachers of the higher level of primary education with the title of “Notions of geometry and further drawing applied to the tasks (“labores”).

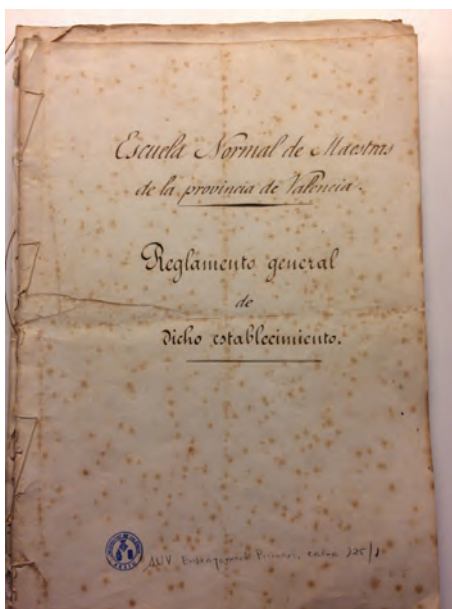


Fig. 2. Cover of the manuscript of the General regulations of the Teacher training school for women of Valencia, 1867

Table 2 shows the part on mathematics of this syllabus. In the case of the teacher training schools for men, there is no change with respect to the program established by the Royal Decree of 1858.

10 Four years before, the Royal Decree for creation of the teacher training school for women of Toledo, which also included geometry linked to the tasks (“labores”) in the syllabus, established that this syllabus would apply to any other school that was created thereafter. However, this Royal Decree did not require schools that had already been created to adopt this syllabus. (Royal Decree March 14, *Gaceta de Madrid*, March 28, 1877, num. 87, pp. 853-854.)

11 A manuscript of this 1867 regulation is kept in the Historical Archive of the University of Valencia (Escuela Normal de Maestras de la provincia de Valencia, *Reglamento General de dicho establecimiento*, AHUV, Primary Education, box 125/1).

Table 2. Mathematics for teacher education in the 1881 Royal Decree’s regulations

Men	Women ¹
Elementary level: <ul style="list-style-type: none">• Arithmetic• Notions of geometry, technical drawing and surveying	1 st Elements of arithmetic applied to the natural numbers, fractions, decimals and the legal system of measures, weights and currencies Drawing applied to tasks (“labores”) with slight notions of geometry
Higher level: <ul style="list-style-type: none">• Complements of Arithmetic and notions of algebra• Elements of geometry, technical drawing and surveying	2 ^d Further arithmetic until proportions, and exercises on problem solving 3 ^d Further arithmetic, including proportions and the applications of this theory

Geometry in textbooks for the training of male primary teachers and female primary teachers

By the end of the nineteenth century, after the introduction of geometry through its link with the tasks befitting the woman teacher was generalised, some other books, intended to cover the syllabus of “Drawing applied to tasks (“labores”) with slight notions of geometry” like the one by Francisca Ferrer, were written. I have been able to find four of them, three written by men, and one written by a man and a woman. The only one I know written by a woman is the one by Francisca Ferrer published in Valencia.

That is not unusual, Ballarín et al. (2000) found in a general survey of textbooks for women education in Spain in the second part of the nineteenth century that fifty six per cent of them were written by men. But it is at least a surprise to find men writing not only on mathematics, geography, history or language, but also on the so-called “tasks befitting the sex”, meaning sewing, embroidering, weaving, cutting, and so on. In the books that I have found, the male authors used different strategies to cope with writing on these themes.

Juan Barceló wrote the book *Nociones de geometría con aplicación a las labores y corte de prendas* [Notions of geometry with application to the tasks (“labores”) and garments cutting]¹² with a woman, Vicenta de Luis, a primary teacher, for a popular series of books called *Biblioteca de las Escuelas Normales* [Teacher training schools library], published by Saturnino Calleja. These books covered all subjects, and were not intended

12 This book has been analysed in Meavilla & Oller (2016).

as textbooks for the teacher training schools, but as texts to prepare the exam for a teaching position¹³ in the Spanish public school system (Barceló & de Luis, 1895).

Crescencio Moles, who was a professor at the teacher training school for women of Barcelona, explains in the introduction of his *Curso de geometría y dibujo lineal aplicado a las labores* [Course on geometry and technical drawing applied to the tasks (“labores”)] his feelings by saying¹⁴:

From the moment that I was entrusted with the teaching of geometry and drawing applied to women’s tasks at the teacher training school for women of Barcelona, I naively confess that I was perplexed at the thought of the special application of geometry I had to do; a work truly somewhat prickly, when there is no deep knowledge of the art to which all my efforts should be directed.

And he adds:

However, the need, and even more, the desire to fulfil my role worthily, made me study all the major tasks (“labores”) in detail; and by this means I was able to overcome certain difficulties [...] (Moles, 1869, p. 5).

Prudencio Solís, who was a professor at the teacher training school for men in Valencia and its director till his death in 1897, does not seem however to have studied the tasks much in order to write and publish his book *Nociones de geometría y dibujo aplicado a las labores para las aspirantes al magisterio* [Notions of geometry and drawing applied to the tasks (“labores”) for the women aspirants to the teaching profession] (Solís, 1889). He had already published a book for the correspondent subject in the syllabus of the teacher training school for men *Nociones prácticas de geometría, agrimensura y dibujo lineal gráfico y a pulso para los aspirantes al magisterio* [Practical notions of geometry, surveying and technical and freehand drawing for the men aspirants to the teaching profession] (Solís, 1881¹⁵), and he just copied literally from this book the first part that contained geometry, omitting some lessons and some items from other lessons, and adding just five pages with some slight indications on ornamental drawings and garment cutting, and the reference to other books where these matters were treated.

This strategy used by Solís to write the book for women repeats somehow the way in which the syllabus for women in Moyano’s Law is presented: it takes what is

13 The degree awarded by the teacher training schools qualifies graduates to work as teachers in any public or private school. To get a permanent position in the Spanish public education system, in which teachers are civil servants, one must pass an exam.

14 I quote from the second edition of 1869. I have not been able to find the first edition.

15 I quote from the second edition of Solís’ book for men and the third edition of Solís’ book for women. I have not been able to find the previous editions.

established for men and removes a part, but not exactly to make room for what is considered appropriate for women. Solís' book for men has 160 pages and his book for women only 85. The book for women is then half the length of the one for men, but it is not only a question of size, what Solís removes from his book for men to compose his book for women is a clear indication of the conception held by the author, and the society of his time, of what parts of mathematics were not suitable or were too difficult for women. The following are a sample of different aspects of mathematics omitted in the book for women.

In one lesson on angles, the Spanish word "bisectriz", meaning "angle bisector", and its definition are removed, and the section only states how to divide an angle in two equal parts.

In one lesson on triangles, the section on Pythagoras' theorem is removed, even though in the book for men Pythagoras theorem appears as a property of rectangular triangles without mention of Pythagoras nor the word "theorem". And, by removing the section on Pythagoras' theorem, the words "hypotenuse" and "catetus" and their definitions are also removed. Figure 3 shows how section 71 from the book for men has been cut out in the book for women and sections 70 and 72 from the book for men have been cut and pasted literally in the book for women, becoming the consecutive sections 64 and 65.

70. Puesto que los tres ángulos de todo triángulo valen dos ángulos rectos, infiérese que de los tres ángulos del triángulo, dos no pueden ser rectos ni obtusos, y por tanto, el triángulo tendrá siempre, bien un ángulo recto y dos agudos, bien un ángulo obtuso y dos agudos, bien los tres ángulos agudos. En el primer caso se denomina *triángulo rectángulo*, como D E F (fig. 43); en el segundo *triángulo obtusángulo*, como C H I (fig. 44), y en el tercero, *triángulo acutángulo*, como A B D (fig. 40).

71. Los lados D F y D E, que forman el ángulo recto D del triángulo rectángulo F D E (fig. 43), toman el nombre de *catetos*, y el lado E F opuesto á dicho ángulo se llama *hipotenusa*.

Entre estos tres lados hay constantemente una relacion dada que constituye una importante propiedad del triángulo rectángulo, á saber:

El cuadrado de la hipotenusa es igual á la suma de los cuadrados de los catetos, esto es: $EF^2 = FD^2 + DE^2$. De aqui resulta que la hipotenusa es igual á la raíz cuadrada de la suma de los cuadrados de los catetos, y un cateto es igual á la raíz cuadrada de la diferencia entre el cuadrado de la hipotenusa y el cuadrado del otro cateto, esto es:

$$FE = \sqrt{FD^2 + DE^2}, \text{ y } FD = \sqrt{FE^2 - DE^2}$$

Si, pues, los catetos de un triángulo rectángulo tuvieran, por ejemplo, 8 metros de longitud uno y 6 metros el otro, la lon-

gitud de la hipotenusa seria igual á $\sqrt{8^2 + 6^2} = \sqrt{100} = 10$.

Si la hipotenusa tuviera 20 metros y un cateto 12, el otro cateto tendria $\sqrt{20^2 - 12^2} = \sqrt{256} = 16$.

72. La construccion de triángulos en general ofrece una gran variedad de casos; pero solo trataremos de los más comunes, á saber:

64. Puesto que los tres ángulos de todo triángulo valen dos ángulos rectos, infiérese que de los tres ángulos del triángulo, dos no pueden ser rectos ni obtusos, y por tanto el triángulo tendrá siempre, bien un ángulo recto y dos agudos, bien un ángulo obtuso y dos agudos, bien los tres ángulos agudos. En el primer caso se denomina *triángulo rectángulo*, como D E F (figura 39); en el segundo *triángulo obtusángulo*, como C H I (figura 40), y en el tercero *triángulo acutángulo*, como A B D (figura 37.)

65. La construcción de triángulos en general ofrece una gran variedad de casos; pero solo trataremos de los más comunes y que el dibujo lineal resuelve con suma sencillez:

Fig. 3. Comparison between Solís' book for men (left) and Solís' book for women (right)

Another significant difference appears in the lesson on areas: in the item on the area of a circle, symbolic formulas are removed, and the letter π does not appear, being replaced by the number 3,14159. Furthermore, in the same lesson, the inverse problem of finding the radius knowing the area is removed as well, in order to avoid a formula that includes a square root. On top of this, almost all three-dimensional geometry is removed.

In summary, technical vocabulary, symbolic formulas, special mathematical signs, complex properties and almost all three-dimensional geometry are removed by Prudencio Solís from his book for men to construct his book for women.

Francisca Ferrer and her book on geometry

Francisca Ferrer, was born in Valencia, Spain, in 1853, and died in 1931 also in Valencia. She studied in the teacher training school for women of Valencia from 1871 to 1873¹⁶. She wrote two books on the teaching of drawing, one for the drawing syllabus of the teacher training school for women of Valencia and another one for the primary education of girls, which was honoured with an award from the University of Valencia and the Department of Public Instruction in 1895. When she wrote her book on geometry for the syllabus of the teacher training school for women of Valencia she was not working as a teacher at this institution, but at a private one called *Institución para la enseñanza de la mujer* [Institution for women's education], founded in 1888 in Valencia¹⁷, an institution in which courses of general culture for women were taught, including courses in arithmetic and geometry. This institution was created by people of krausist ideology¹⁸ with the aim to improve the general education of women.

After publishing her book, Francisca Ferrer worked at the teacher training school for women between October 1899 and January 1902, as a special teacher of drawing and calligraphy. She was not in charge of the subject she treated in her book, but her

16 Courses to get the degree of teacher lasted two years at that time in the teacher training school for women of Valencia.

17 The Institution for women's education was the follow-up of the *Escuela de Comercio para Señoras* (School of Commerce for Ladies) founded in 1883, following a call to expand the pioneering schools for women organized since 1870 in Madrid by the *Asociación para la Enseñanza de la Mujer* (Association for the education of women) (cf. Reig, 2012).

18 Krausism was an ideology, named after the ideas of the German philosopher Karl Christian Friedrich Krause (1781-1832), which was the philosophical foundation of a cultural movement developed in Spain by a group of progressive thinkers, university professors and politicians. Their ideas were put into practice in education through the creation of the *Institución Libre de Enseñanza* (Free Institution of Education) in 1876, when some of their main members got expelled from the university by a conservative government. This Free Institution of Education lasted until the civil war of 1936-1939 and the victory of the dictator Francisco Franco.

book was adopted as an official textbook not only in the teacher training school for women of Valencia, but also in the one of Tarragona.

She was married with a well-known physician, Enrique Pertegás Malvech. She added the name of her husband to her name, Francisca Ferrer de Pertegás¹⁹, and signed her book with her name as a married woman (see the title page of her book in figure 1), instead of Francisca Ferrer Lecha, her maiden name.



Fig. 4. Francisca Ferrer Lecha (1853-1931)

Her book is presented explicitly in the introduction as an answer to the official syllabus of the teacher training school for women of Valencia. However, she adds that

19 In Spain we receive two family names, one from each of our parents. Until recently, the order of the family names had to be the father's in first place and the mother's in second place. Women don't lose their family names when they get married. However, it was common until recent times that they dropped out their second family name, the one from their mother, adopting the first family name of their husband preceded by the preposition "of", "de" in Spanish, meaning "dependent of" or "property of".

in anticipation of the contingencies that might arise from the change of [the syllabus], I have added what, in my opinion, they lack in order to complete the study of this subject; but attending to the published ones, I exclude in the part corresponding to Geometry all mathematical demonstration, allowing only the indication of some simple general formulas of essential necessity. (Ferrer, 1897, pp. III-IV)

Francisca Ferrer separates clearly the parts that answer the official syllabus and the added ones, by two devices. First, the lessons that answer the official syllabus are numbered with roman numerals and the added lessons are unnumbered. Second, the text that corresponds with added parts is printed in a smaller font both in the numbered lessons and in the unnumbered ones (these last ones are then completely printed in the smaller font).

The added lessons are the following:

- Curves composed by arcs of circumference
- Graphic problems of these curves
- Curves described by points or by a continuous movement
- Graphic problems of these curves
- Equal figures
- Equivalent figures
- Similar figures
- Symmetrical figures
- Areas of plane surfaces
- Areas of surfaces of solids
- Volumes of solids
- Solid nets

But this is only a part of what Francisca Ferrer added to the official syllabus. In the lessons that answer the official syllabus there is a considerable amount of added issues. A sample of the most significant is the following:

- Proportional segments
- Division of a segment in extreme and mean ratio
- Number of diagonals of a polygon, number of diagonals from a vertex
- Division of a polygon in triangles
- Measure of the interior angle of a polygon
- Drawing of polygons
- Star polygons

In the added paragraph on “proportional segments”, Francisca Ferrer explains the idea of proportional segments and relates the graphical methods to construct proportional segments to four statements. She explicitly calls these statements “theorems”, although she states them without proof.

With all these added to the syllabus, Francisca Ferrer’s book looks more like Solís’ for men than Solís’ for women. A close comparison of Francisca Ferrer’s book with Solís’ book for men (made in Cucala, 2017) shows that Francisca Ferrer’s book covers, in its first part on geometry, even more than Solís’ book for men²⁰.

On top of this, Francisca Ferrer’s book articulates the content of the first part on geometry with the second part on tasks (“labores”) of the women teacher by mutual references between the content of the lessons in the part on geometry with the content of the second part, something that is lacking in Solís’ book for women, which simply adds some a few pages on the tasks (“labores”). A sample of these references is shown in Table 3.

Table 3. References between the lessons on geometry and the tasks in the second part

Lesson in the part on Geometry	Task in the second part
General issues on the straight line	Main tasks of folding in straight line
Operations with straight lines	To draw simple laces or pointelles (fig. 5)
Angles	Drawing of angles for a frieze (fig. 6)
Scales	Application to the cutting of garments (fig. 7)
Circumference	Drawing of simple festoons
Arcs of circumference	Drawing of all kind of festoons (fig. 5)
Measure of angles	Main crochet tasks
Triangles	Procedure to square a piece of cloth: its foundation
Polygons	Geometric figures in the cutting of garments

²⁰ One may wonder where Francisca Ferrer got her knowledge of mathematics. As far as I know she was an autodidact.

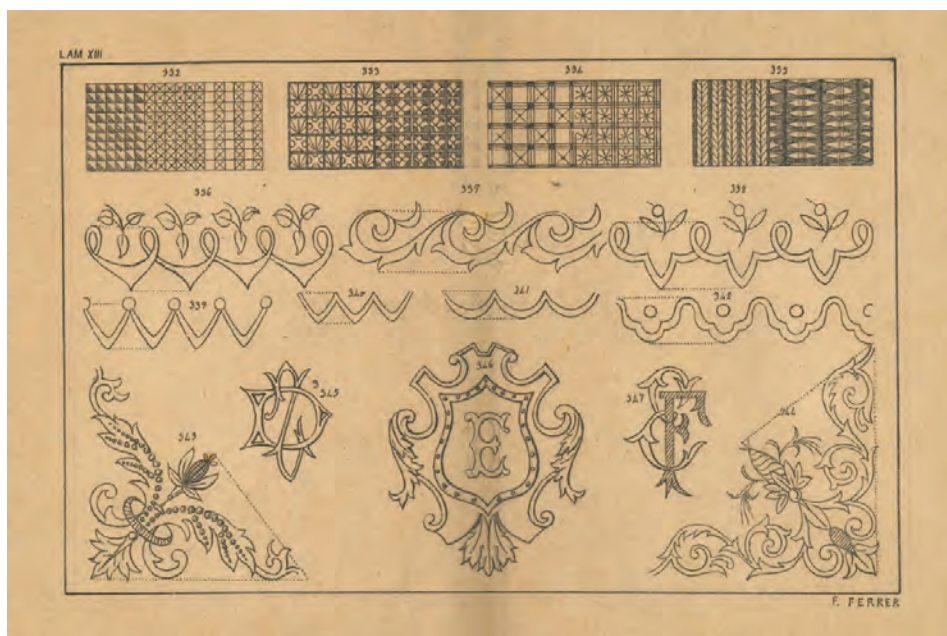


Fig. 5. Pointelles and festoons

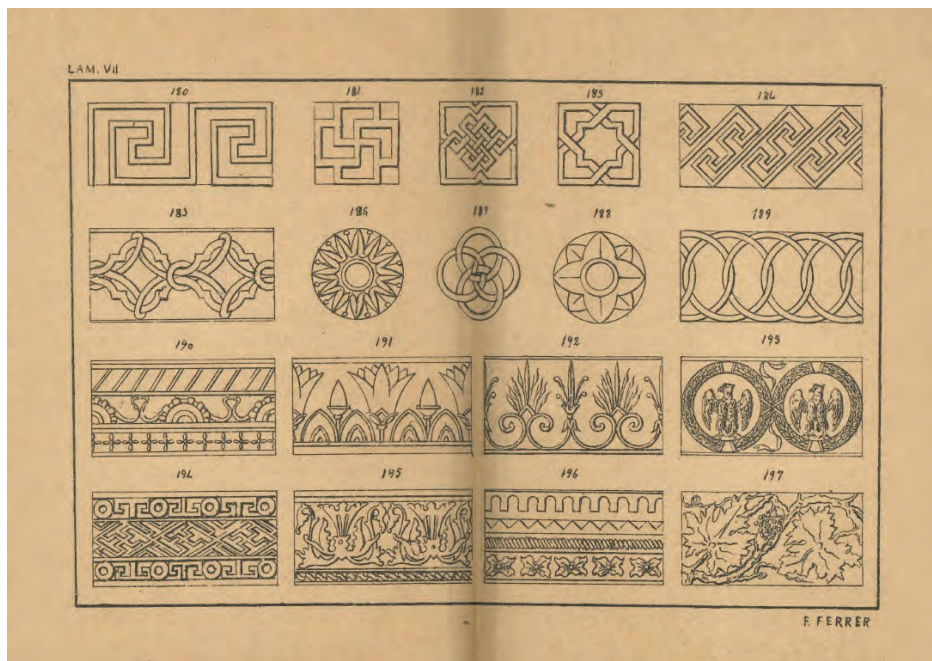


Fig. 6. Friezes

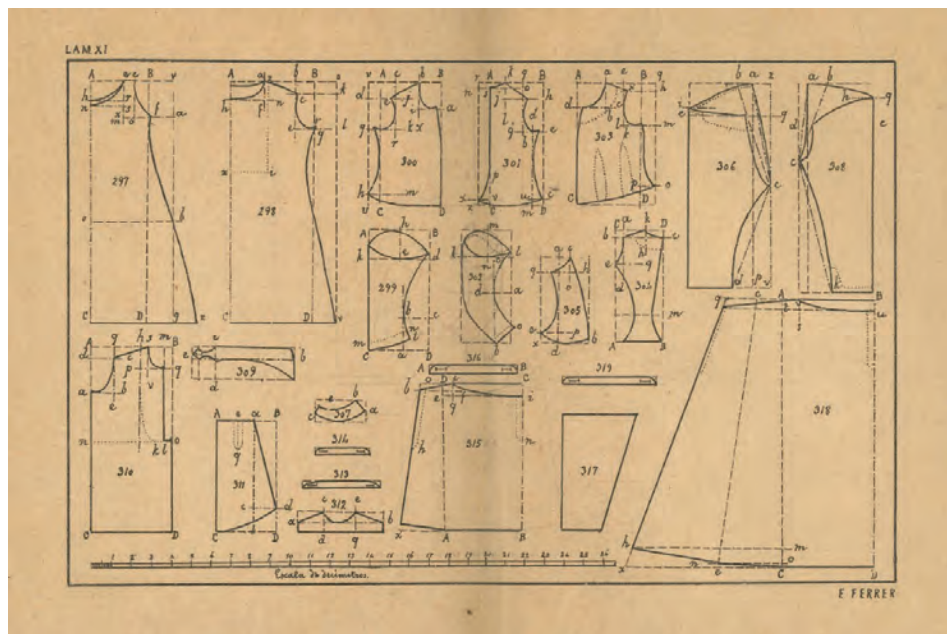


Fig. 7. Cutting of garments

Ferrer's book is an example of how the exclusion of geometry from the teaching of mathematics to girls to give room to the tasks befitting their sex could be restored paradoxically through the use of what had been the cause of its exclusion.

