Solving subtraction problems by adding on: special education students' flexibility in doing subtractions up to 100

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Abstract In this study, we examined special education students' use of indirect addition (subtraction by adding on) for solving two-digit subtraction problems. Fifty-six students (8- to 12-year-olds), with a mathematical level of end grade 2, participated in the study. They were given a computer-based test on subtraction with different types of problems. Although most students had not been taught indirect addition for solving subtraction problems, they frequently applied this procedure spontaneously. The item characteristics were the main prompt for using indirect addition. Context problems that reflect an adding-on situation and problems that have a small difference between the minuend and subtrahend most strongly elicited the use of the indirect addition procedure. Moreover, indirect addition was identified as a highly successful procedure for special education students, and the best predictor of a correct answer was found in combination with a stringing strategy.

Keywords *Primary special education, Flexible computation, Indirect addition, Information and communication technology (ICT), Assessment*

1. BACKGROUND

At the end of primary school, many special education (SE) students are considerably behind on the topic of subtraction with numbers up to 100 compared with their peers in regular education (Kraemer, Van der Schoot, & Van Rijn, 2009). To support low-performing students and to give them confidence in carrying out subtraction problems, it is suggested, for example by the U.S. National Mathematics Advisory Panel (2008), that these students would

benefit from being taught one prescribed way of solving calculations. This opinion is also expressed in the Netherlands.[1]

However, the idea of teaching only one method goes against the goal of developing numeracy in students. This goal implies that students should be able to choose a suitable method when solving number problems (Treffers, 1989; Van den Heuvel-Panhuizen, 2001; Warry, Galbraith, Carss, Grice, & Endean, 1992). A further objection against the one-method approach is that if students would have to restrict themselves to only one way of solving problems, many problems would require an unnecessarily long solution path (see, e.g., Torbeyns, De Smedt, Stassens, Ghesquière, & Verschaffel, 2009). Finally, using prescribed methods can lead to a "didactical ballast" (Van den Heuvel-Panhuizen, 1986) for students. Following a prescribed solution method can be a source of error for students because this method is not grounded in their own thinking, i.e., the ownership is completely on the side of the teacher or textbook author.

In sum, we can say that there are many disadvantages to teaching one fixed solution method for solving calculations. Moreover, studies (e.g., Torbeyns, De Smedt, Ghesquière et al., 2009) have shown that flexible adaptation of the solution method can make problems easier for students. Therefore, one might decide to teach even students who are weak in mathematics the flexible use of solution methods. However, the critical point for making this decision is whether these students are able to operate in such a flexible way. Some studies (see, e.g., Milo 2003; Timmermans, 2005) have indicated that SE students with learning difficulties in mathematics have trouble in choosing a solution method in a flexible way. In the study reported in this paper, we further investigate whether SE students are able to adapt their solution methods to the nature of the problems presented to them. The focus of the study is on subtraction up to 100.

1.1 Strategies and procedures for solving addition and subtraction problems up to 100

Generally, three different types of strategies can be distinguished for solving addition and subtraction problems with numbers up to 100: splitting, stringing, and varying (Van den Heuvel-Panhuizen, 2001). These strategies, of which examples are given in Figure 1, have in common that they describe how we deal with the numbers involved (in splitting both numbers are decomposed in tens and ones, in stringing one number is kept as a whole number, and in varying one or both numbers are changed in order to get an easier problem).

		Strategies Number perspective		
		Splitting	Stringing	Varying
Procedures Operation perspective	DS Direct Subtraction	**63–31=** * 60–30=30 3– 1= 2 30+ 2=32	**63–47=** 63–40=23 23– 3=20 20– 4=16	
	IA Indirect Addition	**67–52=** ** 50+10=60 2+ 5= 7 10+ 5=15	**62–58=** 58+ 2=60 60+ 2=62 2+ 2= 4	
	IS Indirect Subtraction	**67–52=** ** 60–10=50 7– 5= 2 10+ 5=15	**62–58 =** 62– 2=60 60– 2=58 2+ 2= 4	
	MO Multiple operations			**77–29 =** 77–30=47 or 78–30=48 47+ 1=48

* The problem can be solved by the following calculation steps. The description of these steps does not necessarily reflect how the problems are or should be notated by students. The students' use of materials and models is left out as well.
** These calculation steps are not very common to solve this problem; they are only given to explain this particular combination of procedure and strategy.

Figure 1. Relation between procedures and strategies illustrated with problems

A different way of describing a calculation is by focusing on how the operation is carried out. From this perspective, two main procedures for solving subtraction problems can be distinguished: (1) direct subtraction (DS), which means taking away the subtrahend from the minuend (e.g., solving 62–58 by 62–50=12; 12–2=10 and finally 10–6=4), and (2) indirect addition (IA), which means adding on from the subtrahend until the minuend is reached (e.g., solving 62–58 by 58+2=60 and 60+2=62). A less common procedure is indirect subtraction (IS), which means taking away from the minuend until the subtrahend is reached (e.g., solving 62–58 by 62–2=60 and 60–2=58).

Together, the strategies and procedures offer a complete framework for describing how students solve additions and subtractions up to 100. Figure 1 illustrates the strategies and procedures by prototypical examples of subtraction problems in which the numbers are likely to elicit particular strategies and procedures. The framework reflects how these are related. A DS procedure often

goes together with splitting or stringing. For IA and IS, stringing is the most obvious strategy; although splitting can be applied as well. Finally, when a varying strategy is applied, multiple operations are required.

1.2 Solving subtraction problems by indirect addition

Connected to the earlier described debate about whether or not teaching SE students one fixed method for solving number problems, there is also controversy on whether SE students are able to solve subtraction problems by applying IA. For example, a few recent intervention studies concluded that even students in regular primary education hardly ever use IA to solve subtraction problems (Torbeyns, De Smedt, Ghesquière et al., 2009; De Smedt, Torbeyns, Stassens, Ghesquière, & Verschaffel, 2010).

However, these studies are challenged by other intervention studies that support the claim that, already in the first grades of primary mathematics education, students with a wide range of mathematical abilities can learn to flexibly solve subtraction problems by applying IA (Klein, Beishuizen, & Treffers, 1998; Menne, 2001).

1.3 Factors influencing students' procedure use

Important factors that may influence students' procedure use when solving a subtraction problems are: (1) student characteristics, such as their general mathematical ability, age, and grade level (see, e.g., Torbeyns, De Smedt, Ghesquière et al., 2009), (2) teaching characteristics, for example, whether or not students have been taught a particular procedure, turned out to play a role, although not all researchers found this (see Section 1.2), and (3) problem characteristics. With respect to the latter, the influences of the following two features of subtractions problem are discussed: (a) the numbers involved and (b) the problem format (context problems or bare number problems).

1.3.1 Influence of numbers involved

Several studies (e.g., Menne, 2001; Torbeyns, De Smedt, Stassens et al., 2009) have indicated that subtraction problems that require crossing the ten and that have a small difference between the minuend and subtrahend (e.g., 62–58) may evoke the use of IA. However, IA could also be an efficient procedure for solving large-difference subtraction problems with a relatively small difference around the tens and requiring crossing the ten (Torbeyns, De Smedt, Stassens et al., 2009). For example, 82–29 may be easily solved by IA (i.c., 29+1=30;

30+50=80 and 80+2=83, so 1+50+2=53). Finally, research suggested that small-difference problems that do not require crossing the ten (e.g., 47–43) may also evoke the use of IA (Gravemeijer et al., 1993).

1.3.2 Influence of problem format

Two didactical phenomenological interpretations of subtraction are: (1) subtraction as taking away and (2) as determining the difference. In the first interpretation, the matching operation is that of taking away the subtrahend from the minuend. However, in the second interpretation, the difference is determined by bridging the gap, which can be done in two ways: by adding on from the subtrahend until the minuend is reached and by decreasing the minuend until the subtrahend is reached.

To contribute to this broad understanding of subtraction, students should be given more than just bare number problems. Several studies (e.g., De Smedt et al., 2010; Van den Heuvel-Panhuizen, 1996) revealed that bare number problems hardly evoke the use of IA. Context problems, on the contrary, have the possibility to open up both interpretations of subtraction (Van den Heuvel-Panhuizen, 2005).

1.4 The present study

The present study was set up to investigate whether and under which conditions SE students are able to use IA for solving subtraction problems up to 100, and whether they can solve subtraction problems correctly when applying this procedure. The purpose of the study was to clarify the role of the numbers involved, the format of the problem (context or bare number problems), and the occurrence of prior instruction in IA. We formulated the following research questions:

1. Can SE students make spontaneous use of IA for solving subtraction problems up to 100, and which conditions influence the use of IA?