We do not know to what extent the parts of her book that go beyond the official program were used in teaching, but their mere presence in her book had the effect of being a statement of what women are able to study, and of what should be in the syllabus for women. And the fact that her book was adopted as an official textbook in the schools of Valencia and Tarragona guarantees that her book was in the hands of women students who might have had the curiosity to learn the added parts even if they were not taught.

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- Royal Decree September, 20. *Gaceta de Madrid*, September 23, 1858, num. 266, pp. 1-2.
- Royal Decree March, 14. *Gaceta de Madrid*, March 28, 1877, num. 87, pp. 853-854.
- Royal Decree August, 17. *Gaceta de Madrid*, August 31, 1881, num. 243, p. 614.
- Escuela Normal de Maestras de la provincia de Valencia, *Reglamento General de dicho establecimiento*, AHUV, Primary Education, box 125/1.

(Footnotes)

- 1 In the case of the teacher training schools for women the syllabus was presented in the Royal Decree organised by years instead of by levels.

Building new mathematical discourse among practitioners in restoration and enlightenment England 1650 – 1750

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Abstract

By the mid seventeenth century England was becoming a more diverse society where new political developments fostered a more secure and forward-looking society, and the idea of universal education was developing. From the late 16th century a community of mathematical practitioners flourished, and from these, specialist instrument makers had arisen so that by the early 17th century this community had grown considerably. The education system was unable to provide the 'useful' knowledge required for the new demands of an emerging practical society, so independent teachers began to provide the mathematical skills required. From the mid-17th century 'Dissenting Academies' were established to provide Nonconformist students, excluded by their religious beliefs from Oxford and Cambridge, with higher education. These became places where people could learn more 'useful' subjects; mathematics, foreign languages, geography and natural science, thus usurping and expanding the role of the universities. Practical instruction on a range of mathematical and astronomical instruments and new concepts and techniques provided the public needs by authors, showing their readers how to deal with Compound Interest and Annuities, and how to keep accounts "after the Italian manner" with exercises to improve the organisation of trade and business. New publications were intended to persuade the middle classes that the work of the artisan was to be valued. Joseph Moxon's Mechanick Exercises (1703) encouraged interest in the work of the artisans like smiths, builders, and others. Moxon believed in the inclusiveness of types of knowledge where the skills of the Blacksmith could be seen involved in a range of related occupations including those required in working with fine metals in watch-making, and supported his claim by reference to Francis Bacon's mechanical philosophy. This paper presents a discussion of the varieties of access and involvement in mathematics with related artisanal and industrial activities, as seen by the participants themselves, involved in the emergence of a technical curriculum, where mathematics is applied to, and becomes a result of, many different activities.

Keywords: industrial, discourse, practical, gentlemen, culture

The legacy of Francis Bacon (1561-1626)

The intellectual atmosphere of the sixteenth century aided by new kinds of scientific endeavour and founded on the new developments in astronomy and mathematics, had its most popular and powerful expression at the beginning of the next century in Francis Bacon's *Great Instauration* of 1620, containing the *Novum Organon*. In this reorganisation of the search for knowledge Bacon developed a new system of Natural Philosophy that was coherent with the social ethic of Protestantism. He paid attention to the sociology of knowledge by focusing on ideological obstacles,

barriers that were preventing the transformation of intellectual values from one area to another. Bacon produced a model for a reform of natural philosophy and other sciences where technological case-histories were used to demonstrate that the intelligent application of advances in one area, could lead to economic progress and intellectual advancement in another. In this way, Bacon issued a ‘manifesto’ for the experimental method. By attacking preconceived ideas scientific work should not seek ‘final causes’ but use logic and rhetoric in organising what we already know, so the advancement of the ‘mechanical sciences’ was attributable to the accumulation of small improvements; essentially empirical, collaborative and democratic, in contrast to the authoritarian beliefs passed from master to pupil. (Rogers, 2016). The inductive method was consistent with the democratic methods of the mechanic because the only sure means of a secure understanding was an experience with the object or situation itself. By the rigorous application of ‘experimental method’¹ it was hoped to ascertain conditions necessary and sufficient for the production of natural phenomena, whereby it would be possible to imitate nature by producing a grand catalogue of phenomena enabling new theories to explain the confusing data of our senses. This encouraged people’s active participation in empiricism by bringing the natural philosopher and the craftsman together.

Mathematics, instrument makers, and the restoration mathematical community²

The mathematical community of the late 17th and early 18th century England was far broader than hitherto portrayed in general histories of mathematics; it involved a wide range of people with very different backgrounds and very different approaches and understandings of mathematics. An increasingly sophisticated level of practical mathematical activities grew in accountancy, commerce, navigation, and instrument making. New mathematical learning was discussed and employed in workshops, warehouses, dockyards, and coffee houses, and was disseminated in printed books, leaflets, journals and letters, so the proliferation of these works enabled cross-fertilisation of ideas, particularly ideas about the establishment of scientific beliefs and technical practices.

By ‘mathematical community’ (Wenger, 1999), I indicate that group of people who were connected with mathematical practices or study in any way. It includes mathematical practitioners (Rogers, 2012) such as surveyors, navigators and astronomers;

1 Bacon’s ‘experimental philosophy’ included craft skills as legitimate areas of study from which ‘natural philosophy’ could learn and improve its own methods. He is often regarded as the first English empiricist philosopher.

2 The Restoration was the period from 1660 to about 1720 when the Scottish and Irish monarchies were restored under Charles II, until the accession of the Hanoverian dynasty.

teachers of practical mathematics, from schools to Dissenting Academics³; makers of mathematical instruments and booksellers publishing mathematical texts, and a large number of amateurs among the gentry and nobility, who patronised the mathematical sciences. This included the local conditions, the espoused and enacted beliefs of individuals, and the social, political, and economic influences on the establishment of particular kinds of knowledge. It is important to realise that this also included well-known mathematicians of the time who often took part in discussions and collaborative activities.

In the study of the practice of mathematics in 17th and 18th century England, the emerging epistemological varieties appear in the views of the nature and processes of formation of the mathematics and its uses that were espoused by the different agents whose views depended on their ideologies and were inevitably involved in the production of their texts. The purposes of mathematics that were perceived, and the uses to which it was put, to some extent determined the kind of mathematics that an individual was motivated to learn and to apply, and these needs were felt in different ways in particular sections of society. A significant aspect of this was a deep reflexive relation between the Master craftsman and the Instrument. Customers asking for instruction on and improvements to instruments prompted new developments, and craftsmen talking to each other and their clients were driving modifications that challenged the user to explore connections between the practical use of instruments, and the supporting mathematics. In that 'Baconian' discourse, geometry, trigonometry, logarithms and algebra were among the more theoretical aspects of mathematics; while astronomy, navigation, surveying (both civil and military), geography, gunnery, dialling, accounting and book-keeping, gauging, and other forms of mensuration, all fell within the field of practical mathematics.

Familiar books from the 16th century were reissued with additions and modifications: Billingsley's (1570) edition of Euclid, with Dee's *Mathematicall Praeface* was still being read (Ramping, 2011), both Richard Norwood's *The Sea-Mans Practice* and Edmund Wingate's *Arithmetique made easie, including the use of logarithms*, appeared in many different editions and were still being printed well into the eighteenth century. At another level appeared instructions on the use of particular instruments, books containing mathematical tables, or any kind of simple 'ready-reckoners' for measurement. For example, Mr Hoppus' *Measurer Practical measuring made Easy to the Meanest Capacity* (1736) contained instructions for reducing mensuration to the simplest measurements, rules and tables thus avoiding difficult calculation. Extensive proliferation of 'pseudo-applications' of mathematics and science appeared in Almanacs, in

3 The 'Dissenting Academies' provided university-level education for those protestants of various persuasions, who were not members of the established church, barred from entering the universities of Oxford and Cambridge (Parker, 1914).

calendars with astronomical calculations, planetary motions and horoscopes, mixed with medical and religious information.

Expanding subject areas

Arithmetic found its most frequent new application in accounting, book-keeping and calculating interest; older texts were re-edited to include additional sections. Edward Hatton edited the last issue of Recorde's *The Ground of Arts* in 1699, and then went on to produce *An intire system of arithmetic, or Arithmetic in all its parts: containing Vulgar, decimal, duodecimal, sexagesimal, political, logarithmical, lineal, instrumental and algebraical arithmetic* in 1721. This included negative numbers, 'approximation' (using converging series) "the whole intermix'd with rules new, curious and useful", thus advertising the contents. Cocker's *Arithmetic* edited by Hawkins (1703) has tables of interest and rebate, logarithms, and 'algebraical' arithmetic, but new areas of interest and applications of mathematics began to appear with John Graunt's *Natural and Political Observations Made upon the Bills of Mortality* (1662/3), and William Petty's inventions in *Political Arithmetic* (1690).

Geometry, and its applications were being used in surveying and architecture; trigonometry appeared in navigation and astronomy using constructions with sine, cosine and tangent lines on sectors and similar instruments. Geometry traditionally dominated the discourse, particularly when trigonometry was incorporated, and was still the most important part of mathematical theory related to applications.

Algebra rarely found any real applications in mathematics at this time. There was a belief that algebra was important, and there were publications that 'explained' algebra and began to include new useful subjects in the same volume. John Ward produced in 1698 a *Compendium of Algebra, Exemplifying Plain and Easie Rules for the Speedy Attaining to that art . . .* including Equations, Squaring the Circle, making Sines, Tangents and Logarithms with an Appendix on Compound Interest and Annuities. However, William Webster dismissed algebra entirely:

Algebra, indeed is not properly a Science distinct from Arithmetic, but is only a different Method of Computation, performed by substituting letters in the Place of Figures, and expressing the several Parts of the Operation by Symbolical Characters But however useful it may be to those who understand it, it is certainly an unintelligible jargon to the mere Numerist, and can give him very little satisfaction when laid before him as a Demonstration (Webster 1751 Preface A3).

Others like Edward Wells (1714) in *The Young Gentleman's Course of Mathematicks Containing the more Useful and Easy Elements . . .* neglected algebra almost entirely, dealing with the more popular or 'mechanical' subjects like optics and dialling. In most

contexts, virtually all of the practical problems that might have been susceptible to algebraic equations were still being solved by the application of proportional analysis.⁴

Joseph Moxon (1627-1691) was a prolific author of the later 17th century, taking up Bacon's theme that craft knowledge was equally as valuable as 'theoretical' knowledge. He produced the first English mathematics dictionary and the first detailed instructional manual for printers, together with other books on a wide range of subjects, from mathematics, astronomy, and architecture, to works about artisans and craftsmen. He regarded the 'trades' as legitimate occupations and worked to break down social and political barriers, providing a unifying account of people working together for the common good, following Baconian principles. Typical of this position was his *Doctrine of 'Handy-Works'* where he described smithing as an important craft skill which encompassed "not only the Black-Smiths Trade but takes in all Trades which use either forge or file, from the Anchor-Smith to the Watch-Maker." thus including all types of metals in use, and justifying his claims:

The Lord Bacon, in his Natural History, reckons that Philosophy would be improv'd by having the secrets of all Trades lye open; not only because much Experimental Philosophy is coucht amongst them; but also that the Trades themselves might, by a Philosopher, be improv'd. Besides, I find, that one Trade may borrow many Eminent Helps in Work of another Trade (Moxon 1703 Preface).

In 1678, he became the first tradesman to be elected as a Fellow of the Royal Society.

Tools for practical mathematics

Edmund Gunter (1581-1636) is remembered for developing many practical instruments which include the 'Gunter's Chain', 'Gunter's Quadrant', and 'Gunter's scale'. In 1619 Sir Henry Saville, wishing to improve the state of mathematical studies in England, funded the first two chairs of astronomy and geometry at Oxford University. Gunter applied to become professor of geometry, bringing some of his instruments to the interview, but Saville, requiring the applicants to be acquainted with traditional sciences, dismissed him, accusing Gunter of showing 'tricks', not true geometry.⁵ This attitude, also expressed by William Oughtred (1535-1660) showed an innate suspicion of the technically skilled:

⁴ For example; The Rule of Three, or double and triple proportion.

⁵ There is a well-known story (see MacTutor History of Mathematics archive) about how Gunter took his instruments to demonstrate but Saville, a rather opinionated character, rejected him since he had no obvious knowledge of traditional geometric reasoning. Successful use of an instrument does not necessarily mean the operator knows the theory behind it.

That the true way of Art is not by Instruments, but by Demonstration: and that it is a preposterous course of vulgar Teachers, to begin with Instruments, and not with the Sciences, and so, in-stead of Artists, to make their Scholers only doers of tricks, and as it were Iuglers: to the despite of Art, losse of precious time, and betraying of willing and industrious wits, vnto ignorance, and idleness. That the use of Instruments is indeed excellent, if a man be an Artist: but contemptible, being set and opposed to the Art (Oughtred 1632 sig.A3 verso)

In the same year, supported by Henry Briggs, Gunter was appointed professor of Astronomy at Gresham College⁶. In 1620 he published his *Canon triangulorum*, a method of calculating logarithmic tangents and in 1624 a collection of his mathematical works entitled *The description and use of sector, the cross-staffe, and other instruments for such as are studious of mathematical practise*. It was a manual for sailors and surveyors in the real world and published in English. Gunter wrote:

Not that I think it worthy either of my labour or the publique view, but to satisfy their importunity who not understand the Latin yet were at the charge to buy the instrument.

He invented the terms cosine and cotangent and some of his most notable practical inventions were the Gunter chain, the Gunter Quadrant and the Gunter scale. The ‘Gunter chain’ was meant for surveying and triangulation with intermediate measurements indicated, giving a new unit easily converted to area⁷. Gunter’s quadrant contained a stereographic projection of the sphere on the plane of the equinoctial. With the eye placed in one of the poles, the tropic, ecliptic, and horizon form arcs of circles. This is used to find the hour of the day, the sun’s azimuth, etc. and to find the altitude of an object in degrees. Gunter’s scale: on one side are placed the lines of chords, sines, tangents, and rhumbs, etc. on the other side the corresponding artificial or logarithmic lines. Using this instrument questions in navigation, trigonometry etc., are solved with the aid of a pair of proportional compasses. This is a predecessor of the slide rule, commonly used from the 17th to the mid-20th century. His instruments were commonly used and they reflected the practical nature of his teaching that was linked to the more scholarly work of his time.

Advertising your wares

Social gatherings spread information as new instruments became available; advertisements were carried in books or by pamphlets produced by the instrument makers themselves. Whether the purveyor was expert enough in different areas to explain

⁶ Gresham College was founded in 1597 to give free lectures to people of the city of London. It was the first ‘university’ outside Oxford and Cambridge.

⁷ Traditional English units: 66 feet or 20 m long. An area of land of 10 square chains gives 1 acre.

the technical details to a customer is an open question. Seth Partridge (1603-1686) described himself as a surveyor, offering land-measuring, but he seems to have been involved in various branches of mathematics and its applications. In the end-page of his *Rabdologia: or The Art of Numbring by Rods* (1648), his self-praise is typical: declaring himself capable of offering instruction in arithmetic (whole numbers and fractions, decimals, roots, astronomical fractions, algebra, arithmetical rods [i.e. Napier's bones]; geometry (principles, gauging, surveying, use of the plane table, circumferentor, theodolite, circular scale, quadrant, semicircle, peractor, sector, circles of proportion and Wingate's lines of proportion); trigonometry (use of logarithmic trigonometric tables, measuring of heights etc., doctrine of triangles; navigation (including the use of instruments, maps and charts); cosmography (use of globes) the armillary sphere, the astrolabe, and dialling.

John Blagrave (1558-1612) was a gentleman⁸ and self-taught professional mathematician. His work included land-surveying and the design and erection of sundials. He became involved in inventing and modifying new mathematical and navigational instruments. His books accompanying his *Familiar Staffe* (1584) (proportional compasses) and the *Mathematical Jewel* (1585) (a modified armillary sphere) contain instructions for operating the instruments, with basic geometrical and astronomical information on the setting up and use of the instrument in a variety of situations. Typically, these instruments, although often capable of producing more accurate results, never became established being complicated and difficult to set up. His *Mathematical Jewel* appears in Seth Partridge's advertisement in the *Rabdologia* in 1648.

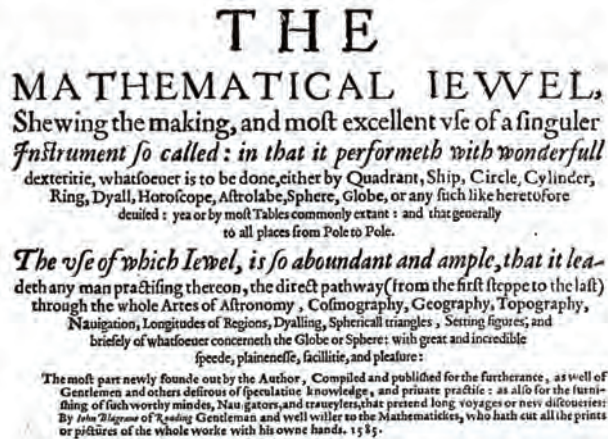


Fig. 1. John Blagrave's Advertisement from his *Mathematical Jewel*. Upper half of title page from Blagrave, 1585, recto.

⁸ The term Gentleman (sometimes 'the gentry') originally referred to anyone who had a private income, owned land, or was an ordained member of the Church of England. Later, merchants owning property became included.

Edmund Wingate (1596-1656) went to Paris, where he became teacher of English to Princess Henrietta Mari⁹. He had learned the ‘rule of proportion’ (the logarithmic scale) recently invented by Edmund Gunter which he communicated to mathematicians in Paris. He quickly published a description, *L’usage de la règle de proportion en arithmétique*, Paris, 1624, and an English version as *The Use of the Rule of Proportion*, London, 1626, demonstrating the principle of the slide rule. Later, he produced *Logarithmotechnia, oringate. The construction, and use of the logarithmeticall tables by the help of which, multiplication is performed by addition, division by subtraction, the extraction of the square root by bipartition, and of the cube root by tripartition* 1635, to secure priority of publication. His most significant work was *Of Natural and Artificiall Arithmetique*, London, 1630, 2 parts. Part I. had been designed “onely as a key to open the secrets of the other, which treats of artificial arithmetique performed by logarithms”¹⁰, it was re-issued and published in 1652 as *Arithmetique made easie*. This book reappeared in many versions until 1753.

Henry Sutton (1624-1665) was a typical example of an English instrument maker working in London from ca. 1650 to 1661. His work was advertised mostly by word of mouth, and it was known that he could make all of the usual mathematical instruments. Known for his high-quality engravings of scales and quadrants, he sold books, and understood the mathematics enough to collaborate with mathematicians in the design of new instruments. He was reputed to be the most talented and original mathematical instrument maker in London in the middle of the seventeenth century and was mentioned in Samuel Pepys’ Diary. His place of work and range of stock is shown from an advertisement, written on a quadrant dated 1658, “This Instrument or any of the Mathematiques are made in Brass or Wood by Henry Sutton Instrument maker behind the royall exchange”.¹¹ One of Sutton’s most important projects was engraving the plates for John Collins’s *The Sector on a Quadrant* (1658) with the design prints of the ‘Sutton quadrant’.

This quadrant, made by Henry Sutton (1657), is a conveniently sized and high-performance instrument. The front is designed as a ‘Gunter’ quadrant and the obverse as a trigonometric quadrant. The side with the astrolabe has hour lines, a calendar, zodiacs, star positions, astrolabe projections, and a vertical dial. The side with the geometric quadrants (shown here) features several trigonometric functions, rules, a shadow quadrant, and the chord line. This ‘Sutton-type’ quadrant, for astronomical calculations was similar to existing quadrants, but extensively promoted,

9 Princess Henrietta Maria of France, a Catholic, became the wife of Charles 1 in 1625. She bore him two children, Charles, (to be Charles II) and Princess Henrietta.

10 Interesting to observe here that that the use of logarithms for calculation made arithmetic ‘artificial’.

11 The Royal Exchange is located in the City of London near St Paul’s and the Bank of England. This area was frequented by many who were making or purchasing mathematical instruments in the 17th and 18th centuries

so by the end of the 17th century it was one of the most common astronomical calculating devices (Jardine, 2006). Sutton died of the plague in 1665.



Fig. 2. Astronomical Quadrant made by Henry Sutton 1657. MHS. Oxford

Promotion and dissemination

Edmund Stone (1700–1768) and Nicolas Bion¹² (1652–1733).

Nicolas Bion's *Traite de la Construction et des Principaux Usages des Instruments de Mathematique* (1709, 1752) was also published in German. Edmund Stone published his first translation of Bion's work in 1723: *The Construction and Principal Uses of Mathematical Instruments Translated from the French of M. Bion, Chief Instrument-Maker to the French King. To which are added, The Construction and Uses of such INSTRUMENTS as are Omitted by M. BION*, particularly those invented or improved by the *ENGLISH*. By EDMUND STONE. The whole book is illustrated with thirty Folio Copper Plates containing the Figures of the several INSTRUMENTS. (2nd Edition 1758).

12 Very little is known about Edmund Stone or Nicolas Bion. Stone was the son of a gardener and taught himself to read and write and learnt Latin and French at an early age. His Patron was the Duke of Argyll, who supported his talents. There are records of his works in the Royal Society, but outside his publications, nothing seems to be known. Bion is equally unknown. As an instrument maker with a workshop in Paris he advertised himself as an instrument maker to the King, but as far as my French colleagues can determine, there are no supporting records.