2. Does the use of IA help SE students solve subtraction problems up to 100 successfully, and under which conditions does IA use lead to successful problem solving?

2. METHOD

2.1 Participants

In total, 56 students from 14 second grade classes in three Dutch SE schools participated in the study. The participating students (39 boys, 17 girls) were 8-12 years old, with a mean age of 10 years and 6 months (SD=10.4 months). In regular education, 8- to 9-year-olds are in grade 3 and 11- to 12-year-olds in grade 6. This means that the students in our study were 1 to 4 years behind in mathematics compared with their peers in regular primary school.

The students' mathematical ability level was established with the Cito Monitoring Test for Mathematics End Grade 2 (Janssen, Scheltens, & Kraemer, 2005). The standardization of this test was based on a representative sample of Dutch second grade students in regular primary education whose average ability score was 56,4 (SD=14,6). The ability scores of the students in our sample ranged from 32 to 56 with an average of 47,8 (SD=6.8) which is a considerably lower score (*d*=–.59) than that of the students in regular primary school.

2.2 Materials

2.2.1 ICT-based test on subtraction problems

An ICT-based test[2] was developed in which item characteristics were varied systematically over 15 items. These characteristics include number characteristics and format characteristics. The number characteristics refer to the size of the difference between the minuend and subtrahend (small means <7 or large means >11), whether the tens have to be crossed (e.g., 61–59) and whether or not the minuend and the subtrahend are close to a ten (<3). The format characteristics refer to whether or not the items are presented as a bare number problem (BN) or as a context problem. The latter can describe a taking-away situation (ConTA) or an adding-on situation (ConAO). Figure 2 shows an example of a ConAO item.

The 15 items were displayed one per screen. The students could click to continue to the next item. The accompanying text was read out by the computer. By clicking on the ear button, the student could hear the spoken text again. After a short introduction, the students worked individually on a touch-screen notebook. Students were told that they were free to choose any solution method. After filling in an answer, they reported verbally how they found this answer.

The students' on-screen work was recorded by Camtasia Studio software. All students and their parents gave their permission for collecting these records.



Figure 2. *Album* item; the accompanying read aloud instruction is: "The album has space for 51 cards. 49 are already included. How many more cards can be added?"

2.2.2 Online teacher questionnaire

To collect data about the students' prior instruction on subtraction problems, we asked their mathematics teachers which procedures they had taught their students for solving these problems. An online questionnaire was developed for collecting these data. The link for the questionnaire was sent by email to the 14 teachers of the students. The teachers received the questionnaire shortly after their students were administered the ICT-based test. All 14 teachers filled in and submitted the questionnaire. Apart from a few general questions the questionnaire contained a specific question on the topic of "subtraction up to 100" to collect data on the procedures (DS or IA) that the teachers had taught their students for solving subtractions up to 100.

2.2.3 Analysis

The students' responses were classified on the basis of the screen videos which captured the students' answers to the test items and their verbal reports. The students' answers were coded as correct or incorrect. The verbal reports were used to classify the students' strategies and procedures. The responses were coded by two raters independently. There were only a few cases of disagreement (<5%). After discussing these cases, full agreement was reached.

Item responses of students (procedures, strategies, and success rate) were collected at case level, and the cases (students x items) are on the one hand nested within students (who are in turn nested within teachers) and on the other

hand nested within items. This structure enabled the use of cross-classified multilevel models with predictors at case, item, student, and teacher levels. We estimated the models in WinBUGS (Lunn, Thomas, Best, & Spiegelhalter, 2000). Because of the dichotomous nature of the dependent variables, we made use of multilevel logistic regression models.

In addition to the cross-classified multilevel analyses, we also used logistic regression models in which neither student, teacher, nor item effects were included. We did this by making use of generalized estimating equations (GEE). In this approach, standard errors of regression coefficients are adjusted as a result of the cross-classified data structure (Halekoh, Højsgaard, & Yan, 2006).

3.RESULTS

3.1 Frequencies of procedures and strategies

The data analysis was based on all cases in which the students gave an answer to a particular item. Of the 840 possible cases (56 students doing all 15 items each), 72 cases were missing. This resulted in 768 cases to be analyzed. DS was used in 63% of the total cases and went together almost equally often with a stringing or a splitting strategy. IA was used in 34% of the total cases of answered items; in 87% IA was applied in combination with a stringing strategy.

3.2 SE students' spontaneous IA use

Of the 15 subtraction problems, the total number of times the students applied IA to solve an item ranged from 0 to 8 items (M=4.6 and SD=1.9).

3.2.1 Different conditions and IA use

Numbers involved

Figure 3 shows that IA was most frequently applied in small-difference problems without and with crossing the ten (A and B, respectively). DS appeared to be the most popular procedure in large-difference problems (D and E), even in large-difference problems that have the minuend and subtrahend both close to a ten (C). The more frequent use of IA in A and B than in C, D, and E appeared to be significant in a GEE logistic regression (b=1.24, SE=.16, p<0.05).

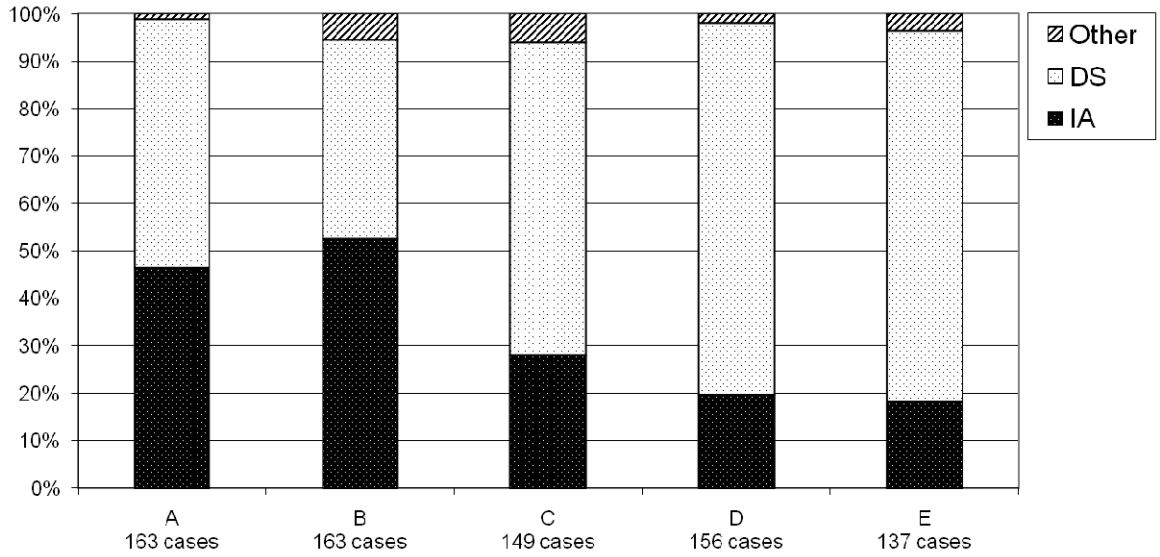


Figure 3. Percentage of procedure use related to number characteristics of the items

Problem format

Figure 4 shows that IA mainly appeared in the items with an adding-on context (ConAO) and that DS was most often used in items with a taking-away context (ConTA). Moreover, when solving bare-number problems (BN), the students preferred DS. A GEE logistic regression showed a significant difference in IA use between context and bare number problems (b=2.38, SE=.25, p<0.05).