Thanks to this translation, and the influence of his patron, the Duke of Argyll, Stone was made a Fellow of the Royal Society in 1725. In 1758 he published a further edition, including *A Supplement concerning a further account of the Most Useful Mathematical instruments as now Improved*. This is a remarkable book, with 326 pages of descriptions and 30 engravings of instruments including many geometrical diagrams, giving the method of construction and mode of operation of every measuring and calculating instrument in use during the 17th and 18th centuries. Nowhere else in this period can be found such a wide-ranging, and detailed collection showing the geometrical principles necessary for their construction. Stone regarded mathematics as a useful and necessary part of a proper education, defining mathematics both as a science, with regard to the theory, and as an art, with regard to the practice. Mathematical instruments connected these important and inseparable sides of mathematics:

Mathematical Instruments are the Means by which those Sciences are rendered useful in the Affairs of Life. By their assistance, it is that subtle and abstract Speculations reduced into Act. They connect, as it were, the Theory to the Practice, and turn what was bare Contemplation, to the most substantial Uses. The Knowledge of these is the Knowledge of Practical Mathematics: so that the Descriptions and Uses of Mathematical Instruments, make, perhaps, one of the most serviceable Branches of Learning in the World (Stone 1758 Preface. [v])

John Collins (1625-1683) was an important figure in the mid-17th century¹³. He is most known for his extensive correspondence with many leading scientists and mathematicians in England and on the continent, providing details of the many discoveries and developments made in his time. He was exchanging notes on mathematical books, problems or inventions with many people who had an interest in both theory and applications of mathematics. Apprenticed to a bookseller, and later employed as a clerk, he obtained some basic instruction in mathematics. He went to sea, where he devoted his leisure to the study of mathematics and merchants' accounts. On leaving his ship in 1649 he taught in London. In 1653 he published *An Introduction to Merchants' Accounts*, republished in 1664 and 1674. He next wrote *The Sector on a Quadrant, or a Treatise containing the Description and Use of three several Quadrants*¹⁴ (1658). In 1659 his *Geometricall Dyalling, or Dyalling performed by a Line of Chords only*, appeared, and *The Mariner's Plain Scale new Plained*, a treatise on navigation, these were written for the East India Company.

After the Restoration, Collins was appointed accountant to the excise office, and accountant to the Royal Fishery Company. Collins helped support many

13 For a comprehensive account of Collins' life and work, see Beeley 2017.

14 A note includes the inscription: "To be sold by George Hurlock book-seller at Magnus Corner, by William Fisher at the Postern near Tower-Hill, and by Henry Sutton mathematical instrument maker, at his house in Threadneedle Street behind the Royal Exchange."

important publications and was particularly active in seeing Jeremiah Horrocks¹⁵ work *Astronomical Remains* through the press. He contributed to *An Account concerning the Resolution of Equations in Numbers*, a survey of recent algebra improvements made in England, completed by John Kersey in 1685. His *Arithmetic in whole Numbers and Fractions, both Vulgar and Decimal, with Tables for the Forbearance and Rebate of Money, &c.*, was published in 1688. For his zeal in collecting and diffusing scientific information, Collins was styled the 'English Mersenne.' He was constantly stimulating others to become involved in useful inquiries and pointing out defects in different branches of science. For this work, Collins was elected a fellow of the Royal Society in October 1667.

Qualitative language, new investigations and new applications

John Graunt (1629–1674) developed early statistical and census methods that later provided a framework for modern demography. Graunt's book *Natural and Political Observations Made upon the Bills of Mortality* (1662/3) used the records of deaths in London in an attempt to create a warning system for the onset and spread of plague in the city. As a result, Graunt produced the first Life Insurance Table recording the likelihood of human survival at each age. His work also resulted in the first statistically based estimation of the population of London. Graunt's life tables presented mortality in terms of survivorship, where he predicted the percentage of persons that will live to each successive age and their life expectancy year by year. He managed to produce a realistic estimate for the population of 17th century London as about 600,000 persons. For this work he was elected to the Royal Society in 1662.

William Petty (1623–1687) was an important figure in the history of social science, a proponent of the new empirical and mechanical philosophy and one of the founders of the Royal Society. Petty's invention in *Political Arithmetic* (posthumously published in 1690) was an early form of economic theory that applied the methods and concepts of seventeenth-century natural philosophy to the challenges of Great Britain as a monarchy governing a colonial empire, consisting of English rule in Ireland, colonies in the Americas, trade in the East Indies, and the politics of religions in England, Scotland and Ireland. Petty developed a sophisticated monetary theory; he reduced political, religious, and ethnic differences to demographic measures. After Petty's death his 'instrument of government' through demographic engineering was rearticulated as a mode of statistical analysis, an early form of social science.

These new areas with applications of mathematics were growing out of the newly perceived needs of a more sophisticated population. People were involved

¹⁵ Jeremiah Horrocks (1618–1641) was a British astronomer who predicted, and with his friend William Crabtree, were the first to view the Transit of Venus in 1639.

in developing new ways of talking about problems and using mathematics to solve them. It is interesting to note that a parallel movement was happening in early education theory. George Snell produced *The Right Teaching of Useful Knowledge to fit Scholars for some honest Profession*. (1649) Snell and others argued that instead of taking so much time over teaching of the classics, 'useful' knowledge should be taught and there was a deliberate movement to establish what knowledge is useful for a contemporary society in order to develop a new, relevant curriculum.

Technology, culture and the Royal Society

From the beginning of the Civil War (1642) many Oxford academics lost their posts for their political and religious views, and replacements were sent to match the current government's agenda¹⁶. The New Commonwealth government was ineffective, purges were badly organised, so Oxford became a place where different opinions, ideas and personalities were brought together in a potentially dangerous situation. John Wilkins (1614-1672) became Warden of Wadham College in 1648 and managed to encourage political and religious tolerance by drawing people into a group that became known as the Oxford Philosophical Club meeting regularly to discuss natural philosophy and experimental science. This group included Robert Hooke, Christopher Wren, Seth Ward, and John Wallis. After the Restoration (1660) there were similar regular meetings at Gresham College¹⁷ in London. Wilkins brought together those studying different disciplines and encouraged them to work collaboratively, forging friendships, intellectual connections, and a spirit of enquiry that would foster the formation of the Royal Society in 1660 as a 'College for the Promoting of Physico-Mathematical Experimental Learning', an important group for discussion and promotion of scientific ideas.

Robert Hooke (1635-1703) had experimented with the new microscope constructed by Christopher White of London¹⁸. Hooke used this microscope for the observations that formed the basis of his *Micrographia*, published in January 1665, the first major publication of the new Royal Society.

16 A complicated area of English history covering the period of the Royalist government, the rise of the Parliamentarians and the Commonwealth of England after the execution of Charles 1.

17 The Will of Sir Thomas (1519-1579) an English merchant and financier, provided for the setting up of Gresham College in Bishopsgate as a place for free public lectures on scientific subjects and frequent discussion of new ideas. The college played an important role in the formation of the Royal Society.

18 Compound microscopes, with an objective lens and an eyepiece appeared in Europe in the early 17th century. A Dutch inventor Cornelis Drebbel is said to have had one in London in 1619. In that context, Christopher White could easily have built and improved the instrument.

This remarkable book containing Hooke's drawings of flies, fleas and lice, revealed an unknown world, inspiring a wide public interest in the new science of the microscope. The popularity of the book helped further the society's image and mission as England's leading scientific organisation, and the illustrations of the miniature world captured the public's imagination in a radically new way. Hooke's Preface to the *Micrographia* explains his rationale and offers some fascinating insights into the scientific method of the day, where he talks about an "inlargement of the Dominion of the senses":

The next care to be taken, in respect of the Senses, is a supplying of their infirmities with Instruments, and, as it were, the adding of artificial Organs to the natural; this in one of them has been of late years accomplished with prodigious benefit to all sorts of useful knowledge, by the invention of Optical Glasses. By the means of Telescopes, there is nothing so far distant but may be represented to our view; and by the help of Microscopes, there is nothing so small, as to escape our inquiry; hence there is a new visible World discovered to the understanding.

And he declares the new kinds of knowledge gained from the experimental methods:

From whence there may arise many admirable advantages, towards the increase of the **Operative, and the Mechanic Knowledge**, to which this Age seems so much inclined, because we may perhaps be enabled to discern all the secret workings of Nature, (Hooke *Micrographia* (1665) Preface) (emphasis by author)

The scientific method encouraged technical advances and supported empiricism. Optical instruments were improved, and new applications developed. Newton invented the reflecting telescope and investigated the use of a prism to produce his *Opticks* (1704), a record of experiments and deductions made from them that were originally reported to the Royal Society in 1672. The significance of this work was that, by experimental method and scientific induction the dogma originating from Classical Philosophy was overturned. Light was a series of colours emanating from the physical property of the phenomenon itself, depending on the angle refracted through the prism, so that colour is a sensation of the mind, not an inherent property of the material objects.

However, once the basic ideas of empiricism had been set out by Francis Bacon, philosophers began to explore and advance the study of the human mind and its behaviour. The experiences of the instrument makers, reported and recorded in publications, and the discussion of the means and consequences of actions and observations provided the data for more philosophical discussion. In 1690 John Locke published his *Essay Concerning Human Understanding*, a seminal source that concerned the foundation of human knowledge. He described the mind at birth as a blank slate to be filled later through experience. However, Bishop Berkeley's *Theory of Vision*

(1709) and *Principles of Human Knowledge* (1710) refuted Locke's theory of human perception and sought to prove that the outside world (the world which causes the ideas one has within one's mind) is also composed solely of ideas. David Hume in his *Treatise of Human Nature* (1739) adopted Locke's theory of ideas but disputed them on the grounds that there was no perceptual experience that conveys the idea of self. 'Common sense' certainty of one's existence as found in Descartes and Locke was disputed and Hume declared that experience of causality arises only through the 'observed sequencing of events'. But the 'observed sequencing of events' was the foundation of the experimental evidence. At this time, apart from Berkeley's *The Analyst*, there was little direct effect of these ideas on the practice of mathematics or its applications.

Discussion and conclusion

In this period, re-examination of classical concepts emphasized the fact that reason requires knowledge, and that reasoning skills are necessary for success in any society by building necessary knowledge and testing emergent hypotheses as entrepreneurial activity develops. A major aspect of this paper is the way in which we see craftsmen become innovators, as many 'traditional' activities like ship-building (Rogers, 2012) and monument-building (Moxon, 1673) become a design activity and develop new ideas by forward planning using mathematics and scientific acumen. Clearly, this applies to the generations of Instrument-makers whose innovative and cooperative work was driving forward the development of science and many different new technologies. The basis for a large part of these advances are the improvements in technologies themselves (Rogers, 2016) Metal-working of all kinds, from new furnace design, smelting, and fashioning to engraving metals is not specifically mentioned here, but as crafts emerge from cottage industries we see the slow, steady early progress towards industrialisation that demands attention from the population.

It became clear that the few educational institutions were quite unable to deal with the demand for appropriate education, to adapt to the changing economy and expanding culture. There was a particular demand for mathematics and its developing specialisations; new methods of calculating and applying mathematics in novel situations and ways of measuring all kinds of objects and events. The political and religious circumstances drove the foundation of the Dissenting Academies to satisfy the needs for the expanding economy, offering occupational and new useful professional training, based on rational philosophy and egalitarian principles,¹⁹ adopting English as the new language of science.

¹⁹ In effect, these Academies, growing outside the universities, became the Mechanics Institutes and Polytechnics of the late 18th and 19th centuries.

While it seems that ‘pure’ mathematics has been ignored in this account, it cannot be neglected, where people like John Collins, Edmund Gunter, and Henry Sutton were living in a community that included people like Harriot, Oughtred, Wallis, Hooke, Wren, and others who were clearly present and involved in exchanges of information and useful techniques, to a greater or lesser degree. Exposing this mathematical community focuses on the changing discourse as we move from individual craft-apprentice through master technician and business organiser to instrument user, recognising the development of professional practice, and the acquisition of new forms of knowledge and their consequent epistemological basis together with the development of the language involved. Between each occupation there can be seen a symbiotic relation where the experience of one informs and improves the other (Meli Bertoloni, 2006). Using the instrument develops knowledge, and in turn, the instrument inspires new ideas. Augmenting the senses develops operative knowledge (Hooke) and people are beginning to recognise that theory and practice lie on a continuum so that each individual can choose where to reside. Overall, we see the gradual evolution of a new curriculum of useful knowledge that laid the foundations of the technical and school curriculum from the early 19th century.

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²⁰ EEBO stands for Early English Books Online. This is found through the Bodleian Library Oxford, (solo.bodleian.ox.ac.uk), <https://eebo.chadwyck.com/>

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Norms and practices of secondary teachers' formation. The Portuguese case (1915-1930)¹

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Abstract

This paper discusses norms and practices of secondary teacher education in mathematics in the beginning of the 20th century in Portugal. Following the educational proposals of the new republican order, Normal Higher Schools (Escolas Normais Superiores), created in 1911 and operational from 1915 until 1930, play a central role in the training of professionals for teaching secondary disciplines, particularly mathematics. For the first time at the secondary level, teacher education comprises a theoretical study of subjects akin to their future profession and a practical initiation at the school level. We focus on the analysis of norms and practices leading to the professional knowledge formation of secondary school mathematics teacher. Norms are assessed through the study of legislative documents, regulations and bureaucratic materials. Practices are determined following the analysis of several kinds of students' work required for the diverse disciplines (exercises, tests, conferences, written assignments, etc.) and the extensive reports produced at the end of their internship. We found these schools fostered the development of a pedagogical knowledge influenced by the New School movement which was based on the ideas of Pestalozzi, Rousseau, etc.; the innovative international mathematical tendencies supported by ICME; and the appreciation of the social utility of school mathematics

Keywords: history of mathematics education; teacher education; internationalization

Introduction

In Portugal, from the beginning of the twentieth century, access to teaching positions at secondary schools required specific training. In 1901 a Secondary Teaching Formation Course [Curso de Habilitação para o Magistério Secundário] was created with a duration of four years (Pintassilgo, Mogarro & Henriques, 2010). This training model required the future teachers to obtain the scientific training at the university level during the first three years and, in the fourth, a pedagogical training took place in Lisbon. Until then, access to the secondary teaching profession was done through examinations for which a university education was not necessary.

The model was further developed after the implantation of the Republic in 1910 with the creation in 1911 of two Higher Normal Schools [Escolas Normais Superiores, ENS] attached to the Faculties of Letters of Coimbra and Lisbon, which

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were completely functioning in 1915 (Gomes, 1989). Born of the republican desire to value education, these schools intended to give the dignity of higher education to professional teacher education and played a central role in the formation of professionals to teach mathematics in secondary schools (, 2014). Later, the dictatorial regime created by Oliveira Salazar (1926-1974) would take a set of measures that lead to their extinction from 1930.

The ENS were initially studied by Joaquim Ferreira Gomes (1989) who presented an exhaustive survey of documentation concerning the two schools, in particular Coimbra. A more analytical approach was developed by Joaquim Pintassilgo, Maria João Mogarro and Raquel Henriques (2010) who related the legislative intentions and other documentation with the tendencies of pedagogical thought of the time. Maria João Mogarro (2012) also studied the relations between the formation in the ENS and the New School trend. This movement gathers a cluster of ideas disseminated from the end of the eighteenth century that essentially put the student at the centre of the educational process. It intended to base teaching and learning in children's experiences and valued the importance of active methods rejecting authoritarian pedagogical approaches centred on the transmission of abstract knowledge. The proposals of Jean-Jacques Rousseau, Johann Heinrich Pestalozzi, among others, are associated with this movement. Our previous work on this topic focused on the development of teachers' professional knowledge (Matos, 2017) and on the ENS's role in the circulation of ideas (Santiago & Matos, 2018). ENS represent a significant point in the development of the autonomy of school knowledge (Julia, 1995) in Portuguese secondary education. For the first time, contexts for reflections and practices focusing on specific contents of school mathematics and the related teaching methods were available (Matos, 2015).

The purpose of this text is the analysis of norms and practices (Julia, 1995) leading to the formation of professional knowledge of secondary school mathematics teachers in Portugal from 1915 until 1930. Norms include knowledge to teach and behaviours to be inculcated and they will be assessed through the study of legislative documents, regulations and bureaucratic materials. Practices that allow the transmission of this knowledge and the incorporation of these behaviours will be assessed through the analysis of several kinds of students' work required for the disciplines (exercises, tests, conferences, written assignments, etc.) and the extensive reports produced at the end of their internship. This paper is based on documentation preserved at the archives of the University of Coimbra, at personal archives, and in Biblioteca Nacional in Lisbon that was not previously studied, as far as mathematics education is concerned. This research is somehow unbalanced, as the materials found come mainly from the ENS in Coimbra.

We will survey the path students took as they entered these institutions: entrance examinations, first year and its disciplines, teaching practice of the second year, final

examination and the required dissertations. For each of these steps we will analyse their norms and practices.

Entrance examinations

To be admitted to the ENS, prospective secondary teachers should have had a college degree and have passed an entrance examination. In the case of mathematics, candidates should have had at least a “Bacharelato” in mathematics (3 years). This degree had a roughly similar structure at the University of Coimbra and at the two polytechnic schools of Lisbon and Porto. It included the following disciplines (Quadro de equivalências, 1915):

- Higher Algebra, Analytic Geometry, Spherical Trigonometry
- Differential, Integral, and Variations Calculus, Probability,
- Rational Mechanics,
- Descriptive Geometry,
- Astronomy, Geodesy,
- Physics, Chemistry, Mineralogy, Geology, Zoology, Botanic,
- Drawing.

Entrance examinations were composed of two parts. An eliminatory general part, common to all prospective teachers, intended to verify the degree of general culture of the candidates (Decree n. 2.646, 26/9/1916). It was composed of a written essay about Portuguese history (3 hours) and a translation of a text from Portuguese to French (1 hour). These examinations also included a special part akin to the discipline the candidates intended to teach. In the case of mathematics, this part was composed of a practical test, which involved solving a problem of algebra or geometry and two oral examinations on analysis and geometry. Candidates could also present other elements (books, texts...) and some in fact wrote special ‘mémoires’ for the exam on such topics as set theory, conics, complex numbers, and complex functions.

The archives keep some of the written tests including students' answers. Examinations for the special part in mathematics that were found included problems of geometry, analytic geometry, analysis, algebra, and numerical computations with a difficulty level near to the one they had experienced at the Bacharelato. We present some of them below.

Discuss the curve represented by the equation $y = x^3 - 2x^2 + 3x + 1$ and show the form of the curve. (Entrance examination, Special part, 7/12/1917)

Given the equation of the ellipse $y^2 - 3xy + 5x^2 + 2y - 3x - 5 = 0$, determine the diameter of the cords parallel to the line $y = 2x - 1$ and the curve's axes. (Entrance examination, Special part, 26/1/1918)²

Calculate with the support of logarithmic tables the expression

$$\frac{4.5832}{\sqrt[5]{28 \sin 24^\circ 15' 8''}} : \frac{\sqrt[3]{\pi}}{e}$$

(Entrance examination, Special part, 20/3/1920)

A folder of documents was found (Box 1) that allows us to understand the procedures for the entrance examinations of the special part of mathematics for the school year 1925-26. Although the school year should have started in October, these exams only took place between 23/11/1925 and 12/12/1925 and the jury convened several times to prepare lists of problems and questions that were later distributed among the candidates. Classes for the first year must have taken place between January and June of 1926.

One handwritten document lists the problems of analysis and algebra that were given in the afternoon of 2/12/1925. These included properties of polynomials and proofs involving series and integrals which the candidates had two hours to complete. Below are two examples of these questions:

If P and Q are two relatively prime polynomials, having degrees m and n respectively, there are two other polynomials P_1 and Q_1 , of degrees $m_1 < m$ and $n_1 < n$, such that

$$PQ_1 + P_1Q = 1.$$

The system P_1, Q_1 is unique.

(Entrance examination, analysis part, 2/12/1925)

Any fraction smaller than the unity can be univocally developed in a series of

the form $\frac{1}{2^{\alpha_1}} + \frac{1}{2^{\alpha_2}} + \dots + \frac{1}{2^{\alpha_n}} + \dots$ with different integer numbers α_n .

Prove this proposition and deduce a rule to approximately evaluate the side of the regular pentagon, using only the subdivision of segments in 2 equal parts.

(Entrance examination, analysis part, 2/12/1925)

A typewritten document listing the distinct geometrical problems proposed to the candidates in the afternoon of 3/12/1925 was also found containing problems of

2 Diameter of the cords may refer to the family of cords parallel to the given line.

descriptive, projective, and analytic geometry. Candidates had two hours to solve one of these. Below are two examples.

[Trace perpendiculars from one point to two straight lines.] Find the nature of the locus of the points such that the distance between the intersecting points of the perpendiculars to the straight lines is constant.

(Entrance examination, geometry part, 3/12/1925)

Show that, if an angle of fixed size is moving on a plane, such that the vertex describes a fixed straight line and a side passes a fixed point, the other side has a parabola as its envelope.

(Entrance examination, geometry part, 3/12/1925)

We also found another typewritten document listing the questions posed at the oral examinations of 1925, which took place after the written parts. It includes the following topics: analytical study of conics with centre; analytical study of the circle and its relationship with conics; homogeneous coordinates in the plane and “in the star”³, transformation formulas of these coordinates; polarity in the fundamental forms of the second kind, applications to the study of conics; classification and diametric properties of quadrics; tangent planes to the revolution surfaces by the methods of descriptive geometry. In the middle of December the process was concluded. Figure 1 shows the results of the oral and written tests of the four candidates and the final results.