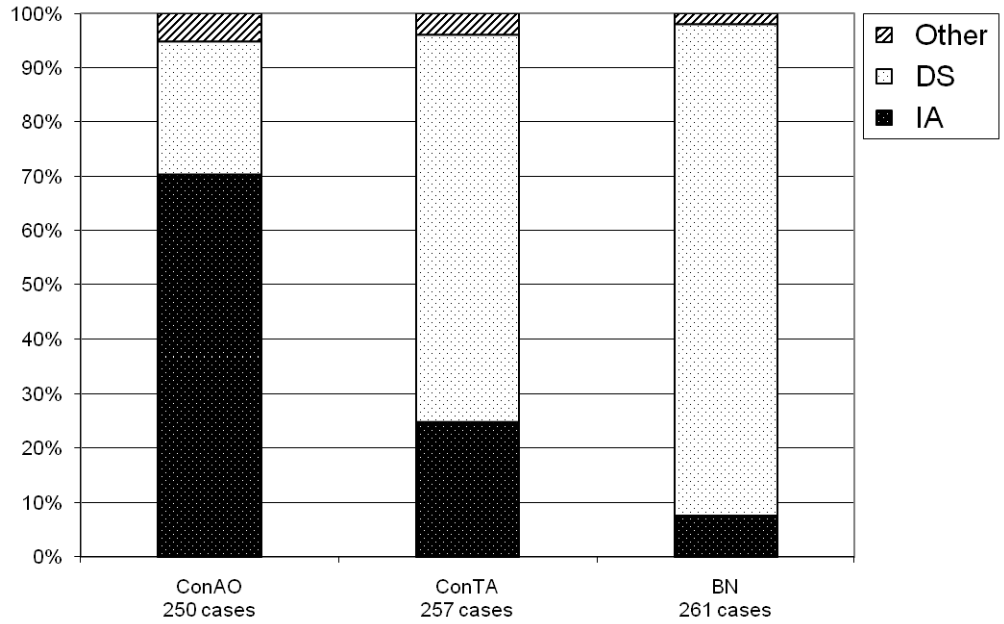


Figure 4. Percentage of procedure use related to the problem format

Prior instruction

The teachers' responses to the online questionnaire revealed that two different textbook series were used in the 14 classes. Although these textbook series each contain some missing addend problems, they do not explicitly address the inverse relation between addition and subtraction.

Because teachers could have paid attention to IA without it being addressed in the textbook series, we asked them which procedures they taught their students for solving subtraction problems. Their answers made it clear that all teachers taught DS. Only three teachers responded that they taught both DS and IA. Therefore, the students of these three teachers, 16 in total, were taught both procedures. These 16 students applied IA in 29% of the total of 209 cases (16 students answered 15 items each, minus 31 missing cases). The other 40 students who were not taught IA applied this procedure in 36% of the total of 559 cases (40 students answered 15 items each, minus 41 missing cases).

3.2.2 Multilevel analysis with IA use as dependent variable

To examine the influence of the different conditions on IA use, we carried out a multilevel analysis in which we specified a cross-classified multilevel model containing an empty model 0 and a model 1 with predictors (see Table 1).

In model 0, only random effects of items, students, and teachers are specified. The intercept represents the average use of IA transformed onto the logit scale of the multilevel logistic regression model. The intercept (b= –1.47, SE=0.82) is smaller than zero, which implies that IA is applied in less than half of the cases.

The large SD of the random item effect (SD=2.83) compared with the random student effect (SD=0.93) indicates that IA use is mainly an item characteristic. This means that the application of IA is elicited by the nature of an item rather than by the specific preference of a student. Thus, students seemed to apply IA in a flexible, item-specific way.

The SD of the teacher component (SD=0.41) is also small compared with the SD at the item level. Nevertheless, it should be noted that there is a substantial variation between teachers whose instruction might have consequences for students' IA use.

Table 1. Multilevel logistic regression model with IA use as dependent variable

	Model 0		Model 1		
Fixed Part	*b*	SE	*b*	SE	
Intercept	-1.47	0.82	–1.05	1.73	
Item level					
Numbers involved A			–0.45	1.06	
Numbers involved C			–1.73	1.11	
Numbers involved D			–3.35*	1.26	
Numbers involved E			–2.98*	1.17	
Problem format ConAO			4.74*	0.93	
Problem format ConTA			1.42	0.99	
Student level					
Cito ability score			–0.01	0.03	
Male student			0.44	0.41	
Teacher level					
IA taught			–0.44	0.55	
Random Part	*SD*	*SE*	*SD*	*SE*	R^2 (Model 0 – Model 1)
Item level	2.83	0.72	1.19	0.53	0.83
Student level	0.93	0.19	0.99	0.19	0^
Teacher level	0.41	0.28	0.44	0.33	0^

* $p<0.05$

^ R^2 was negative; therefore, it was set to 0

The intercept of the fixed part with respect to:

- item level corresponds to numbers involved B and problem format bare number (Model 1)
- student level corresponds to average Cito score and female student (Model 1)
- teacher level corresponds to not IA taught (Model 1)

In model 1, the *numbers involved* and the *problem format* are included as predictors at the item level. Here, all categories except a reference category of these variables are dummy coded (1=item possesses the property, 0=item does not possess this property). The regression coefficients of categories A, C, D, and E of the predictor numbers involved represent their contrast with the reference category B. In Table 1, the negative regression coefficients for numbers involved categories A, C, D, and E indicate that the frequency of IA use for the items belonging to these categories is smaller than for items in category B. However, we only observed a significant difference for numbers involved categories D (b=–3.35, SE=1.26, p<.05) and E (b=–2.98, SE=1.17, p<0.05) and not for numbers involved category A (b=–0.45, SE=1.06, p>0.05) and C (b=–1.73, SE=1.11, p>0.05). That the regression coefficient of A is close to zero

indicates that in items like 47–43 (category A) and 61–59 (category B), IA is equally frequently used. For C the regression coefficient suggests that students applied IA less frequently in category C than in B, but more often than in categories D and E.

With respect to the problem format, the regression coefficients of the categories of the predictor problem format (ConAO and ConTA) represent their contrast with the category BN problems. We found that IA was significantly more often applied for items that involve a context problem that reflects adding on (ConAO) than for BN problems (b=4.74, SE=.93, p<0.05). Such a significant difference was not found between items that involve context problems that reflect taking away (ConTA) and BN problems (b=1.42, SE=.99, p>.05). To investigate whether there is a difference in IA use between the context problem types ConAO and ConTA, we created a new variable which is defined by the difference of the regression coefficients of the two context problem types. Based on the WinBUGS output we computed the distribution of this new variable, which revealed that IA use occurred significantly more often for ConAO than for ConTA items (b=3.33, SE=.86, p<0.05).

When examining whether there is a difference in IA use between context problems (ConAO and ConTA) and BN problems, we found that IA was significantly more used in context problems (b=3.08, SE=.86, p<0.05).

The SD of the item effect in model 1 (SD=1.19) was substantially smaller than the corresponding SD in model 0 (SD=2.83). This means that a large amount of item variance in IA use is explained by the item predictors numbers involved and problem format. The explained variance at the item level (R^2_{Item}=0.83) corresponds to the reduction of variance from model 0 to model 1.

At the student level, neither *gender* (b=.44, SE=.41, p>0.05) nor the *Cito ability score* (b=–.01, SE=.03, p>0.05) turned out to be a significant predictor for IA use. At the teacher level, we found that despite the variation between the teachers, the variable *IA taught* (b=–.44, SE=.55, p>0.05) is not significant. Both for the teacher and student level a small increase of SD is observed in model 1 compared to model 0.

3.3 SE students' success rate in IA

Of the 15 subtraction problems, the students solved between 1 and 14 items correctly (M=7.7, SD=3.5). In 68% of the 260 cases in which IA was applied

and in 51% of the 480 cases in which DS was applied, the students' answers were correct. The higher success rate when using IA appeared to be significant in a GEE logistic regression model (b=.82, SE=.17, p<0.05).

3.3.1 Different conditions and success rate in IA

Number involved

Figure 5 shows that for items in category B, students' success rate when using IA is 87%, whereas it is 39% when using DS. This positive difference of 48 percentage points in success rate between applying IA and DS deviates from the negative difference of 11 and 4 percentage points found in the categories D and E respectively. This difference in success rate between IA and DS for the different categories of numbers involved appeared to be significant in a GEE logistic regression model (b=2.64, SE=.56, p<0.05).