All the candidates passed the test but their final grades (in the scale 0-20, in which a grade below 10 means a failure) were not high. We are aware of 47 prospective secondary mathematics teachers that passed these exams but we do not have information of how many candidates failed the tests.

	<i>Written</i>	<i>oral</i>	<i>Result final</i>
<i>Luís de Castro Marques</i>	<i>11</i>	<i>15</i>	<i>13 (these)</i>
<i>M.^o Augusto Rabaca</i>	<i>8</i>	<i>15</i>	<i>12 (done)</i>
<i>Manoel Baptista Costa</i>	<i>10</i>	<i>13</i>	<i>12 (done)</i>
<i>Manoel de Santa Helena</i>	<i>10</i>	<i>9</i>	<i>10 (done)</i>

Fig. 1. Summary of admission examinations 1915.

Apparently, even if these tasks included standard problems studied at the universities, which we did not verify, the examiners intended to check mathematical knowledge of the candidates beyond an elementary level.

3 May refer to coordinates used in cosmography.

The disciplines of the first year

According to the founding decree of the ENS (Decree with force of law, *Diário do Governo*, 129, 1911, 2081-3), after entrance, students attended a two-year course that included an initiation to pedagogical practice in the secondary schools (Liceus). Given the shortage of teachers in some areas, it is likely that some of the candidates already had teaching experience.

The curriculum of the first year was composed of the following disciplines: Pedagogy (with exercises in experimental pedagogy) (annually), History of Pedagogy (annually), Child Psychology (semesterly), Theory of Science (semesterly), General Methodology of Mathematical Sciences and Natural Sciences (annually), Organization and Comparative Legislation of Secondary Education (quarterly), General Hygiene and especially School Hygiene (semesterly), Moral and Higher Civic Instruction (semesterly).⁴

This curricular plan valued the general and specific pedagogical training on the basis of the assumption that education and psychology are experimental sciences. Hygiene and moral education, the topics dear to the republican spirit of the time, were also included; the latter was considered an important element of instruction of republicans and secular citizens (Pintassilgo, Mogarro, & Henriques, 2010). The General Methodologies were new because they focused on the specific professional knowledge of each school discipline.

Legislation included recommended teaching methods for the ENS. The instruction was supposed to include traditional magisterial lessons, conferences followed by discussions or sets of practical assignments, especially written exercises to be done in class, exercises on experimental pedagogy, and studies of children's pedagogy.

The archives provide us with information regarding the practices of these disciplines in Coimbra. As we are looking for the professional knowledge developed in these schools, we will focus firstly on Pedagogy and History of Pedagogy considering especially the ways in which these were shaping pedagogy as a new area of knowledge and secondly on the content of Methodology of the Mathematical Sciences.

Pedagogy

The disciplines of Pedagogy and Psychology in the ENS of Coimbra were taught essentially by Augusto Joaquim Alves dos Santos (1866-1924) until his death and later by José Joaquim de Oliveira Guimarães (1877-1960). The former was a pioneer

4 The curricular plan underwent minor changes three years after schools started, abolishing the Theory of Science, dividing the discipline of Legislation in two semesters and creating a discipline of General Methodology of the Sciences of the Spirit (Decree n° 4,649, *Diário do Governo*, 157, 1918, pp. 1311-4).

of Experimental Psychology in Portugal. A colleague of Édouard Claparède at the Jean-Jacques Rousseau Institute in Geneva, he created the first laboratory in this subject area in the country and was well acquainted with the leading European psychologists of his day.

The discipline of Pedagogy contained what was believed to be the core of knowledge related to education: its foundations, curricular organization, and teaching methods (Pintassilgo, Mogarro, & Henriques, 2010). In 1917 and 1930, topics of students' written essays and conferences encompassed themes such as measurement of memory and auditory attention, ludic activities, psychological conditions of attention, pedagogical optics. Most notably, some of these essays report exercises of experimental pedagogy. For example, the essay *Measurement of auditory attention* written in 1917 by Francisco Ferreira Neves (Box 1) describes how he measured reaction times of blindfolded "subjects" hearing a series of strokes. He explains how the experiment was designed and reaction times to six series of strokes varying in rhythm (constant/varying) and warning (given or not given prior to the beginning of the experiment) were measured. Distractions were added. Several tables and graphics were included and a descriptive statistical analysis supported the conclusions. After discussing the limitations of the study, he concluded that warnings seemed to shorten reaction times.

History of Pedagogy

Luciano Pereira da Silva (1864-1926), a mathematician, and Joaquim de Carvalho (1892-1958), a philosopher, taught History of Pedagogy in the ENS of Coimbra. The essays kept at the archives address topics as History of instruction in Portugal, Religious congregations in the 17th century, The New School, Russian Pedagogy after 1918. The ideas of pedagogues were also discussed, namely: Pestalozzi, Rousseau, Rabelais, Montaigne, João de Barros. In a given school year, the same topic was assigned to all students in the class.

There are differences between these two disciplines. On the one hand, an enthusiastic teacher, who sought to introduce students to the ways in which a scientific (experimental) methodology could be used in an innovative field, gave the discipline of Pedagogy a very advanced status. On the other hand, History of Pedagogy was a topic not even close to the scientific interests of the teachers in charge, and appears to be much more conservative approach.

Methodology of Mathematical Sciences

The discipline Methodology of Mathematical Sciences is of special interest. It was taught in the ENS of Lisbon by Eduardo Ismael dos Santos Andrêa (1879-1937), a teacher in secondary schools who taught calculus in the University of Lisbon

and authored successful textbooks. In the ENS of Coimbra Methodology of Mathematical Sciences was taught by Luciano Pereira da Silva (1864-1926), also a teacher of calculus at the University of Coimbra, author of books about history of mathematics, astronomy and navigation, and Director of ENS, and by João Pereira Silva Dias (1894-1960), a teacher of Calculus, Geometry and Physics at the University of Coimbra who was appointed in the 1940s to official posts related to education and culture.

We investigated the essays produced in Coimbra in detail. Prospective science secondary teachers (Physics and Chemistry) and future teachers of mathematics in normal schools also attended these classes, but the content was essentially focused on mathematical topics. Most of them discussed methods of the mathematical sciences such as inductive, deductive, analytic, graphical, and other methods and a special attention was given to the laboratory method. For example, in 1917, by the end of May, a written exercise on “The laboratory method” was proposed to the class, probably as a take-home exam. The three handwritten essays in the archives (ranging from 4 to 16 pages) focused on the contemporaneous trends in school education in England and the United States, on how a logical approach to mathematics was being changed to a method grounded in Psychology and taking into account children’s interest (today we would say ‘motivation’). All supported the importance of planning mathematical instruction as a move from the concrete to the abstract and stressed the importance of linking together distinct mathematical topics and of integration of mathematics with other subjects. In 1921 another series of essays and conferences was devoted to “The value of graphics in the laboratory method”.

Mathematical methods *per se* were also discussed. For example, in 1923, essays written in class focused on the distinction between the inductive and the deductive method. In the same year, the laboratory approach was also a topic for the take-home essays. In 1924, several methods were discussed.

The proposals of the New School movement underlay these essays. In Portugal, this perspective was widely spread from the end of the nineteenth century as Education (or Pedagogy) and Psychology started seeking acceptance as scientific fields. The republican movement used the New School banner (“intuitive teaching”) as a perspective that would improve schools (Mogarro, 2012).

The second year, pedagogical practice

Under the legislation, the second year was occupied by an initiation to the pedagogical practice developed at the Liceus accompanied by a secondary school teacher. Until December candidates had to assist to the classes of their advisers and teach occasionally. The advisers had to teach them the special methodology of the discipline. During the rest of the year candidates would have full teaching responsibilities

supervised by their advisers. They also had to participate in other school activities. University professors also had to participate in these educational activities.

In Coimbra, pedagogical practice was conducted at Liceu José Falcão. Alberto Álvaro Dias Pereira (1891-1984) was the adviser for most of the time. He was a former teacher of mathematics in several schools, a congressman, and also the Rector of the Liceu. He was dismissed as a teacher for political reasons in 1936. Occasionally other teachers took this role: António Tomé e Aníbal do Amaral Cabral, José Custódio de Moraes.

In Lisbon the pedagogical practice was conducted in three Liceus and the advisers were: Adolfo Bernardino de Sena Marques e Cunha (1872-1927) at Liceu Pedro Nunes (he had a background in medicine and was a prominent member of a religious group); Domitila Hormizinda de Carvalho (1871-1966) at the female Liceu Maria Pia who graduated in mathematics and philosophy, and had a doctorate in medicine (she was the first woman to officially graduate in the University of Coimbra and one of the first women deputy at the national assembly); and José Ferreira de Carvalho e Santos at Liceu Passos Manuel.

We do not have much information about the actual procedures of the pedagogical practice (class attendance, interactions between advisor and students, etc.), nor do we have any evidence that university teachers actually participated in the activities of the second year. We only know that prospective teachers took the classes of the advisors and their attendance was monthly reported to the ENS, and, at least from 1927, students were required to write their teaching plans for specific topics.

The dissertations and the final exam

After finishing the pedagogical practice of the second year of the course, State Examination [Exame de Estado] took place and the jury appreciated the overall merit of the candidate's work. The exam included: two discussions of half an hour each, a lesson given to a class in the Liceu followed by its pedagogical discussion, and the presentation of a dissertation on a topic of didactics of secondary teaching at the candidate's choice (Decree 2646, 26/9/1916). The dissertation was one of the significant elements together with the performance during the teacher practice. From 1927 the dissertation was replaced by a report on the pedagogical practice (Decree 13296, 17/3/1927) of which we only have an example (Santos, 1929).

Joaquim Ferreira Gomes found the titles of numerous dissertations, essentially produced by ENS students from Coimbra (1989). This listing was enriched with other titles collected through the consultation of legislative acts and of the existing documentation in Biblioteca Nacional.

Themes of the dissertations

We are currently familiar with the titles of 42 dissertations related to the teaching of mathematics at ENS in Coimbra — which correspond to almost all of those presented in State Examinations — and two from ENS in Lisbon (table 1).

Table 1. Themes of the titles of the dissertations.

Theme	Number
School mathematics topics	
Arithmetic	5
Algebra	2
Geometry	5
Analysis	8
Analytic Geometry	4
Cosmography	3
Other	6
General educational topics	10
Teaching methods ⁵	1 (+4)
Total	44

As might be expected, two-thirds of the papers have titles that indicate the development of topics in school mathematics (*The Teaching of Derivatives in Liceus* or *On Extending the Idea of Number*, for example) and were aggregated according to mathematical themes of the time. Some remaining titles were placed under the theme “Other topics”. These are subjects such as trigonometry, logarithms, complex numbers, indeterminate analysis or numerical approximations. Most of the rest are titles of a general scope (*Mathematics in secondary education*, etc.). Table 1 also highlights titles that denote a concern with teaching methods (e.g., *The heuristic method in the study of fractions* or *The slide rule*).

We found 15 of these dissertations. Physically, they are texts between 20 and 110 pages of various sizes, some printed in book printing shops, others typewritten. Students assumed their choice of the theme, sometimes manifesting a strong conviction about its importance. In none of the texts do we find a reference to either a professor of the ENS or their adviser at the Liceu. Apparently the students chose the topic with a reasonable degree of freedom. These dissertations allow us to go beyond the titles and actually study their content.

⁵ We include here four dissertations on mathematics topics that were counted above because they also explicitly propose a teaching method.

Development of mathematical topics

About one third of the 15 dissertations that we found were focused on mathematical topics, usually at the level of an undergraduate university course. Usually a justification of the author's choice of topic was provided. For example: authors argue that analytic geometry can better be taught if it integrates functional, algebraic, and geometrical dimensions; graphics allow the use of geometrical intuition; transformation geometry provides a better system of axioms.

Some authors are aware of the international centrality of the study of analysis as they develop mathematical texts on functions or on differential and integral calculus. They usually include references and quotations from contemporary texts by Henri Poincaré, Jules Tannery, Emanuel Beke, Félix Klein, Bertrand Russell, and Federico Henriques. In fact, this was a subject being discussed in the Commission Internationale de l'Enseignement Mathématique (CIEM) (Zuccheri & Zudini, 2014) and partially integrated in the Portuguese high school programs since 1905 (Aires & Santiago, 2014). These dissertations draw on the work of Félix Klein, a known advocate for introducing the topic in the secondary curriculum, and Jules Tannery, one of the proponents of the introduction of analysis in French curricula. Henri Poincaré is also referred to, to justify the need for an approach that, while using a logical treatment, does not let students forget intuition.

Rego (1918) in particular develops the theme supported by numerous quotations from the report presented by Emanuel Beke to the 'Conference internationale de l'enseignement mathématique⁶' organized by the CIEM in Paris four years earlier, which discussed precisely the progress of the reforms of the teaching of the analysis underway in several countries. His work is important to us, as it is the only dissertation for the ENS of Lisboa found so far. We know that his teacher of Methodology of Mathematical Sciences, Eduardo Andréa, was the author of one of the rare articles on mathematical education (1905), precisely supporting the teaching of analysis in secondary schools. Apparently, the topics discussed in Lisbon were not far from the ones discussed in Coimbra.

The geometry of transformations is another innovative topic that can be found in three dissertations. Henriques Júnior (1921) and Tavares (1927) explore alternative axiomatic systems for geometry based on geometric transformations and, as in the previous case, this theme has been debated internationally (Barbin & Menghini, 2014). Much influenced by the proposals of Bertrand Russell and Federico Henriques, Tavares (1927) in his dissertation, written as a college level mathematics textbook, is aware of its philosophical implications and identifies his study as metageometry. Henriques Júnior (1921), in a text closer to secondary school mathematics, but still

⁶ According to Zuccheri and Zudini (2014), this report was published in *L'Enseignement Mathématique*, 16, 285-89.

containing the definitions, axioms and theorems relevant to support a perspective of geometry based on transformations, presents the ‘gonio’, a mathematical instrument (also known as the compass of parallels) which he developed at the Physics Laboratory of the University of Coimbra during his training at ENS, and his dissertation is part of a book dedicated to the subject. Also the work of Silva (1920) briefly discusses, among other subjects, the controversy about the foundations of geometry and explores some elementary transformations.

Mathematics and its methods

One third of the dissertations found do not focus upon a mathematical topic. Instead we could say they discuss philosophically the science of mathematics and its methods. The centrality of mathematics among the sciences is highlighted, but the dominant theme of several dissertations is the presentation of arguments to counter the idea of the primacy of logical deduction in the production of mathematical knowledge, an idea they associate with current teaching methods at Liceus. As Gersão (1917) says, to recommend the mathematicians to follow the deductive method would be to suggest that they search for a needle in a haystack. An inductive path is much more productive. Neves (1919) deepens this idea:

Mathematical sciences are commonly characterized by their exclusively deductive method. Thus, as deduction is a reasoning that goes from the general to the particular, it would result that mathematics would teach little (...). But on the other hand, we constantly hear that the tendency of mathematics is generalization. In effect it is: for example, from the fundamental notion of integer, we pass to that of a fractional number; then that of an irrational number. (...) This is how primitive simple science, *arithmetic*, undergoes a generalization: *algebra*; which then undertakes a new extension: *analysis*; and who knows what will be the last word in mathematics? (Neves, 1919, pp. 31-2, italics in the original).

Neves continues and, based on Poincaré, argues that along with deduction, mathematical induction and reasoning by recurrence are equally fundamental mathematical methods. As for the nature of the axioms, he distinguishes the positions of Kant (axioms are evident truths or a priori), Stuart Mill (experimental acts), and Poincaré (conventions).

Silva (1920) and Santos (1929) produce similar arguments. Santos adds that the methods of mathematical investigation are: the analytical, the synthetic, the reduction to the absurd, the deductive, the inductive, that of the indeterminate coefficients, and that of the infinitely small, thus not limiting mathematics to a by-product of logic. Many dissertations state similar classification of mathematical methods, argue that mathematics is not exclusively deductive, and stress intuition, observation, and experimentation. Immanuel Kant, John Stuart Mill, and Henri Poincaré are

often referred. This concurs with the contents of students' essays for the discipline titled Methodology of Mathematical Sciences.

Teaching methodologies

One third of the found dissertations explicitly address teaching methodologies. Almost all argue that no desirable learning comes from an accumulation of notions, and on the contrary, they are generally concerned with students' tendency to memorize concepts they forget on the first occasion. Such is the case of Silva (1920) who believes that basing teaching on definitions would be counterproductive. Alternatively, a justification part should precede and prepare the general notions that would only be presented after the consideration of several particular cases. Similar opinion has Beirão (1921) for whom

the abstract notions of mathematics must rest on sensible notions which aid their understanding; and thus, by geometric processes we will interpret the different notions ... and in this concrete way we can be able to arouse the interest of students, who feel a natural repugnance for abstract notions, for them always vague and indefinite. (Beirão, 1921, p. 8)

Five dissertations, with a remarkable uniformity of views, distinguish three teaching methods. The first, the didactical method, is described as "the teacher exposes the questions, directs them to show clearly what he has in view. Students listen, follow the teacher's reasoning and arrive at the same time as him to the conclusions" (Gersão, 1917, p. 6). In the heuristic method, the teacher "expose truths inductively and lead students through a series of well-directed questions to discover the truth" (Gersão, 1917, p. 7).

The student walks by himself, feeling the ground, standing here and there, but supported by the teacher, who guides him, overcoming the main obstacles, when he alone is unable to do so. By means of insinuations, suggestions, remembrances of principles forgotten by the teacher, the pupil will follow the path of truth in front of the teacher, and, once the purpose of the work was achieved, the student is left with the salutary and stimulating impression that he discovered it; (...) the key is the method or the know-how that the student grasped. (Neves, 1919, p. 40)

Some students believe this method came from Immanuel Kant, Jean-Jacques Rousseau, and Johann Heinrich Pestalozzi. Finally, comes the laboratory method, which:

seeks to establish an intimate approximation between Mathematics and other sciences, using natural phenomena, which gave rise to the emergence of certain theories, now purely abstract, to achieve them. (Abreu, 1922, pp. 45-6)

With regard to mathematics teaching, the graphic process is presented as an example of using the laboratory method. The millimetre paper, “which students should always have at hand” (Abreu, 1922, p. 47) allows pupils to know “by sight, relations of greatness existing between certain objects, relations that would have no representation in their spirit, if were given by means of numbers” (Abreu, 1922, p. 47). The graphs are pointed out as an essential element for this understanding, which allows abstract notions, such as function, to have a concrete meaning (Neves, 1919). One of the students (Abreu, 1922) explicitly associates the laboratory method with the term “lessons of things” [lições das coisas], referring to one of the key terms popular among republican educators sympathetic of the New School movement (Mogarro, 2012). Some of these dissertations ascertain the origin of the laboratory method in the works of Eliakim Hastings Moore, John Perry, and Jacob William Albert Young.

Neves (1919), Abreu (1922), and Santos (1929) agree that in the first three classes of Liceus the laboratory method should be preferred, in the fourth and fifth the heuristic method and, in the sixth and seventh, the didactic method, which should also be used in the University. Abreu justifies this sequence with what might be called a “theory of the mental development of the child”:

Teaching methods must be in accordance with the stages that the child goes through during her development, from childhood to puberty. During the time from infancy to puberty, they must give education a purely concrete feature; during puberty, the concrete must operate with the aid of the abstract, but it must predominate; from puberty teaching must be characterized by the mastery of the abstract. (Abreu, 1922, p. 43)

The social utility of mathematics

One of the main advocates of the laboratory approach is John Perry. He envisioned mathematics laboratories where problem-based approaches could incorporate the technological developments associated with graphical analysis and the use of slide rules. He proposed the regular use of graph paper, in which the integrating element would be the concept of function. This movement has expanded in several countries and has naturally been reflected in the work of ICME (Furinghetti, Matos and Menghini, 2013).

At the basis of this position was the idea that the teaching of mathematics should primarily concern itself with utility. Mathematics should therefore be taught experimentally in laboratories and not through abstraction (Howson, 1984). We find a reflection on this theme in three dissertations (Guardiola, 1921, Neves, 1919, Rego, 1918). Neves explains his position upfront:

In writing this work, one idea constantly guided me: this idea is that man must be educated in order to be **socially efficient**. (...) Man is an element of society, to whom is distributed a function that he will perform without friction and with a maximum income to be distributed by himself and by society. (Neves, 1919, p. v, bold in the original)

It is therefore incumbent upon the educator to ensure that the share of education made at the expense of those sciences is maximal, so that together with the results obtained by other means, the individual is enabled to produce a maximum revenue in his work, compatible, of course, with the general conditions of existence. (Neves, 1919, pp. vii)

A similar position is adopted by Guardioli (1921). Rego (1918) completes this idea, arguing that the secondary school should provide elements of a general culture, without losing sight of the usefulness of the knowledge imparted.

Conclusion

ENS schools were intended to replace the system under which access to the position of secondary teachers depended solely on approval on a state exam. These new schools included demanding entrance requirements both in general knowledge and mathematical proficiency. We also found that the disciplines of the first year paved the way for the development of a pedagogical knowledge, shown especially in the dissertations, based on:

- 1) the ideas of the New School movement that looked for the educational enterprise from children's perspective, valued the need for a concrete basis for the abstract knowledge (and thus the rejection of logic as a starting point for mathematics teaching), supported the centrality of "intuitive teaching", the importance of relating teaching to the real world ("lições das coisas"), and channelled these ideas into a laboratory approach to mathematics;
- 2) the innovative international mathematical tendencies supported by ICME, especially related to the importance of the inclusion of analysis in the mathematics curricula and on the axioms for geometry;
- 3) and for some of the prospective teachers, the appreciation of taking into account the social utility of school mathematics

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Archive of the University of Coimbra

Box 1: Depósito IV, Secção 2ª.E, Estante 9, Tabela 5, nº 19. Escola Normal Superior de Coimbra. Archives of the University of Coimbra.