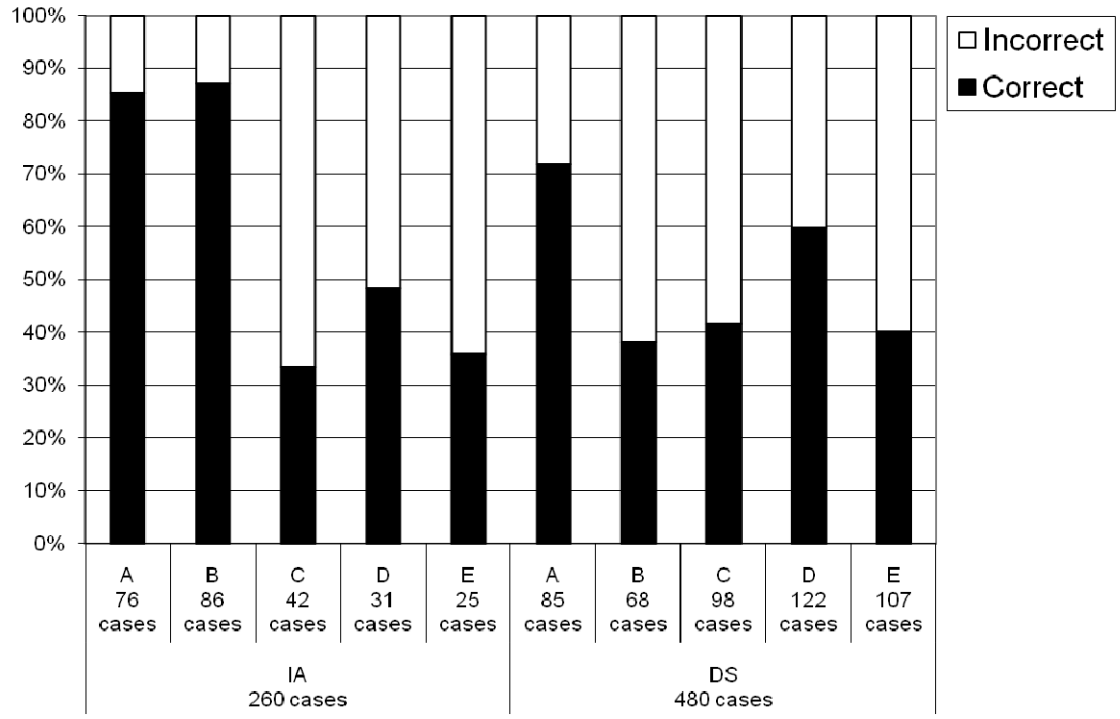


Figure 5. Percentage of correct answers related to number characteristics of the items

Prior instruction

The students who had received IA instruction correctly solved 77% of the 61 total cases for which they used IA. The students who did not receive IA instruction correctly solved 67% of the 199 total cases in which they applied IA.

The difference in these percentages did not appear to be significant in a GEE logistic regression (b=0.55, SE=0.34, p>0.05).

3.3.2 Multilevel analysis with success rate as dependent variable

To examine the influence of the conditions on success rate, we carried out a multilevel analysis in which we specified a cross-classified multilevel model containing a model 0 and model 1 (see Table 2).

In model 0, the SD of the random student effects (SD=1.20) is larger than the SD of the random item effects (SD=1.12) which indicates that correctly solving an item is more student-related than item-related. In addition, the SD of the random teacher effect (SD=0.31) is quite small compared with the SD at the item level.

Table 2. Multilevel logistic regression model with success rate as dependent variable

	Model 0		Model 1		
Fixed Part	*b*	*SE*	*b*	*SE*	
Intercept	0.36	0.38	0.14	0.58	
Case level					
IA use			–0.40	0.52	
Stringing use			0.72*	0.28	
IA use and Stringing use			1.17*	0.55	
Item level					
Numbers involved A			0.89	0.56	
Numbers involved C			–1.39*	0.52	
Numbers involved D			–0.24	0.55	
Numbers involved E			–1.30*	0.55	
Problem format ConAO			0.36	0.43	
Problem format ConTA			0.05	0.45	
Student level					
Cito ability score			0.12*	0.03	
Male student			–0.20	0.39	
Teacher level					
IA taught			–0.09	0.44	
Random Part	*SD*	*SE*	*SD*	*SE*	R^2 (Model 0 – Model 1)
Item level	1.12	0.29	0.53	0.25	0.78
Student level	1.20	0.18	1.07	0.18	0.20
Teacher level	0.31	0.24	0.26	0.20	0.30

* p<0.05

In model 1, several predictors at the case, item, student, and teacher levels are included. At the case level, the predictors *strategy use* and *procedure use* are included to investigate their influence on success rate. Because our focus is on the IA procedure, which was mostly combined with a stringing strategy, we used IA and stringing use as the dummy variables. Although there is a positive relation between IA use and success rate, IA use did not significantly predict success rate (b=–0.40, SE=0.52, p>0.05). The use of the stringing strategy increases success rate significantly (b=0.72, SE=0.28, p<0.05). However, the best predictor of a correct answer is the combination of IA and stringing (b=1.17, SE=.55, p<0.05). This finding was obtained even after controlling for all the other predictors at item, student, and teacher levels.