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Let them speak; hear them speak — old Chinese wisdom on mathematics education

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Abstract

Since the mid-1990s there has been an upsurge of interest in the process of learning and teaching in a classroom environment dominated by the so-called Confucian heritage culture (CHC), which is brought into focus in the form of two paradoxes, namely, the CHC Learner Paradox and the CHC Teacher Paradox. This paper takes a look at how in the past Chinese scholars and mathematicians viewed the subject of mathematics and what kind of underlying philosophy in learning and teaching of the subject they believed in, by studying some selected passages from mathematical texts from very early times to the nineteenth century.

Keywords: Confucian-heritage-culture classroom; Chinese mathematical classics

Introduction

Since the mid-1990s there has been an upsurge of interest in the process of learning and teaching in an environment dominated by the so-called Confucian heritage culture (CHC), which is brought into focus in the form of two paradoxes (Leung, 2001; Watkins & Biggs, 1996, 2001; Wong, 1998), namely, (i) The CHC Learner Paradox: CHC students are perceived as using low-level, rote-based strategies in a classroom environment which should not be conducive to high achievement, yet CHC students report a preference for high-level, meaning-based learning strategies and they achieve significantly better in international assessments! (ii) The CHC Teacher Paradox: Teachers in CHC classrooms produce a positive learning outcome under substandard conditions that Western educators would regard as most unpromising! A more detailed discussion on these two points can be found in Chapter 1 of (Watkins & Biggs, 2001).

It would therefore be of interest, both for mathematics educators and historians of mathematics, to take a look at how in the past Chinese scholars and mathematicians viewed the subject of mathematics and what kind of underlying philosophy in learning and teaching of the subject they believed in. To this end selected mathematical texts, particularly prefaces or accompanying commentaries or essays, from very early times since the second century B.C.E. to the late-nineteenth century before China more or less fully adopted the Western way and content of mathematics in education, will be quoted. If a Chinese text is quoted, the source of the English translation will be noted, but to keep to the length of the paper the original Chinese

text will not be shown. If no source of translation is noted, the translation is that of the author's, may not be the best but hopefully adequate to convey the meaning. Many of the original Chinese mathematical books are collected in (Guo, 1993).

An Early Discussion on Learning Mathematics

Let us begin by looking at a passage taken from *Zhou Bi Suan Jing* [周髀算經 The Arithmetical Classic of the Gnomon and the Circular Paths], which is believed to be compiled between the fifth and second centuries B.C.E., and is perhaps the earliest extant Chinese treatise on astronomy and mathematics. In the book there is an interesting dialogue between Rong-fang [榮方] and Chen-zi [陳子]:

Rong-fang: [...] Can a fellow as stupid like me learn the way of mathematics?

Chen-zi: Of course you can. What you have learnt in elementary arithmetic is sufficient to let you go on to learn it. But you must be willing to think continually in earnest.

[A few days later, *Rong-fang* came back to *Chen-zi*.]

Rong-fang: I cannot figure it out. May I enquire again?

Chen-zi: This is because you have thought about it but not yet to the point of maturity [...]. You cannot yet generalize what you have learnt. [...] The mathematics is simple to explain but has wide applications. After understanding one category of problems one can infer the reasoning for a variety of other categories. [...] What makes it difficult to be well versed in the way of mathematics is that when one has learnt it one worries about a lack of breadth; when one has attained breadth, one worries about a lack of practice; when one has attained practice, one worries about a lack of ability to understand. To be able to compare and contrast different categories of problems, that is the mark of an intelligent person.

Another prominent Chinese mathematical classics is *Jiu Zhang Suan Shu* [九章算術 Nine Chapters on the Mathematical Art], which is believed to be compiled between 100 B.C.E. and 100 C.E. with part of the content already quite well-known much earlier, as evidenced by another book (written on bamboo strips) *Suan Shu Shu* [算數書 Book of Numbers and Computation] dated to 200 B.C.E, excavated in Zhangjiashan in the Hubei Province in 1983 (Peng, 2001). In the mid-third century the noted Chinese mathematician LIU Hui [劉徽 ca. 225-295] wrote a detailed commentary on *Jiu Zhang Suan Shu*. In his preface he said, "I studied *Jiu Zhang* [*Suan Shu*] at an early age and perused it when I got older. I see the separation of the *Yin* and the *Yang* and arrive at the root of the mathematical art. In this process of probing I comprehend its meaning. Despite ignorance and incompetence on my part I dare expose what I understand in these commentaries. Things are related to each other

through logical reasoning so that like branches of a tree, diversified as they are, they nevertheless come out of a single trunk. If we elucidated by prose and illustrated by pictures, then we may be able to attain conciseness as well as comprehensiveness, clarity as well as rigour. Looking at a part we will understand the rest.”

What LIU Hui meant by “elucidated by prose and illustrated by pictures” points out a special feature in traditional Chinese mathematics, namely, that geometry and algebra/arithmetic, or that shapes and numbers, are intimately integrated. How these two aspects come together will be best seen through examples taken from the commentaries of LIU Hui, so let us take a more detailed look at Chapter 9 of *Jiu Zhang Suan Shu*.

The design of this final chapter (titled *Gou Gu* [勾股 Right Triangles]) well exemplifies what is known as the variation theory in teaching/learning (à la Ference Marton and his team) in the Western community of educators, and as the *bian shi* method [變式] (à la GU Ling-yuan [顧冷沅] and his team) in the Eastern community of educators. For instance, the first three problems in the chapter, which simply ask for one side of a right triangle given the other two sides, are set at “level zero” of variation! By moving to some word problems in different contexts learners are urged to move to a higher level in Problem 4 (about cutting a rectangular plank out of a circular log) and Problem 5 (about a winding vine around a tree). Learners are further urged to move to yet higher and higher levels of variation in subsequent problems that place a higher demand on making use of knowledge about a right triangle, at the same time unfolding a systematic theory according to a well-designed framework. (A more detailed discussion on this framework is offered in (Li, 1990).)

Let us see how shapes and numbers go hand in hand in traditional Chinese mathematics by studying the solutions of these problems. For fun (and as a token of thanks to the host of this 5th ICHME) allow me to make use of a recruitment advertisement that appeared in a Dutch magazine (April 27, 1999) with the title “Ben jij ook altijd zo benieuwd naar de ontknoping?” The advertisement shows six amusing mathematical problems, one of which is actually a problem with a rich heritage. It is Problem 13 in Chapter 9 of *Jiu Zhang Suan Shu*, which also appeared (with different data) in *Lilavati* written by the Indian mathematician Bhāskara II (1114-1185) in the twelfth century, and in a European text written by the Italian mathematician Filippo Calandri in the fifteenth century. The problem asks: “Given a bamboo 1 *zhang* high, which is broken with its tip touching the ground 3 *chi* away from the base. What is the height of the break?” In modern day mathematical language it means that we want to calculate the side b of a right triangle given a and $c + b$, where c is the hypotenuse.

The answer given in *Jiu Zhang Suan Shu*, expressed in modern day mathematical language, is given by the formula

$$b = \frac{1}{2} \left[(c + b) - \frac{a^2}{(c + b)} \right]$$

A school pupil of today would be able to obtain this formula by invoking Pythagoras' Theorem and solving a certain equation. However, how was it done more than two thousand years ago, when the Chinese did not yet have the facility afforded by symbolic manipulation at their disposal? Both LIU Hui, and later another noted Chinese mathematician YANG Hui [楊輝 1238-1298], explained how they obtained the answer in their commentaries. Their method is ingenious. Not to spoil the fun for readers who like to seek their own methods I will leave this ingenious explanation to a *Geogebra* applet which may be accessed at the link <https://ggbm.at/2772025>.

Let us now move to Problem 15 in Chapter 9 of *Jiu Zhang Suan Shu*. "Given a right triangle whose *gon* is 5 *bu* and whose *gu* is 12 *bu*. What is the side of an inscribed square? The answer is 3 and 9/17 *bu*." (See Fig. 1.)

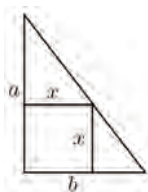


Fig. 1. Inscribed square in a right triangle.

The text follows with a method: "Let the sum of the *gon* and the *gu* be the divisor; let the product of the *gon* and the *gu* be the dividend. Divide to obtain the side of the square." In modern day mathematical language it means the formula $x = ab / (a + b)$.

Again, a school pupil of today can handle this problem easily using similar triangles. However, unlike the ancient Greeks the Chinese did not develop a geometric theory involving the notion of parallel lines and similar figures. How did LIU Hui solve the problem at the time? His ingenious method using dissection-and-reassembling provides a nice "proof without words"! (See Fig. 2.)

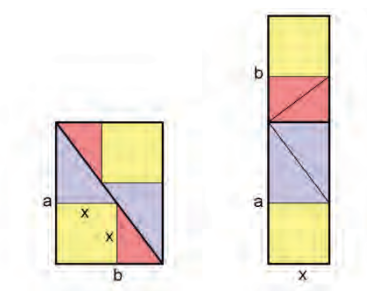


Fig. 2. Calculation of side of the inscribed square.

An animated solution is provided in a *GeoGebra* applet that can be accessed at the link <https://ggbm.at/2812253>.

Readers may like to try their hands in applying a similar method to solve the next problem (Problem 16) in Chapter 9 of *Jiu Zhang Suan Shu*. “Given a right triangle whose *gou* is 8 *bu* and whose *gu* is 15 *bu*. What is the diameter of its inscribed circle? The answer is 6 *bu*.” For a more detailed discussion see (Siu, 1993). An animated solution is provided in a *GeoGebra* applet that can be accessed at the link <https://ggbm.at/697695>.

A Chinese Essay on Mathematics Education in the Song Dynasty

In the latter part of the thirteenth century the mathematician YANG Hui of the Southern Song Dynasty wrote several books that were later compiled as a treatise known as *Yang Hui Suan Fa* [楊輝算法 Yang Hui’s Methods of Computation], which includes the three books with titles *Cheng Chu Tong Bian Suan Bao* [乘除通變算寶 Precious Reckoner for Variations of Multiplication and Division] of 1274, *Tian Mu Bi Lei Cheng Chu Jie Fa* [田畝比類乘除捷法 Practical Rules of Arithmetic for Surveying] of 1275, and *Xu Gu Zhai Qi Suan Fa* [續古摘奇算法 Continuation of Ancient Mathematical Methods for Elucidating the Strange Properties of Numbers] of 1275.

The preface to the first chapter of *Cheng Chu Tong Bian Suan Bao* is perhaps the first paper on mathematics education in China. With its title *Xi Suan Gang Mu* [習算綱目 A General Outline of Mathematical Studies] this preface offers a re-organized syllabus of the traditional curriculum in a comprehensive programme of 260 days, which is comparable to a modern school programme of about 1500 hours (Zhou, 1990).

Let us take a look at some passages in *Xi Suan Gang Mu* to see how the author sequenced the learning of the basics. Some other passages in the other chapters of

Yang Hui Suan Fa would indicate how the author valued comprehension to mere rote learning and drilling¹.

First learn the multiplication tables. < Start with 1 times 1 equals 1 up to 9 times 9 equals 81, from the smaller numbers to the larger ones. Their applications are not shown here. > Learn the rules of multiplication and how to fix the place-values. < Lesson for one day. > [...] Learn the rules of division and how to fix the place-values. < Lesson for one day. > [...] In the chia (addition 加) method the number is increased, while in the chien (subtraction 減) method a certain number is taken away. Whenever there is addition there is subtraction. One who learns the chien method should test the result by applying the chia method to the answer of the problem. This will enable one to understand the method to its origin. Five days are sufficient for revision. [...] In learning the chiu kwei (tables of division 九歸) one will need at least five to seven days to become familiar with the recitation of the forty-four sentences. However if one examines carefully the explanatory notes of the art on chiu kwei in the Hsiang Chieh Suan Fa (A Detailed Analysis of the Methods of Computation 詳解算法 — a lost treatise by YANG Hui), one can then understand the inner meaning of the process and a single day will suffice for committing the tables and their applications to memory. Revise the subject on chiu kwei. < one day. > [...] In mathematics, root-extraction constitutes a major item. Under the headings kou ku, p'ang yao, yen tuan and so chi can be found numerous common examples of root-extraction. [...] Learn a method a day and work on the subject for two months. It is essential to enquire into the origins of the applications of the methods so that they will not be forgotten for a long time. [...] When the methods of multiplication, division, fractions and root-extraction in the 246 problems of the Chiu Chang have been revised, then start with chapters fang t'ien and su mi. These will require one day to know thoroughly. The next chapter ch'ui fên deals with proportional parts. The whole of shao kuang is devoted to the addition of fractions and all of shang kung is on substitution of equivalent volumes. The chapter chün shu employs the cross-multiplication of ordered numbers. Take three days to peruse each chapter. The applications of the methods in the three remaining chapters ying pu tsu, fang ch'êng and kou ku are more complicated, so take four days to study each chapter. Make a detailed study of Chiu Chang [Suan Fa] Tsuan Lei, so that the rules of applications are thoroughly known. Then only will the art of the Chiu Chang be fully understood.

It would be of interest to compare such a curriculum with what students in the state-run School of Mathematics in the Tang Dynasty [唐 681–907] went through. The period of the Tang Dynasty is chosen for its most established form of the curriculum and the state examination system in mathematics, which later dynasties either modelled after it or it was even no longer in place. Beginning with the Sui Dynasty [隋 581–618], further consolidated in subsequent dynasties, a comprehensive official

1 The translated text of the quoted passages are taken from (Lam, 1977).

system of education was established with a well-planned curriculum, including the syllabus and the adopted textbooks, for each of several chosen disciplines. State examinations for these chosen disciplines were held regularly and successful candidates were appointed to official posts according to merit in their performance at examinations. (Siu, 1995; Siu & Volkov, 1999)

At an Imperial Order the Tang mathematician LI Chunfeng [李淳風 602–670] collated *Suan Jing Shi Shu* [算經十書 Ten Mathematical Manuals], which was adopted as the official textbook in the School of Mathematics in 656. This compendium comprised ten classics that were compiled by different authors at different times, listed below roughly in chronological order: (1) *Zhou Bi Suan Jing* [周髀算經 The Arithmetical Classic of the Gnomon and the Circular Paths, ca. 100 B.C.E.], (2) *Jiu Zhang Suan Shu* [九章算術 Nine Chapters on the Mathematical Art, between 100 B.C.E. and 100 C.E.], (3) *Hai Dao Suan Jing* [海島算經 Sea Island Mathematical Manual, third century], (4) *Wu Cao Suan Jing* [五曹算經 Mathematical Manual of the Five Government Departments, sixth century], (5) *Sun Zi Suan Jing* [孫子算經 Master Sun's Mathematical Manual, between fourth and fifth centuries], (6) *Xia Hou Yang Suan Jing* [夏侯陽算經 Xia Hou Yang's Mathematical Manual, fifth century], (7) *Zhang Qiu Jian Suan Jing* [張丘建算經 Zhang Qiu Jian's Mathematical Manual, fifth century], (8) *Wu Jing Suan Shu* [五經算術 Arithmetic in the Five Classics, sixth century], (9) *Qi Gu Suan Jing* [緝古算經 Continuation of Ancient Mathematics, seventh century], (10) *Zhui Shu* [綴術 Art of Mending, fifth century]. The original text of *Zhui Shu* was lost in about the tenth century, with its role in the compendium being subsequently replaced in the Song Dynasty by *Shu Shu Ji Yi* [數術記遺 Memoir on Some Traditions of the Mathematical Art, a book of doubtful sixth century authorship].

Students in the state-run School of Mathematics were divided into two programmes. In Programme A students studied *Sun Zi Suan Jing* and *Wu Cao Suan Jing* for 1 year, *Jiu Zhang Suan Shu* and *Hai Dao Suan Jing* for 3 years, *Zhang Qiu Jian Suan Jing* for 1 year, *Xia Hou Yang Suan Jing* for 1 year, *Zhou Bi Suan Jing* and *Wu Jing Suan Shu* for 1 year. In Programme B students studied *Zhui Shu* for 4 years, and *Qi Gu Suan Jing* for 3 years. In addition to these books, students in each of the two programmes must also study two more manuals, *Shu Shu Ji Yi* and *San Deng Shu* [三等數 Three Hierarchies of Numbers, written not later than the mid-sixth century but was lost by the Song Dynasty].

It is recorded in the chronicles Tang Liu Dian [唐六典 The Six Codes of the Tang Dynasty, 739] and Xin Tang Shu [新唐書 The New History of the Tang Dynasty, 1060] that regular examinations were held throughout the seven years of study, and at the end of each year an annual examination was held. Any student who failed thrice or who had spent nine years at the School of Mathematics would be discontinued. Judging from the age of admission at fourteen to nineteen years

of age, we know that a mathematics student would sit for the state examination at around twenty-two-year-old, meaning that they spent some seven to ten years studying mathematics (Siu & Volkov, 1999). We can now compare the comprehensive programme of 260 days suggested by YANG Hui with the seven-year programme in the official system!

In a later chapter of the book *Cheng Chu Tong Bian Suan Bao* the author said:

The working of a problem is selected from various methods, and the method should suit the problem. In order that a method is to be clearly understood, it should be illustrated by an example. If one meets a problem, its method must be carefully chosen.

In another book *Tian Mu Bi Lei Cheng Chu Jie Fa* the author said:

It is difficult to see the logic and method behind complicated problems. Simple problems are hereby given and elucidated. Once these are understood, problems, however difficult, will become clear.

In the third book *Xu Gu Zhai Qi Suan Fa* the author said:

The author has placed the small diagram on the island in the sea before him, and came to understand a little of the method employed by his predecessors. If the complete method is handed down, is not the secret purport being slighted? And if it is not handed down, then there is nothing to further the good work of his predecessors. [...] Therefore after this one single problem, the learned reader should be able to examine and solve by analogy other remaining problems. What is the need of passing it on so easily to the uninitiated?

Back to the CHC Learner Paradox

What kind of teaching environment would conduce to good learning? In the Western literature it is normally agreed that such an environment should include factors like (i) varied teaching methods, (ii) student-centered activities, (iii) content presented in a meaningful context, (iv) warm classroom climate, (v) high cognitive level outcomes expected and assessed, (vi) classroom-based assessment in a non-threatening atmosphere (Biggs & Moore, 1993). However, in a Confucian-heritage culture (CHC) classroom, Western observers see just the opposite! It would therefore be natural to conclude that (1) CHC classrooms should give rise to low quality outcomes: rote learning and low achievement, and (2) CHC students are perceived as using low-level, rote-based learning strategies. But paradoxically what instead happens is to the contrary! CHC students have significantly higher levels of achievement than those

of Western students (e.g. in IEA, PISA studies)², and CHC students report a preference for high-level, meaning-based learning strategies (Biggs, 1994). A more detailed discussion about this “Western misperceptions of the Confucian Heritage learning culture” can be found in Chapter 3 of (Watkins & Biggs, 1996).

Is it true that rote learning was the norm in education in ancient China? Perhaps a reading of some texts in ancient Chinese classics can shed some light on this question. (These texts are about moral philosophy, but the principle applies to mathematics education as well.) In the well-known Chinese classics *Lun Yu* [論語 Confucian Analects] it is written: “The Master said: Learning without thought is labour lost; thought without learning is perilous.” In another classics *Zhong Yong* [中庸 Doctrine of the Mean] it is written:

He who attains sincerity chooses the good and holds fast to it. This involves the extensive study of it, close inquiry into it, careful deliberation of it, clear distinction of it, and earnest practice of it³.

The noted neo-Confucian Zhu Xi [朱熹 1130-1200] of the Southern Song Dynasty compiled *The Four Books* (the original copies believed to be compiled between the 6th and 5th century B.C.E) and one passage reads:

In reading, if you have no doubts, encourage them. And if you do have doubts, get rid of them. Only when you’ve reached this point have you made progress.

Another passage reads:

Generally speaking, in reading, we must first become intimately familiar with the text so that its words seem to come from our own mouths. We should then continue to reflect on it so that its ideas seem to come from our own minds. Only then can there be real understanding. Still, once our intimate reading of it and careful reflection on it have led to a clear understanding of it, we must continue to question. Then there might be additional progress. If we cease questioning, in the end there’ll be no additional progress.

Still another passage reads:

Learning is reciting. If we recite it then think it over, think it over then recite it, naturally it’ll become meaningful to us. If we recite it but don’t think it over, we still won’t appreciate its meaning. If we think it over but don’t recite it, even though we might understand it, our understanding will be precarious. [...] Should we recite it to the point of intimate familiarity, and moreover

² In recent years there have emerged views which question the effectiveness and validity of such kind of international assessment studies, but that is not what we wish to enter into discussion here.

³ Translated texts are taken from (Legge 1893/1960).

think about it in detail, naturally our mind and principle will become one and never shall we forget what we've read⁴.

All these passages indicate clearly that repetitive learning is not to be equated with rote learning.

Let us further illustrate this point with what was supposed to be tested in the State Examination in Mathematics in the Tang Dynasty. In the state examination in mathematics for either Programme A or Programme B candidates were examined on two types of question. The first type was described in *Xin Tang Shu* as:

[The candidates should] write [a composition on] the general meaning, taking as the basic/original task a problem and answer. [They should] elucidate the numbers/ computations, [and] construct an algorithm. [They should] elucidate the structure/principle of the algorithm in detail.

For Programme B there was added the remark,

If there is no commentary, [the candidates should] make the numbers/computations correspond [to the right ones?] in constructing the algorithm.

The second type of question was known as *tie du* [帖讀 strip reading] in which candidates were shown a line taken from either *Shu Shu Ji Yi* or *San Deng Shu*, with three characters covered up and were asked to answer what those three missing characters were, that is, what nowadays we call “fill in the blank”. It is interesting to note that *Shu Shu Ji Yi* is a short text with only 934 characters, which could be committed to memory with reasonable ease (not to mention that a candidate had seven years to do it!). *San Deng Shu* was probably a book of similar nature (Siu & Volkov, 1999).

What is meant by that added remark? Since no trace of any examination question is extant, we can only attempt to “re-construct” an examination question, with an eye to supporting the thesis that the curriculum in mathematics in ancient China was not so elementary nor was mathematics learnt by rote. It is hard to believe that a group of selected young men spent seven of their golden years in simply memorizing the mathematical classics one by one without understanding them, just to regurgitate the answers in the state examination at the end! (Siu, 2004) Let us illustrate with one “fictitious” examination question that might appear in a state examination, that of calculating the volume of a truncated pyramid with a rectangular base. For convenience we will describe it using modern day mathematical language: Compute the volume of an “(oblong) *ting* [亭 Pavilion]” of height h with bottom and top being rectangles of sides a_1, a_2 and b_1, b_2 respectively ($a_1 \neq a_2, b_1 \neq b_2$) (Siu, 2004).

Problem 10 in Chapter 5 (titled *Shang Gong* [商功 Consultation of Engineering Works]) of *Jiu Zhang Suan Shu* is about the volume of a “*fang ting* [方亭 Pavilion with

4 Translated texts are taken from (Gardner, 1990).

square base]”, which is a truncated pyramid with square base. If a and b are respectively the side of the bottom and top square and h is the height, then the volume is given in the text as

$$V = \frac{1}{3}(a^2 + b^2 + ab)h.$$

In his commentary LIU Hui explained how to arrive at this formula by an ingenious method of assembling blocks of standard shapes, called by him “*qi*” [碁 chess piece], including three standard types, namely, (i) *li fang* [立方], which is a cube of side a , with volume a^3 , (ii) *yang ma* [陽馬], which is a pyramid of square base of side a and one vertical side of length a perpendicular to the base, with volume (iii) *qian du* [堐堵], which is a triangular prism with isosceles right triangle of side a as base and height a , with volume . (See Fig. 3.)

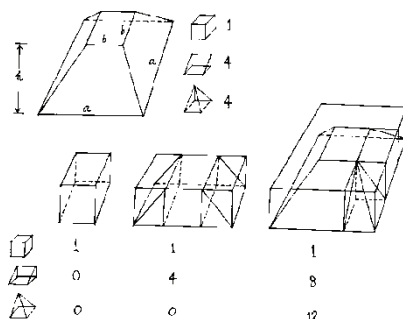


Fig. 3. Assembling a truncated pyramid.

He also explained how to arrive at an alternative formula by using another way of dissection. To keep to the length of this paper we omit all the detailed calculation. Interested readers may like to consult (Siu, 2004), or go to an animated solution provided in a *GeoGebra* applet at the link <https://ggbm.at/6829110>.

A student who understands the argument by LIU Hui can easily modify either method to arrive at the correct answer, which is

$$V = \frac{1}{3} \left[a_1 a_2 + b_1 b_2 + \frac{1}{2} (a_1 b_2 + a_2 b_1) \right] h.$$

But if a student merely memorizes the formula given in the textbook by heart, it is not easy to hit upon this correct formula. In view of practical need a candidate in the School of Mathematics might face in his subsequent career as an official engineer, such a demand imposed on the candidate in the state examination is quite reasonable.

Interested readers may go to an animated solution provided in a *GeoGebra* applet at the link <https://ggbm.at/6834973>.

Some Thoughts on Mathematics Education in the IT Age

My university (University of Hong Kong) adopts as its motto SAPIENTIA ET VIRTUS in Latin, rendered in Chinese as “明德格物”, which is derived from the text of the very ancient Chinese text *Da Xue* [大學 the Great Learning]. It is meaningful to quote part of the full text:

The point where to rest being known, the object of pursuit is then determined; and, that being determined, a calm unperturbedness may be attained to. To that calmness there will succeed a tranquil repose. In that repose there may be careful deliberation, and that deliberation will be followed by the attainment of the desired end.

This ancient passage motivated me to raise a set of questions in a lecture given in the 3rd International Congress of Chinese Mathematicians held in December of 2004 in Hong Kong, particularly because after reading a cover story of *Newsweek* (Aug.25-Sept.1, 2003) that featured “Bionic Kids: How technology is altering the next generation of humans”, in which one of the featured articles with the title “Log on and learn” mentions two points:

Children’s brains are growing adept at handling a variety of visual information. [...] Kids are getting better at paying attention to several things at once. But there is a cost, in that you don’t go into any one thing in much depth.

We need to recognize, like it or not, that the IT age breeds a generation with a different working habit and a different learning habit, even a different mentality, from that of their parents and teachers. The set of questions I like to raise is, in my opinion, what mathematics educators would do well to deal with in this IT age. How should IT be employed to enable students to learn better but not to limit their ability to think critically and in depth? How can we ensure that a discovery approach is not to be equated with a hit-and-miss tactic? How can we ensure that imaginative thinking is not to be equated with a cavalier attitude, that multi-tasking needs not be identified as sloppy and hasty work, and that the use of IT is not to be identified as following instructions step by step without thinking? (Siu, 2008)

A Chinese Essay on Learning Mathematics in the Late Nineteenth Century

I like to conclude with an account on the personal learning experience described by the nineteenth century Chinese mathematician Hua Heng-fang [華蘅芳 1833-1902],

a well-known mathematician of the latter part of the Qing Dynasty who also translated many Western books of science and mathematics in collaboration with the English missionary John Fryer (1839-1928). In two books he wrote, *Xue Suan Bi Tan* [學算筆談 Essays on Studying Mathematics] of 1882 and *Suan Zhai Suo Yu* [算齋瑣語 Trivial Talks in the Mathematics Study Room] of 1896, he offered a perceptive and detailed discussion on how he studied mathematics and acquired new learning when Western mathematics was introduced into China.

In *Xue Suan Bi Tan* he wrote:

If computation and reasoning are two different activities, then those who compute need not reason. But can those who reason compute? And is what they compute the same as what those who do not reason compute? Perhaps those who reason do not prefer to compute, not that they cannot compute. If they are forced to compute, what they perform is exactly the same as what those compute will perform. However, since those who compute have carried out the activity with such familiarity, they compute with celerity and accuracy, usually not to be surpassed by those who prefer to reason.”

In *Suan Zhai Suo Yu* he wrote:

I hold the view that in writing a book in mathematics one must offer discussion in words besides writing in symbols and formulas and drawing figures. Symbols, formulas and figures can only explain what is in a problem but not necessarily what is in the periphery of the problem. Confined to the problem one can only explain what one understands rather than what one has not yet understood. Furthermore, while reading the book and finding every page covered with 1, 2, 3, 4 or a, b, c, d , one may inevitably feel bored and tiresome. Insertion of discussion would facilitate the flow of ideas and offer a new and refreshing appearance, thus open up one's thought.

The more senior Chinese mathematician LI Shan-lan [李善蘭 1811-1882] was a mentor of Hua. LI translated in collaboration with the English missionary Alexander Wylie (1815-1887) a textbook on calculus *Elements of Analytical Geometry and of Differential and Integral Calculus* (1850) written by the American mathematician Elias Loomis in 1859, and later presented a copy of the translation to Hua, who worked hard on studying this branch of mathematics new to him. Hua recounted his learning experience in perusing the translation which bears the Chinese title *Dai Wei Ji Shi Ji* [代微積拾級 Geometry and Differential and Integral Calculus Step by Step] in Chapter 5 of *Xue Suan Bi Tan* that ends with a rather poetic metaphor:

[...] After browsing through a few pages I was at a loss, not comprehending what the book said. [...] I asked Master Li about it, and he told me it was not easy to express in words the subtlety contained therein, but everything had been written down in the book with nothing hidden from the reader. One has to read it many times to acquire the meaning. How can one expect to

accomplish full understanding overnight? I believe in what he said and study the content of the book repeatedly. I began to get somewhere, just like seeing stars appearing at dusk. At first I only saw one, then a few more, then several tens more, then several hundreds more, and finally the starlit sky was full of them!”

An Epilogue

It is mentioned in the introductory section that China more or less fully adopted the Western way and content of mathematics in education since the late nineteenth and early twentieth centuries. However, it seems that the Chinese maintain a traditional attitude to learning despite such an adaptation, which makes the so-called CHC classroom (in all subjects besides just mathematics) a topic which attracts so much attention of some scholars that its study leads to the rich discussion of the two CHC paradoxes. In Chapter 1 of (Watkins & Biggs, 2001) several positive features of this tradition of the CHC classroom are highlighted and explained, grouped into six categories:

- (1) memorizing and understanding,
- (2) effort versus ability attributes,
- (3) intrinsic versus extrinsic motivations,
- (4) general patterns of socialization,
- (5) achievement motivation: ego versus social,
- (6) collective versus individual orientation.

Readers can consult (Watkins & Biggs, 1996, 2001) for a more complete discussion on all these aspects, while in this paper we focus on (1) and (2), in that repetitive learning is different from rote learning, and that understanding is regarded as a long process which requires considerable mental effort. All in all, the issue is deeply entrenched in a social and cultural setting so that merely replicating on the surface the way of teaching, either the Western world from the Eastern world or vice versa, without considering the social and cultural setting in which the learners are brought up since childhood may not give a satisfactory solution.

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Dutch mathematics teachers, magazines and organizations: 1904-1941

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Abstract

The group of teachers of mathematics at Dutch gymnasia, and the magazine of the general organization to which the group belonged, played an important role in Dutch mathematics education, but their history is hardly described so far. This paper will focus on this group and the magazine. We will compare their activities with those of the legal forerunner of the present day organization for mathematics teachers, the NVvW and its magazine Euclides and we will demonstrate that in the prewar years this almost forgotten organization and the magazine were in fact more active and interesting. The history of Dutch mathematics education during the first half of the twentieth century is usually considered as a rather dull period. The study of relatively neglected elements from that period can add some nuances to that picture.

Keywords: mathematics teacher associations; HBS; gymnasium; teacher journals

Introduction

Two years ago, the Dutch Association¹ of Teachers of Mathematics celebrated its 90th anniversary. In Dutch, its name is the Nederlandse Vereniging van Wiskundeleraren, usually abbreviated as the NVvW. That association, or at least its legal forerunner Wimecos², was founded in 1925. Nowadays, the NVvW is the only representative for teachers of mathematics for all types of secondary education. It also represents mathematics teachers active in vocational colleges for tertiary education, but not those at universities.

Its official organ, the magazine *Euclides*, dates already from 1924³. As it becomes immediately clear from that date, the magazine did not start its life as the organ of Wimecos. It was founded as an independent magazine, owned by the publishing

1 We will use the word ‘association’ for an organization whose members are of the same profession, focussing on the advancement of the professional activities and practices of these members in the first place. With a ‘trade-union’, or in short a union, we mean an organization that focusses on the improvement of the conditions of employment of its members, such as salaries and pensions. Sometimes an organization combines both aims.

2 Officially, it was an association for teachers of Mathematics (**W**iskunde), **M**echanics and **C**osmography, hence the name *Wimecos*

3 The name *Euclides* was used from volume 4 onwards, before it was an appendix to the *Nieuw Tijdschrift voor Wiskunde* [New Journal for Mathematics]

company P. Noordhoff in Groningen. It was only after the war that Wimecos/NVvW and *Euclides* became closely connected.

Recently, a number of publications was devoted to the history of journals for teachers of mathematics. For instance, both Furinghetti and Pizarelli published articles on the role of teacher journals in Italy (Furinghetti, 2017; Pizarelli, 2017). Furinghetti and Somaglia published an article about a journal on teaching mathematics for primary schoolteachers (Furinghetti and Somaglia, 2018). Krüger published an article on two Dutch journals for mathematics teachers; one of them, the *Wiskundig Tijdschrift* [Mathematical Journal] is from the same period that we describe (Krüger, 2017). A bit older are the publications of Van Hoorn about the pre-war years of *Euclides*, the journal already mentioned before (Van Hoorn, 2008^a, 2008^b, 2008^c).

So far, less attention has been paid however to the history of mathematics teacher associations. A remarkable exception is *Mathematics for the Multitude?* by Michael H. Price, that describes the history of the English Mathematical Association (Price, 1994). Fujita and Jones published an article about the same association and its endeavor to redesign the English geometry curriculum in 1902. (Fujita and Jones, 2011). Eileen F. Donoghue devoted some pages in the *History of School Mathematics* on associations and their publications in the early years of the 20th century in the USA (Donoghue, 2003). Johan Wansink published an article about the history of the first fifty years of Wimecos/NVvW (Wansink, 1976).

Descriptions of the history of these associations can sometimes be found on their websites, or in publications on the occasion of a jubilee and the like. For instance, the NVvW published, on occasion of its 75th anniversary, a jubilee book titled *Honderd jaar Wiskundeonderwijs* [Hundred years of Mathematics Education], in which one chapter is devoted to the history of both the NVvW and the magazine *Euclides* (Maassen, 2000).

As a consequence, the historiography of these associations is mostly devoted to those who still exist. The history of associations that have disappeared is often neglected, although they may have played an important role in mathematics education during the years they existed. Such was certainly the case in the Netherlands, and focusing on the Wimecos/NVvW only results in an incomplete picture of the history of Dutch mathematics education.

In this paper we will pay attention to one of these forgotten associations in particular, the “group”⁴ for mathematics teachers at the gymnasium. As the main source for its history we will make use of a magazine that was not especially intended for

4 It is usually called a “group”, since it was not an independent association but part of a larger organization.

teachers of mathematics only, but that published much interesting information for those teachers nevertheless. Moreover, that was already the case in the seventeen years of its existence before the founding of the group of gymnasium mathematics teachers.

That magazine, the *Weekblad voor Gymnasiaal en Middelbaar Onderwijs* [Weekly Journal for Classical and Secondary Teaching] was founded in 1904, so we start our research in that year. It existed until 1950. The group for gymnasium mathematics teachers ceased to exist in 1972. We will however end our research in 1941. In that year the German occupants took control over the magazine, and soon afterwards also over the general organization of gymnasium teachers, to which the group of mathematics teachers belonged. That created a completely different situation. After the war the gymnasium teachers organization, including its group of mathematics teachers, and the magazine were restored, but the educational scene in the Netherlands was soon the object of major changes. So, the year 1941 is a natural endpoint for our research.

To facilitate the reading of the paper, we summarize the most relevant Dutch organizations and journals. Some we have already mentioned above, some will be discussed below.

- 1830 Founding of the Society for Teachers at Dutch Gymnasia.
- 1864 Founding of the General Association for Teachers at Secondary Education (the AVMO). (mainly teachers at the HBS⁵)
- 1904 Founding of the *Weekblad*, a weekly magazine, the official organ the Society as well as the General Association.
- 1921 Founding of Liwenagel, a group within the Society, with four subgroups: for mathematics, physics, chemistry and biology.
- 1924 Founding of *Euclides*, an independent journal for the teaching of mathematics.
- 1925 Founding of Wimecos, an independent association for teachers of mathematics, mechanics and cosmography at the HBS.

The Society for teachers at the gymnasia

In 1830 a ‘Society for Teachers at Dutch Gymnasia’ was founded by a small group of teachers. Its first objective was the improvement of the conditions of employment for the teachers, such as salaries and pensions, but gradually the improvement

⁵ This name is an abbreviation of *Higher Burger School*. The school type was created in 1863 and can be considered as the Dutch variant of the German *Realschule*.

of teaching gained more importance. The Society was both a trade-union and an association.

In the early years, its members were most likely only the teachers of Latin and Greek. When in 1876 the gymnasia were thoroughly modernized, and other school subjects, including mathematics, became more important, teachers of other subjects will have joined the Society⁶. But at the schools as well as in the Society however, the teachers of Latin and Greek remained the largest group, and no doubt they were the most influential.

In 1907 Dr. H.W.C. Bückmann, a teacher of mathematics at the municipal gymnasium of Amsterdam, made the following remark to a younger colleague from another school:

We, teachers of mathematics, are working in an isolated position. That should change. Why don't you try to establish some kind of an association? (Verrijp, 1930, p. 99).

That younger colleague, Dr. D.P.A. Verrijp, admitted later that this remark was not taken very seriously by him then, and nothing came of it. Thirteen years later however, Verrijp had not only become older and wiser, but he had also experienced that pleas for mathematics teaching by an individual teacher were not very influential. He proposed therefore to the board of the Society to form a committee of mathematics teachers, which committee should busy itself with problems and questions concerning the teaching of mathematics.

The board answered positively and pointed out that under the new rules of the Society such a committee, as a "group" within the Society, could have an official, permanent and relatively independent status, with its own board and regulations. The board also suggested to include the teachers of the sciences within the group. One year later, in 1921, such a group was indeed formally established, including the formation of four subgroups: for mathematics, physics, chemistry and biology. Its first chairman was Verrijp (*Weekblad*, 17, 32, 1341-1344).



Fig. 1. Dr. D.P.A. Verrijp (1869-1937).

⁶ Data for those years are not available.

L.i.W.e.N.a.G.e.L.

The newly established group had the curious name of L.i.W.e.N.a.G.e.L. It was an acronym for *Leraren in Wiskunde en Natuurwetenschappen aan Gymnasia en Lycea* [Teachers in Mathematics and Natural Sciences at Gymnasia and Lycea]⁷. When it started, it had about a hundred members. The Society of gymnasium teachers to which it belonged had around 750 members.

The direct incentive for the creation of the group had been rather defensive. The government had changed the distribution of the number of hours devoted to mathematics in the lower classes of the gymnasia. Verrijp opposed this measure and wrote a letter to the government to persuade it to revoke the measure. He, not surprisingly, achieved nothing and therefore tried to involve the board of the Society, which eventually led to the foundation of Liwenagel.⁸

There was in those years however a more general feeling of discontent among mathematics teachers. After World War I, mathematics and sciences had lost much of their credit. The 'anti-mathematische Bewegung' [anti-mathematical movement], active in Germany in the first place but also not without influence in the Netherlands, forced mathematics teaching in the defensive – at least, that was how it was felt by its teachers (Van Berkel, 1996, pp. 81-82). The formation of this gymnasium mathematics and natural sciences teacher group in 1921, and also of the association for mathematics teachers at the HBS in 1925, and the foundation of the magazine *Euclides* in 1924 were partly reactions to this movement. They wanted to defend and protect the interests of mathematics teaching, which were threatened in the opinion of many of its proponents.

One of the first actions of Liwenagel was to try to maintain the written final exam for mathematics for the literary branch of the gymnasium. Its attempt was unsuccessful, the written exam was soon replaced by an oral one. Also, the endless discussions about the number of hours devoted to mathematics, and most of all about their distribution over the various classes, was only partly successful. Verrijp soon experienced that not only as an individual teacher, but also as chairman of Liwenagel, his influence was limited.

But although Liwenagel had a defensive or conservative incentive, it was certainly not a purely conservative group. Its formal aim was of course to promote the interests of the teaching of mathematics, but that did not mean only a defense of the status quo. It was also concerned about the modernization of the mathematics teaching at the gymnasia.⁹

7 A *Lyceum* was a new type of school, a combination of a gymnasium and a HBS.

8 We will use from now on the notation Liwenagel, which came in use soon after 1921.

9 We won't take the activities of the subgroups for the natural sciences into consideration.

That implied not only the modernization of the curricula, but also of the didactics of teaching. One of the methods that, according to the first regulations of the group, should be employed, was to “open the opportunity for the members to exchange views about all topics belonging to their subject”. Liwenagel indeed organized these discussions, and on account of these discussions, Verrijp wrote that it was important to see that within the group there were two points of view. One, he said, was more conservative and laid the emphasis on mathematical rigor and the historical development of mathematics. The other one attached more value to modern psychological and didactical insights and stressed the consequences these should have for mathematics teaching. Verrijp concluded that

although it is well known what I prefer [no doubt the conservative side], I was always of the opinion that it is an advantage that both points of view were represented in the discussions in our group. (Verrijp 1930, p. 103).

The group was embedded in the general Society of the gymnasium teachers, which of course limited its space for maneuvering. It could hardly proclaim opinions completely contrary to those of the general Society. Heavy attacks on the interests of other school subjects and colleagues would not have been appreciated. Liwenagel also abstained from discussions about salaries, pensions or the teaching load and the like, such questions were handled by the Society as a whole.

But of course, that was not only a disadvantage. Within the group, contact with science teachers who formed the other sub groups, was easy and obvious. The inevitable cooperation in the Society as a whole certainly attributed to the awareness that mathematics was not the only subject with legitimate desires and interests.

There was another, more important advantage of belonging to a more general association: the group could make use of the weekly magazine that was published by that association. It could publish its points of view, papers and reports of its meetings in the magazine. The archives of the group were lost in the Second World War, so we have no direct source for its activities¹⁰. But, since many of its activities were described in the magazine, it is still possible to reconstruct these activities.

A weekly magazine

That magazine, the *Weekblad voor Gymnasiaal en Middelbaar Onderwijs* [Weekly Magazine for Classical and Secondary Teaching] was the official organ of two associations. One was the Society of teachers at the gymnasia that we have discussed already, the other was the Association of teachers at the HBS. That school type, the so called Higher Burgher School, the Dutch variant of the German Realschule, was founded

¹⁰ The secretary of Liwenagel, Dr. A.T.M. Kramer, lived in a neighborhood in The Hague that in March 1945 was accidentally bombed by the RAF.

in 1863. A general organization for HBS-teachers was founded already in 1867, and it also had a double mission: the improvement of the conditions of employment of its members as well as the improvement and advancement of the teaching at the HBS. It was called the *Algemene Vereeniging voor Leraren aan Inrichtingen van Middelbaar Onderwijs* [General Association for Teachers at Schools for Secondary Education], usually abbreviated as the AVMO.

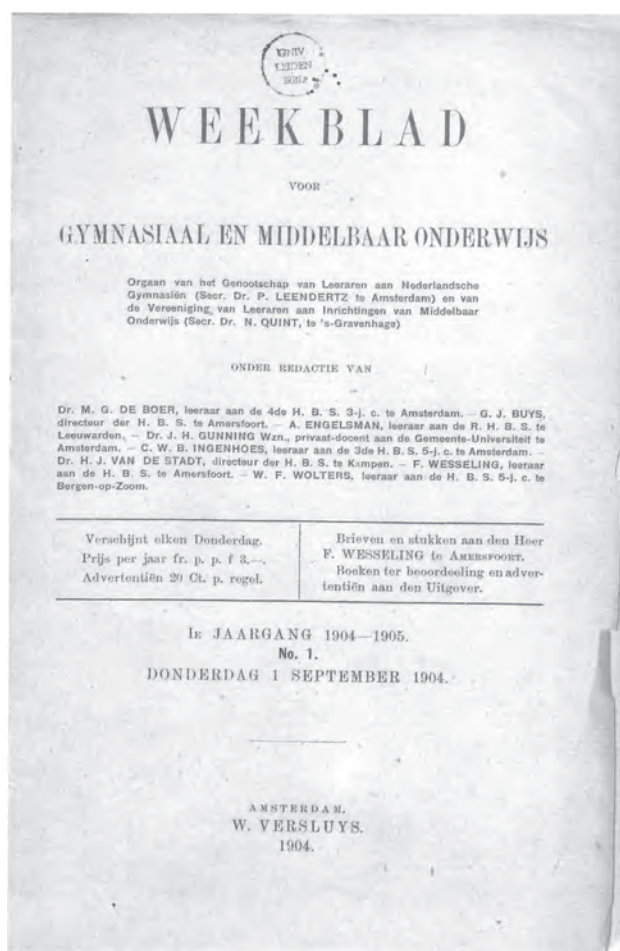


Fig. 2. Front page of the first number of the magazine.

In 1904, both associations started together with the publication of the *Weekblad voor Gymnasiaal en Middelbaar Onderwijs*, a weekly magazine for the teachers of both school types. Before, both associations had published a magazine on their own, but it had been proven to be difficult to publish a magazine of an acceptable level, that also was financially viable. But the *Weekblad* was a success. Although there were

sometimes irritations between the two organizations about editorial matters, as a whole their cooperation went very well. The commercial publishing company behind the magazine, W. Versluys, was well known and most capable, as a publisher of schoolbooks as well as of literary works like novels and poetry. The cooperation of the three parties involved remained successful until 1941.

The magazine was published every week, including school vacations, 52 numbers a year, usually more than 1500, and sometimes even more than 2000 printed pages in a year. So the 37 years we do consider amount to more than 50.000 pages. Of course, most of them had nothing to do with the teaching of mathematics. Many pages were filled with the internal affairs of both associations, or with news and discussions about union-like subjects, like salaries, pensions and the legal position of the teachers. But during all those years, the editors welcomed also contributions with news, opinions and discussions about the teaching of the various school subjects, including mathematics.

Not everybody was so enthusiastic about the latter. In the years 1905-1921, there existed a special journal for teachers of mathematics, called *Wiskundig Tijdschrift* [Mathematical Journal]. Its editor, F.J. Vaes, also played an important role in the AVMO, but he complained several times to the editors of the weekly about articles on mathematics teaching that should, in his opinion, rather have been published in his specialized journal. But the editors of the magazine stuck to their opinion that also articles about specific school topics, like mathematics, should have a place in their weekly.

We checked all the volumes from 1904 until 1941 for articles concerning mathematics education, which amounted to about 400 articles, about 1350 pages. Very short communications such as simple announcements of meetings were not included. An article should have some serious mathematical/didactical content to be counted. Of course, not every number of the weekly contained something about mathematics, but on average every month a mathematics teacher could read something that had to do with the subject he taught. That was already the case long before the founding of the Liwenagel. Maybe the mathematics teachers at the gymnasias felt a bit isolated, but they were not neglected in the magazine of their Society.

The mathematical content of the *Weekblad*

As might be expected, the mathematical content of the *Weekblad* comprises a wide variety of subjects. Nevertheless, it is possible to devise some categories that cover most of the content. We made up six categories: book reviews, (school) mathematics, curricula and exam programs, methodology and didactics, reports of groups outside both associations, and reports from the associations, or groups within the associations themselves.

Book reviews

These could vary from short reviews of new schoolbooks to very long reviews of rather high-brow books on mathematics and its history. The number of reviews varied over the years; some reviewers were very active and wrote a large number of reviews. But there were also years when the number of reviews was considerably less.

It is self-evident that reviews of schoolbooks were something that would interest most mathematics teachers. Usually, these reviews were rather benevolent, with some minor critical remarks, for instance about small errors or sloppy formulations. One can find only a few really critical reviews in which teachers were advised not to use the book. Sometimes such a verdict was based on the serious mathematical errors it contained, but for instance the schoolbooks of J.H. Schogt, impeccable from a mathematical point of view, were considered as too difficult for most pupils.

Not only schoolbooks were reviewed, also books of some scientific significance. As an example: the book *De vierde dimensie* [The fourth dimension], described as an introduction to the comparative study of different geometries by Hendrik de Vries, professor in mathematics at the University of Amsterdam. The review consisted of no less than eight pages, and one can imagine that it went beyond the interest and even understanding of most readers of the Weekly.

Articles about (school) mathematics

This category contains in the first place articles about mathematics itself, often school mathematics, but sometimes also about mathematics of a higher level. Articles about school mathematics could lead to lengthy discussions. For example, an article of J. Wansink about the remainder theorem sparked several reactions, which in turn led to responses by Wansink (*Weekblad*, 37, 36, 851-853).

Another topic that could lead to heated discussions were exam problems. Teachers argued that problems did not fit into the program, were ambiguous or unclearly formulated, too difficult or even wrong. For instance, a complaint about a question of the exam for trigonometry for the HBS in 1935, evoked no less than nine reactions (*Weekblad*, 31, 40, 1173-1174).

The fact that the magazine was published every week, and was produced and printed by a professional publisher who could work with a short production time was highly conducive for such discussions. After the war, when *Euclides*, which was a monthly with a much longer production time, became the official organ of Liwenagel and Wimecos, such discussions became impossible. Of course, nowadays they rage on the internet!

Curricula and exam programs

When F.J. Vaes and C.A. Cikot suggested in 1904 to introduce calculus in the HBS-curriculum, they could publish and defend their proposals in the recently founded *Weekblad*. Their proposals were submitted in 1907 to the mathematical section of the general meeting of the AVMO, were subject of ample discussions there and in the end rejected, which in turn provoked new reactions. The weekly published detailed reports about these discussions and the final vote. Publication of and discussion about proposals concerning changes in curricula and exam programs were recurring topics in the magazine. It happened with the proposals for a so called ‘normal-program’ for the HBS in 1917, with gymnasium programs in 1919 and 1922, with the so called “Beth-Dijksterhuis” report for the HBS in 1926, and with the new HBS-program in 1937. Members of the AVMO and the Society could therefore know all the ins and outs of the proposed changes in the curricula and exam programs, and, if they wanted so, also join in the discussions.

Methodology and didactics

The editors of the weekly conducted a generous policy concerning articles about the methodology and didactics of the teaching of mathematics. There were articles advocating the main stream ideas of those days, underlining the need to acquire the necessary skills and routines, or defending the status quo against all proposals for renewal. But there was also a place for contributions which diverged largely from the standing practices and opinions. An interesting example of this category is Tatyana Ehrenfests “About the role of axioms and proofs in geometry” from 1915 (*Weekblad*, 12, 13, pp. 457-470).

Mrs. Ehrenfest - Afanassjewa (1876-1964) was a Russian mathematician and physicist who had studied with Klein and Hilbert, who was very interested in mathematics education. She had been active in teaching mathematics and teacher training in Russia. few years before, when her husband had been appointed as professor in physics in Leiden, she had moved from Petersburg to the Netherlands, but she could not yet write in Dutch. The article, her first in Dutch, was a translation of one of her lectures held in Russia for mathematics teachers and was intended, as she later wrote, to introduce herself to a Dutch audience. That was certainly successful. The article provoked some reactions, which were answered by her. A few years later she started a didactical discussion group at her house. On the long term that group became highly influential. After the war it was attended and eventually chaired by Hans Freudenthal, whom she greatly influenced (Smid, 2016).

Reports from outside

The magazine published also reports of activities from groups and organizations that did not belong to one of the organizations. In 1936, the didactical discussion group of Tatyana Ehrenfest for instance got some official status as the Mathematics Working Group, belonging to the Dutch branch of the New Education Fellowship. The magazine published a report of the first official meeting of this working group, followed by six other reports of activities of the group. (*Weekblad*, 33, 12, 381-382) Reports of the pre-war activities of this group are rare, so these give valuable information about the group.

While the Mathematics Working Group can be considered as a progressive one, the independent association of teachers of mathematics at the HBS, Wimecos, was more conservative. Its founding must have been a setback for the AVMO that had tried to create a group like Liwenagel within the association. Nevertheless, not only the formation of Wimecos was mentioned, but also reports of other activities by Wimecos were occasionally published in the magazine (*Weekblad*, 26, 20, 621-623).

Reports from inside

The weekly magazine published of course many reports and much information concerning the activities of both organization. Until the founding of Liwenagel and Wimecos, matters concerning mathematics education were usually discussed on the yearly meetings of the Society and the AVMO, we have mentioned already some reports of these meetings.

Later these matters were mainly left to Liwenagel and Wimecos; although the latter was not part of the AVMO. Since Liwenagel was part of the Society for gymnasium teachers, the activities of that group received much more attention in the magazine than those of Wimecos. Not only reports of the yearly meetings of Liwenagel were published in the weekly, also other activities were extensively covered. Since the archives of Liwenagel were lost in the war, these accounts give crucial information about Liwenagel.

The reports in the weekly magazine show that the group became very active soon after its founding. It conducted surveys of its members, it produced reports on curricula and exam programs which were widely discussed, it organized lectures for teachers and it held yearly meetings. In 1926, it took the initiative to organize meetings for all mathematics teachers, including the HBS teachers. From 1932 on, these meetings were meant for teachers in mathematics and the natural sciences, held every two year. The last one of these meetings was held in 1970.

Liwenagel, Wimecos and Euclides

Liwenagel was not the only organization for teachers of mathematics. We have already mentioned Wimecos several times – we will discuss it in more detail later – but there were more. Education in The Netherlands was – and party still is – organized according to religion. Not surprising then, there were also organizations for teachers working at Protestant and at Roman-Catholic Gymnasia and Higher Burgher Schools. Schools for secondary education on a religious basis formed in the beginning of the 20th century a small minority, but their number was growing.

Both organizations had sections for the teachers of mathematics, comparable with Liwenagel. Contemporaries therefore spoke of four organizations for teachers of mathematics: Liwenagel, Wimecos and the two groups on a religious basis. The meetings for all teachers of mathematics that were initiated by Liwenagel, were for instance organized, as the chairman Verrijp said, with the cooperation Wimecos and the mathematical groups of the Protestant and Roman Catholic teachers associations (*Weekblad*, 23, 28, p. 869). The history of these groups however, which disappeared in 1972 with the dissolution of the organizations to which they belonged, remains so far unwritten.

Wimecos, the association for teachers of mathematics at the HBS, the legal forerunner of the present day mathematics teachers association, on the other hand did *not* belong to the general organization for HBS teachers, the AVMO. We can only guess why the founders chose for an independent association. The reason sometimes given, that the rules of the AVMO did not allow such groups is certainly not correct. On the contrary: the board of that association tried in the years before 1925 to establish such groups, and it explicitly pointed to the example of the group of mathematics and science teachers for the gymnasium teachers to illustrate what it had in mind (*Weekblad*, 22, 12, p. 436 & 15, pp. 523-525). Perhaps the founders of the HBS mathematical association in 1925 considered the atmosphere for mathematics in the AVMO as not favorable. Anyhow, the example of Wimecos was followed. In the following years several independent associations for teachers in school subjects were founded: for teachers in physics and chemistry, in biology, in modern languages, history, and more.

By opting for an independent association however, the teachers of mathematics also chose for isolation and a rather powerless organization. Wimecos displayed little initiatives or activities of its own, apart from a yearly meeting that usually attracted hardly twenty participants. It did not have its own magazine, but as the chairman said in his yearly address in 1931, “that was not a problem, since we are not a fighting organization and do not have many announcement to make” (Wansink 1976, p. 9).

That sounds of course not very ambitious, but there is also another interesting aspect to this remark. In the same year Verrijp, the chairman of Liwenagel, had said

in his speech on occasion of the 10th anniversary of Liwenagel, that if necessary, Liwenagel did behave as a fighting organization (*Weekblad*, 27, 35, pp. 1060-1069). So no doubt the remark by Tiddens was a reaction to Verrijps remark, and intended to highlight their differences.

One could suppose that the magazine *Euclides* could in a way be used by Wimecos, but that was not the case. That magazine was owned by the Noordhoff Publishing Company and its founding editor, P. Wijdenes was very keen on maintaining an independent position. *Euclides* sometimes published some of the lectures presented at activities organized by math teachers groups, but Wijdenes did not allow the associations any influence on the policy of his journal. Even the founding of Wimecos was not mentioned in the magazine (Wansink 1976, p.7).

But *Euclides* also had to pay a price for its independence. As we have seen, the *Weekblad* offered a teacher of mathematics quite some interesting information about his teaching subject, and if he was already a member of the Society (and therefore automatically of Liwenagel) or of the AVMO, he received the magazine for free. Joining Wimecos was not expensive – only one guilder a year – but *Euclides* was not that cheap: five, later six guilders a year, for no more than six numbers a year. Why to subscribe to *Euclides* if you were already a member of the Society or the AVMO?

So it is no surprise that *Euclides* had difficulties to survive. In 1939, when the existence of *Euclides* was seriously endangered, the publishing company Noordhoff and the editor in chief P. Wijdenes made a deal with Liwenagel and Wimecos: *Euclides* became their official organ and they acquired the right to publish their official announcements in *Euclides*. In return their members became – there was no choice! – subscribers to *Euclides* for a highly reduced price. But Liwenagel and Wimecos did not yet acquire real influence on the magazine, this would only happen some years after 1950, when Wijdenes retired.

The end of the story

In the sixties, the whole educational landscape of the Netherlands was radically transformed. One of the long term effects of this transformation was a process of merging of societies, associations and unions in the educational field. That resulted in the abolishment of Liwenagel in 1972, when the Society for gymnasium teachers and the AVMO for the HBS teachers merged. In the end, in 1997, only two general organizations for teachers, both for primary as well as for secondary education, were left: one 'neutral', that is to say on a non-religious basis, and one of a protestant orientation. They function, certainly for secondary education, as a trade-union in the first place, leaving the care for the teaching of the various school subjects mainly to the respective subject orientated teachers associations.

One of these is the present Dutch Association for the Teachers of Mathematics, the NVvW, into which Wimecos was transformed in 1968. In that year the mathematics teachers at the gymnasia and at the schools that replaced the former extended elementary schools, were admitted. In 1976 also the teachers of mathematics at vocational schools joined the NVvW. The NVvW was from then on the only association for teachers of mathematics left, with as its official organ the magazine *Euclides*.

Discussion and conclusion

In the introduction, we have mentioned some publications of Furinghetti, Somaglia and Pizzarelli on Italian journals for teachers of mathematics. We have also mentioned two Dutch journals: *Euclides* and the *Wiskundig Tijdschrift*.

Euclides and the *Wiskundig Tijdschrift* were founded by (former) teachers of mathematics, devoted almost only to mathematics and intended for mathematics teachers. The position of the *Weekblad* was quite different, it was a magazine founded by two general teacher organizations, and it comprised a wide range of subjects, for a large part subjects typically for trade unions. It was intended for all teachers at the gymnasia and the HBS.

The journals described by Furinghetti, Somaglia and Pizzarelli also differ markedly from the *Weekblad*. Although these Italian journals show also great differences mutually, some are for instance devoted to mathematics only, others cover a wide range of school subjects, they have in common that they all have an educational mission only.

The wide variety of subjects concerning school mathematics and mathematics teaching published in the *Weekblad* is however highly comparable with the content of the more specialized journals, for instance the *Wiskundig Tijdschrift*, or the *Periodico di Matematica*. The *Weekblad* had no regularly questions and answers column, but the possibility to engage in lively discussions on articles published in the magazine was quite unique.

The long lasting combination of subjects from the sphere of a trade union, and from the field of education in the magazine is not something one would expect.¹¹ That may be a reason why the *Weekblad* has not been systematically used so far as a source for the history of Dutch mathematics education. It has been used occasionally for that purpose, for instance to describe the endeavor to introduce calculus at the HBS in the beginning of the 20th century (Smid, 2000), but as far as we know,

¹¹ The *Weekblad* also published articles about other school subjects, but those on mathematics teaching formed by far the majority. Why this was the case remains unclear.

this is the first time that this magazine is systematically analyzed concerning its content about the teaching of mathematics. By doing that, we could also describe the history of the mathematics subgroup of Liwenagel.

Conclusion

If we describe the history of Dutch mathematics education in the first half of the past century only by looking back to the history of the present day Dutch Association of Mathematics Teachers, its predecessor Wimecos and its official organ, the magazine *Euclides*, we miss an important part of the picture. We have described the gymnasium mathematics teachers group Liwenagel and we hope we have shown that it was an active group, a mixture of more conservative and more progressive teachers, taking full advantage of being a group of a larger association by using its magazine and having the backing of a larger organization. The *Weekblad* published important news about curricula and exam programs for the teachers of mathematics, it offered a variety of articles concerning all aspects of the teaching of mathematics, and it also offered a platform for opinions and discussions both from a conservative as well as a progressive point of view.

The first half of the twentieth century is usually seen as a rather dull and uninspiring period in Dutch mathematics education. And it is true: curricula, exam programs and teaching methods did not change much. If one only looks at Wimecos and *Euclides*, it seems that mathematics teachers were mostly conservative, and showed little interest in renewal, both concerning content matter as well as didactics. That might well be the case of course, but if one looks at Liwenagel and the *Weekblad* however, that picture shows more nuances and is much livelier than is usually taken for granted so far.

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Views on usefulness and applications during the sixties

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Abstract

At the Royaumont Seminar (1959) the New Math reform was officially launched. In the decade between Royaumont and the first ICME congress in Lyon (1969), many mathematics educators were involved in actions to facilitate the implementation of the New Math reform. The New Math advocates were convinced that a deep knowledge and understanding of the structures of modern mathematics was a prerequisite to arrive at substantial applications, but in actual classroom practices the applied side of mathematics was often completely neglected. But already in Royaumont there were alternative voices who pleaded for taking the role of applications seriously. We investigate the arguments for integrating applications in mathematics education, as well as the kind of (new) applications that were envisaged, at the Royaumont Seminar and in the decade thereafter.

Keywords: applications and modelling; Hans Freudenthal; ICME-1; Modern mathematics; Royaumont Seminar

Introduction

The OEEC Seminar, held from November 23 to December 5, 1959 at the Cercle Culturel de Royaumont in Asnières-sur-Oise (France) is considered as a turning point in the history of mathematics education in Europe and in the United States (De Bock & Vanpaemel, 2015a). As Bjarnadóttir (2008, p. 145) stated: “The Royaumont Seminar can be seen as the beginning of a common reform movement to modernize school mathematics in the world”. Or in the words of Skovsmose (2009, p. 332): “After the Royaumont seminar, modern mathematics education spread worldwide, and dominated a variety of curriculum reforms”. The famous slogan “Euclid must go!”, launched at Royaumont by the Bourbakist Jean Dieudonné, became a symbol of the radical modernization of school mathematics. Most of the Royaumont proposals were strongly influenced by Bourbaki, the French structuralist school whose members or adherents, such as Gustave Choquet, Jean Dieudonné, Lucienne Félix and Willy Servais, were well represented at the Seminar. According to these scholars, the basic model for modernizing school mathematics should be the academic discipline of mathematics, as re-constructed and formalized from the late 1930s on by Bourbaki.

Less well known is that alternative reform proposals, emphasizing the role of applications, were also voiced at Royaumont. These too, were inspired by new developments in the field of applications during World War 2. The application-oriented proposals were however less decisive for developments during the 1960s than the dominant structuralist ideas. These ideas, especially on what should be taught at school about the (axiomatic) basis of mathematics, determined the debate. In this chapter, we first discuss the kind of (new) applications that were envisaged by some Royaumont lecturers, as well as their pleas for integrating the applied side of mathematics in secondary school curricula. This discussion will be based on *New Thinking in School Mathematics* (OEEC, 1961a), the official report of the Royaumont Seminar. Second, we examine the views of the more radical New Math reformers on applications or more generally, on the usefulness of mathematics. Third, we follow the debate on applications and modelling in the mathematics education community between Royaumont and the first ICME congress in Lyon (August 24-31, 1969). For several reasons, this decade is less well documented in the history of our field. At that time, *L'Enseignement Mathématique*, the only international journal on mathematics education until the late 1960s, had become a purely mathematical journal (see Furinghetti, 2009) and the international conference series which are now strongly established in our field (ICME, PME, ICTMA, HPM, ...) had not yet started. An exception might be the annual meetings of the International Commission for the Study and Improvement of Mathematics Teaching (CIEAEM), founded in the early 1950s, but CIEAEM only started publishing Proceedings of their meetings in 1974 (Bernet & Jaquet, 1998).

Although there were no strong international communication channels in the mathematics education community during the early- and mid-1960s, it was a very rich period of noteworthy international activity, including several seminars and symposia organised under the auspices of OEEC/OECD, UNESCO or ICMI (see, e.g., Furinghetti et al., 2008). These meetings mainly focused on issues related to the forthcoming introduction of New Math (program development, renewal of geometry teaching, teacher (re-)education and new didactical methods), but occasionally, concerns and proposals about the integration of applications in the curricula were expressed too. On the basis of the Proceedings and other edited documents from these meetings, we more generally review the visions on the role of applications in the international mathematics education community of that time.

By the end of the 1960s the debate on applications and modelling gained momentum. Hans Freudenthal, at that time president-elect of ICMI, organized the international colloquium 'How to Teach Mathematics so as to Be Useful' (Utrecht, August 21-25, 1967). The contributions to that colloquium were published in the first issue of *Educational Studies in Mathematics* (May, 1968). In his introductory address, Freudenthal took the opportunity to explain his views on the colloquium's theme. He argued that students could not be expected to (be able to) apply the

mathematics they had been taught in a purely theoretical way. Instead, to enable students to apply the mathematics they have learned, mathematics education should start from concrete contexts and patiently return to these contexts as often as needed (Freudenthal, 1968). It is the beginning of a new era in which applications and modelling gradually became an essential part of mathematics education. In the Netherlands, the theory and practice of ‘Realistic Mathematics Education’ (RME) were developed and inspired the teaching of mathematics in a large number of countries worldwide (Van den Heuvel-Panhuizen, 2018). We conclude this chapter with a more detailed discussion of Freudenthal’s ideas on applications and modelling in the years preceding ICME1.

The focus of this chapter is on what happened in Europe and North America between 1959 and 1969. As for other aspects of mathematics education, the role of applications during this decade is not well documented, with exception of a paper in French by Hélène Gispert (2003). On one side, Gispert’s contribution has a broader scope: it provides more information about the decade preceding Royaumont, for instance, about the discussion on applications of mathematics during the International Mathematical Union (IMU) congress in Amsterdam (1954). On the other side her contribution goes into more detail about what happened in France, more specifically with respect to Bourbaki, who was mostly interested in the general structural aspect of mathematics and not in mathematics education at the secondary level. With respect to the decade between Royaumont and Lyon she focuses on the OECD conference in Athens (1963), where not much progress was made with respect to the teaching of applications of mathematics. According to Gispert, the reason was that mostly professors in pure mathematics participated and discussed the direction of the New Math reform.

Applied mathematics at the Royaumont Seminar

The Royaumont Seminar was organised by the Office for Scientific and Technical Personnel (OSTP) of the Organisation for European Economic Cooperation (OEEC; later joined by nations outside Europe to form the Organisation for Economic Co-operation and Development, OECD). The Office was created for the purpose of promoting international action to increase the supply and improve the quality of scientists and engineers in OEEC countries (OEEC, 1961a). The main motive for OEEC/OSTP to organise a Seminar aimed at upgrading mathematics education was clearly economic: industry and other branches of economic activity were confronted with new applications of mathematics leading to a demand for more mathematicians with new kinds of skills. Therefore, a re-appraisal of the content and methods of school mathematics was needed. In his opening address, Marshall H. Stone, at that time president-elect of ICMI, formulated the functional argument as follows:

[...] the usefulness of mathematics in practical matters has been an added factor in its vitality as a component of the school curriculum. In this period of history, it is the rise of modern science and the ensuing creation of a technological society which compels us to give increasing weight to the utilitarian arguments for the more intensive teaching of mathematics (OEEC, 1961a, p. 17).

Stone also emphasized the need for a better coordination between mathematics and science teaching: "It is not going to be sufficient to improve the mathematical curriculum as an isolated part (...). It is of the first importance that instruction in mathematics and in the various sciences should be adequately co-ordinated" (OEEC 1961a, p. 21).

In view of the above, the Royaumont Seminar should thus have been a breakthrough of an applied and interdisciplinary perspective in mathematics education, but it turned out differently. Due to the dominance at the Seminar of professional mathematicians, most of them members or adherents of the French structuralist school, pure academic mathematics was de facto adopted as a model for school mathematics and most participants only paid lip service to the active application of mathematics. For instance, Dieudonné admittedly referred to applications to theoretical physics as a main argument for the inclusion of new topics in university courses of analysis, but left open the question whether any kind of 'applied mathematics' should already be integrated in the secondary school programs. Nevertheless, he believed that a favourable consideration of his reform proposals, having a clear Bourbaki orientation, would already provide the theoretical foundations for teaching questions of applied mathematics (OEEC, 1961a).

An alternative voice at Royaumont was that of Albert W. Tucker, a Canadian mathematician at that time working at Princeton. Tucker discussed the aspect of new uses of mathematics and their implication for mathematics education. Rather than study problems which involve two variables – or at most three or four – as most problems in classical physics, new branches of mathematics were developed to deal with complex realities involving several variables, which often occur in the social sciences, for instance in economics and psychology. Within these realities, Tucker distinguished problems of 'disorganised complexity' and problems of 'organised complexity'. The first category referred to problems with numerous variables and asked for techniques of probability theory and statistical interference, being effective for describing 'average behaviour'. Problems of organised complexity involved a sizable number of factors which were inter-related into an organic whole and required, among other things, a knowledge and use of matrix algebra. Tucker exemplified this last category with a problem of linear programming, utilising inequalities, intersections, graphic methods, and unique algebraic procedures for solving equations. According to Tucker, integration in all secondary-school programs of these

newer types of mathematics, in a suitable form, was feasible and desirable. He however acknowledged that an effort was needed to enhance teachers' knowledge about modern mathematics and its applications to teach the subject well (OEEC, 1961a).

Tucker's plea for the integration of probability theory and statistics in secondary-school curricula was supported by Luke N. H. Bunt from Utrecht University (the Netherlands). Bunt presented at Royaumont the outline of a syllabus on this subject matter taught in a Dutch experiment for the alpha streams of secondary schools (for more details on this experiment, see, e.g., Bunt, 1959):

- Some elements of descriptive statistics, such as frequency distributions, histograms, mean, median, and standard deviation.
- 'Classical' probability theory, with proofs of some of the elementary theorems.
- Intuitive treatment of binomial probability distributions; application to physics.
- Testing of a hypothesis (Bernoulli type of distribution); null hypothesis; level of significance; sample space; critical region; confidence limits; sign test; rank correlation. Only Type I errors (accepting a false hypothesis) are considered.

(OEEC, 1961a, p. 91)

For Bunt the problem of estimating some characteristics of a population on the basis of the values of these characteristics in a sample should be the dominant objective of a course in statistics. Bunt's proposal went against the general trend of the Royaumont Seminar because (a) he did not primarily focus on those mathematically gifted students that would become mathematicians or engineers, but on future students in economics, psychology and other social sciences, and (b) his didactical approach was pragmatic rather than mathematically rigorous (see also De Bock & Vanpaemel, 2015b).

New Math reformers' view on applications and modelling

Although at Royaumont and in the decade thereafter, there were several calls for mathematical instruction to take applications of mathematics seriously (Niss et al., 2007), New Math, strongly focussing on theoretical academic mathematics, was – at least in continental Europe – the dominant reform paradigm. Originally, the ambitions of the New Math reformers and practitioners' call for a focus on useful mathematics were not in contradiction, or as stated by Niss (2008):

It is worth noticing that despite the strong theoretical orientation of the New Math movement, its founders insisted that one of the points of the reform was to provide an ideal platform for dealing with the application of mathematics to matters extra-mathematical (p. 72).

Claims about the omnipresence and increased usability of (modern) mathematics can be found in many contemporary sources. In the *Charte de Chambéry*, a main French reform document prepared by the ‘Commission Lichnerowicz’ and adopted by the Association des Professeurs de Mathématiques de l’Enseignement Public (APMEP), the broad usability of modern mathematics is emphasized (and used as a main argument for the reform of mathematical teaching at all educational levels).¹

Contemporary mathematics is useful in many fields: theoretical physics of course, but also computer science, operational research, stock management of companies, organization charts of big administrations, planning for major projects, sociology, linguistics, medicine (diagnosing), pharmacy ... (Charte de Chambéry, 1968).

Georges Papy, the architect of the new mathematical curriculum in Belgium and president of the CIEAEM during the mid-1960s, wrote in the Preface of *Mathématique Moderne 1*, the first volume of his pioneering textbook series:

The scope of the material studied in the first 13 chapters [sets and relations] goes far beyond the boundary of mathematics. The student is initiated into types of reasoning constantly used in all spheres of thought, science and technology (Papy, 1963, p. vii).

In 1968, Papy further clarified his position. In his view, the ‘mathematization of situations’ was the way education should prepare students to applications of mathematics:

Thus, students are immediately accustomed to an approach which is essential for applications: the mathematization of situations. Obviously, it is difficult to predict the kind of mathematics that will be used by the students later. In the modern world, mutations are common. Many people, during their lifetime, have to change of profession several times and, in any case, of technical skills in their own profession. Mathematics does not escape from this phenomenon. [...] We do not know how to predict which situations will be mathematized later, nor which mathematics will be used for that purpose, but we know that the mathematization of situations will remain fundamental. It is therefore essential to accustom our students, from the beginning, to this important strategy of the mind. By the active mathematization of situations, one substitutes ‘learning’ for ‘teaching’. The ultimate goal of teachers is not to teach, but to enhance understanding and to learn learning (Papy, 1968, pp. 7–8).

A closer look at Papy’s approach, as elaborated in his textbooks, revealed that Papy indeed occasionally leaves the pure mathematical path to pay attention to the ‘mathematization of situations’: he regularly presented ‘daily-life’ situations to prefigure new mathematical concepts and structures. However, these situations did not

1 The next four quotations were translated from French by the second author.

incorporate realistic or authentic problem situations to be solved with mathematical tools. Their only purpose was to facilitate comprehension of an abstract formalized definition of the mathematical concept or structure that was targeted. Moreover, the newly learned mathematics was never (re)invested to analyse and to solve new challenging problems outside mathematics.

To better characterize the role of extra-mathematical situations in New Math courses of the 1960s, Hilton's (1973) distinction between 'illustration' and 'application' might be helpful. The point Hilton made is essentially the following. A situation, within or outside mathematics, is an illustration of a mathematical theory if and only if that situation clarifies the theory. A situation is an application of a mathematical theory if and only if that situation is clarified by the theory. For the high-order mathematical structures of the New Math, such as groups, fields or vector spaces, no applications were available for the early-aged students to whom these structures were taught and thus, these structures only could be illustrated with concrete instantiations (e.g. concrete materials or games especially constructed for that purpose). Although New Math advocates often referred to the universal applicability of the powerful structures of the modern mathematics in today's science and technology, they were unable to demonstrate this applicability to their students, for them they were just words, no tools for real problem solving, application analysis or modelling.

[Mathematical] structures are great and admirable machines, but in early mathematics education, they can only produce too little things and too little effects. These little things are the naive examples of structures that embellish textbooks in modern mathematics and are designed specifically for students (Rouche, 1984, p. 138).

Applications and modelling in the post-Royaumont era

Based on a survey of 21 national reports, Kemeny (1964) observed that the main interests of the international mathematics education community in the late-1950s/early-1960s were one-sidedly directed towards pure mathematics. The debate was focussed on the type of new mathematical subjects that could find a place in secondary school programs, on how the teaching of traditional topics could be improved by the adoption of modern ideas, on the 'right' way of teaching geometry, ... With the exception of a widely supported plea for teaching some notions of statistics at the secondary level, relatively little attention was paid to the applied side of mathematics. Also at the international meetings on mathematics education of the 1960s, organised by OEEC/OECD, UNESCO or ICMI, only occasionally ideas for integrating applications of mathematics in secondary school curricula were voiced. In the next paragraphs we briefly discuss three main sources of applications that, aside from statistics, were mentioned at these forums.

First, reference was still made to applications of mathematics to classical physics. The Group of Experts, that met in Dubrovnik (1960) for the purpose of preparing a detailed synopsis for modern secondary school mathematics, as stipulated in one of the Royaumont resolutions (OECE, 1961a), insisted once more on the need for a better coordination between the teaching of mathematics and the teaching of science (particularly of physics), but provided little or no concrete suggestions to put that coordination into practice. An exception might be the early introduction of vectors and the systematic development of their algebraic and geometrical properties in a modern curriculum for school geometry, which they considered, at least potentially, of the greatest use to the students and teachers of physics (OECE, 1961b). From physical scientists, an increasing pressure was felt to teach a more or less intuitive introduction to calculus in secondary schools – which was not the case in many countries – but mathematics education reformers of that time did not have clear ideas how such introduction could be properly integrated in a modern mathematical curriculum (Kemeny, 1964).

Second, there is the mathematics related to the upcoming computing machines which began to fundamentally impact secondary school mathematics. Examples of new computer-related applications and their curricular impact were thoroughly discussed at the OECD conference in Athens (1963). That conference provided a special section on ‘applications in the modernisation of mathematics’ (OECD, 1964) in which Henry O. Pollak (USA), at that time Director of Mathematics and Statistics Research at Bell Laboratories and one of the pioneers in the field of applications and modelling in mathematics education, examined, among other things, new areas of mathematics motivated by computer sciences. He stated that the basic notions of programming, including the use of flow diagrams in the construction of algorithms, should be essential parts of secondary school curricula. In an interview with Alexander Karp, Pollak also mentioned the importance of the relationship between mathematics, computers and computing (Karp, 2007). He further told about his pioneering activities with respect to modelling and applications at the undergraduate level and about his vision on these issues:

The point of all of what we were trying to do in mathematics itself is understanding, understanding when and how and why this stuff works. [...] The point of applications of mathematics is also understanding. The difference is that we’re trying to understand something outside of mathematics rather than inside (Karp, 2007, p. 77).

Pollak and his collaborators collected a series of engineering problems that led to nice mathematical formulations. The problems served as a basis for an early book on applications of mathematics (Noble, 1967). Freudenthal invited Pollak to speak at the ICMI colloquium in Utrecht (1967) and at the first ICME congress in Lyon (see the next section).

Hermann Athen, a German contributor to that Athens conference argued, computers may have a much broader impact on human thinking:

A factor not to be neglected is the technical and economic revolution which is taking place as a consequence of the big automatic computers. This revolution in psychic and intellectual functions of human thinking and computing is continuously leading to new investigations in the fields of logic and the analysis of thinking. There is practically no field of mathematical investigation which is not dependent upon the use of computers, e.g. many problems of the social, behavioural, managerial and economic sciences (OECD, 1964, p. 245).

Other topics, in some way or another related to computers or their use, were suggested at that time, for instance binary representations of numbers, coding, numerical analysis, discrete mathematics, electrical circuits, logic and Boolean algebra.

Third, as a genuine application to economics and other social sciences, linear programming was repeatedly mentioned. The topic fitted well within a modern course of linear algebra, but also could strengthen students' numerical skills related to solving equations and inequalities. Moreover, it opened a window to operational research, a recent field of applied mathematics that deals with the application of advanced analytical methods to help make better decisions given certain constraints. Probably more than other fields of application, optimization involves mathematizing and modelling, i.e. interpreting a real-world situation in terms of a precisely formulated mathematical model (OECD, 1964). Modelling and models were not yet widespread notions during the 1960s, but they gained ground. In his Introduction to the Proceedings of the UNESCO colloquium in Bucharest (1968), Nicolae Teodorescu observed that the notion of 'model' had acquired universal presence and circulation, and already acknowledged the cyclic nature of modelling processes.

Modelling the complex, heterogeneous reality is the deliberate aim of any modern research method in sciences of nature, in social sciences and in humanities. The victorious penetration of mathematics in other scientific domains is accounted for by modelling which, repeated successively, leads to mathematical models (International UNESCO Colloquium, 1968, p. 27).

Freudenthal and the emerging RME movement

The end of the 1960s was characterized by an increased interest for the didactics of mathematics, particularly at the micro level. Not only the purely mathematical subjects, but the way a child learns, became a main guiding principle for developing mathematics education. This new trend was reflected in a growing number of (international) congresses and meetings in the field. In the context of this chapter, the ICMI colloquium initiated by Freudenthal around the theme 'How to Teach

Mathematics so as to Be Useful' (1967), deserves our special attention. It was the first meeting in which an international panel discussed the differences in opinion about the role of the use of mathematics (La Bastide-Van Gemert, 2015). In his opening address Freudenthal sketched, in a general way, his views on mathematics education. He explained that teaching mathematics 'so as to be useful' is not the same as teaching useful mathematics:

Useful mathematics may prove useful as long as the context does not change, and not a bit longer, and this is just the contrary of what true mathematics should be. Indeed, it is the marvellous power of mathematics to eliminate the context. [...] In an objective sense the most abstract mathematics is without doubt the most flexible. In an objective sense, but not subjectively [...] (Freudenthal, 1968, p. 5).

He further argued that we should neither teach 'applied mathematics', nor 'pure mathematics' (and expect that the student will be able to apply it later). Mathematics is rather learned by doing, as a human activity, as a process of mathematizing reality and if possible, even of mathematizing mathematics.

The problem is not what kind of mathematics, but how mathematics has to be taught. In its first principles mathematics means mathematizing reality, and for most of its users this is the final aspect of mathematics, too. For a few ones this activity extends to mathematizing mathematics itself (Freudenthal, 1968, p. 7).

Freudenthal's colloquium was sometimes regarded as the symbolic beginning of a new era in the history of applications and modelling in mathematics education, the so-called 'advocacy phase' (Niss et al., 2007) in which advocates of applications and modelling provided arguments in favour of the serious inclusion of such components in the teaching of mathematics. In several countries (such as, for example, the UK and the US), this phase was quickly followed by a second 'development phase', mainly characterized by the development of new educational materials to put such teaching into practice, sometimes by institutes especially created for that purpose. In the Netherlands, for example, the implementation was driven by the Institute for the Development of Mathematics Education (IOWO) – founded in 1971 by Freudenthal, nowadays the Freudenthal Institute² – which shaped the philosophy and practice of RME.

But these developments did not yet reflect a global trend in mathematics education during the late-1960s. When in 1969 the first ICME congress was held, only two of the twenty published papers were related to applications and modelling (Editorial Board of Educational Studies in Mathematics, 1969; Gispert, 2003). Arthur Engel

² The Freudenthal Institute is nowadays more than the successor of the IOWO. It also includes research groups on Science Education and on History and Philosophy of Science.

(Germany) talked about the relevance of modern fields of applied mathematics, such as operations research, computer science, stochastic and game theory, for the teaching of mathematics. He argued that not the new techniques per se are important, but the new ‘modes of thinking’ they provide to cope with the real world (Engel, 1969). Henry O. Pollak (1969) classified and discussed different types of realistic and not so realistic (word) problems that are used to involve students in applications of mathematics at different educational levels.

Conclusion

The New Math movement, launched in 1959 at the Royaumont Seminar but with roots in the mid-1950s (Gispert, 2003), was dominated by academic mathematicians who had a genuine interest in education, but most of them were rather involved in pure than in applied mathematical research. Applications were not their first concern. Moreover, they were convinced that a thorough insight in the structures of the New Math was a solid and necessary basis for teaching questions of applied mathematics. In daily-school practice New Math adherents rather used real-world situations to illustrate than to really apply mathematical structures. During the 1960s calls for taking applications – and later also mathematical modelling – seriously grew louder and culminated in Freudenthal’s colloquium ‘How to Teach Mathematics so as to Be Useful’. This meeting marked the beginning of a new and more favourable period in the history of the teaching of mathematical modelling and applications. Although the first ICME congress in Lyon (1969) can be seen as a sequel of Freudenthal’s colloquium, the presence of modelling and application was not impressive.

In sum, it can be stated that the teaching of applied mathematics was not a primary concern during the early- and mid-1960s. But although most leading reformers of that time focussed on pure mathematics and only paid lip service to the active application of mathematics, some mathematics educators highlighted the role of (new) applications – and later also that of modelling – and regarded it as an essential element in the modernization of mathematics teaching.

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The First International Conference on the History of Mathematics Education took place in Iceland in 2009. In the Proceedings, the editors expressed their expectation that the research on history of mathematics education would be sufficiently productive to warrant the continuation of this conference every two years. And indeed, participation has grown, from 35 participants and 18 presentations in 2009, to 65 participants and 40 presentations in 2017, at the Fifth International Conference on the History of Mathematics Education. The venue of the Fifth Conference was the historical University Hall of the University of Utrecht in the Netherlands. The Descartes Centre and the Freudenthal Institute, both part of the University of Utrecht, and the Dutch Association of Mathematics Teachers sponsored the organisation and the Proceedings of the conference. The Local Committee, consisting of Heleen van der Ree, Harm Jan Smid, Jan van Guichelaar and Jenneke Krüger, organized the conference in an excellent manner. The final editing of the Proceedings was in the capable hands of Nathalie Kuijpers.

Previous international conferences on the history of mathematics education:

2009 Garðabær (Iceland)

2011 Lisboa (Portugal)

2013 Uppsala (Sweden)

2015 Torino (Italy)

The Sixth International Conference on the History of Mathematics Education will be held in Marseille (France) in September 2019.