At the item level, the predictors *numbers involved* and *problem format* are included. Items belonging to the numbers involved categories C (b=–1.39, SE=.52, p<0.05) and E (b=–1.30, SE=.55, p<0.05) are significantly more difficult than items of category B. Concerning the problem format, we found that both types of context problems (ConAO and ConTA) did not significantly differ from the BN problems (b=.36, SE=0.43, p>0.05 and b=0.05, SE=.45, p>0.05 respectively).

At the student level, it appeared that students' success rate is positively related to *Cito ability score* for mathematics (b=.12, SE=0.03, p<0.05), however, *gender* is not (b=–.20, SE=.39, p>0.05). Finally, at the teacher level we found that *IA taught* is not a significant predictor of success rate (b=–0.09, SE=0.44, p>0.05).

Using the SDs at the item level in model 0 and model 1 we found that the item difficulties are largely explained (R^2_{Item}=0.78) by the item predictors. The explained variance at the student level (R^2_{Student}=0.20) is smaller. Apparently, other student characteristics besides the two included in model 1 are responsible for the variance at student level. The explained variance at the teacher level (R^2_{Teacher}=0.30) is also less than on the item level.

To investigate whether the success rate in case of IA use differed for the different numbers involved, we specified an additional multilevel regression model including the predictors IA use, the categories of numbers involved (which are also used in model 1), and the interactions of IA use with each of these categories. As in model 1, category B served as a reference category. For all interactions of IA use with numbers involved categories, we found significant negative regression coefficients. This means that IA use is most successful when

it is applied in small-difference problems with crossing the ten (category B) compared with all the other categories of numbers involved (A vs. B: b=–1.58, SE=0.67, p<0.05; C vs. B: b=–3.07, SE=0.67, p<0.05; D vs. B: b=–3.14, SE=0.71, p<.05; E vs. B: b=–2.59, SE=0.74, p<0.05).

4. CONCLUSIONS AND DISCUSSION

Our study showed that SE students can indeed make use of IA when solving subtraction problems. The main prompt for using IA turned out to be the item characteristics. Students used IA in a rather flexible item-specific way. With respect to the numbers involved, we found that students mainly used IA in small-difference problems with crossing the ten. With regard to the problem format our study revealed that students most frequently applied IA in context problems that reflect adding on. However, students did not apply IA more often when having received instruction in IA.

Our study showed that the SE students were quite successful in solving subtraction problems when using IA, but the results from the two types of applied analyses were not univocal. In the GEE regression, IA use was found to significantly influence success rate, whereas in the multilevel regression (in which – in contrast to the GEE approach – the student's general ability of solving the subtraction problems in the test is included as a random effect), IA use was not a significant predictor for success rate. Because students were free to choose their solution method, the use of IA might be related to their general ability to solve test items correctly. This explains why the GEE approach and the multilevel approach lead to different results (see also Molenberghs & Verbeke, 2004).

Furthermore, solving the test items by applying IA together with stringing appeared to be more successful than applying DS together with splitting. This finding emphasizes the importance of examining procedures (IA use or DS use) as well as strategies (splitting, stringing, and varying) when investigating students' ability to solve number problems.

Regarding the numbers involved, we found that in small-difference problems with crossing the ten, students were more successful when applying IA. Again, for prior instruction, we did not find an effect of IA use on success rate.

In sum, our study has revealed that: SE students (1) are able to use IA spontaneously, (2) are rather flexible in applying IA to solve subtraction

problems, and (3) are quite successful when solving subtraction problems by IA. These outcomes contrast with some research findings described in Section 1.3 which suggested that weak students have difficulties in applying IA to solve subtraction problems. Our findings made it clear that sensitive assessment tools are needed to reveal students' ability. In our case, test items designed with particular format and number characteristics enabled us to make SE students' ability to use IA visible.

Although the present study confirmed to that SE students are able to use IA to solve subtraction problems up to 100, our results should be handled with care. First of all, our study was limited in number of students and schools. A second drawback of our study was that we did not carry out a detailed inventory of the students' prior instruction in IA, i.e., we only asked whether the students had been taught a particular procedure and not how it was taught. This lack of information on the quality of the instruction might explain why no influence was found of prior instruction on the students' success rate. Finally, the test we used for this study has some shortcomings. Not only did we have no more than a small number of items but we also offered these items to every student in the same order. The latter means that our results could be flawed as a result of order effects in the items. Although we tried to minimize the order effect of the item characteristics by distributing them uniformly over the test item positions, it cannot be guaranteed that the particular sequence of the item characteristics did not influence the outcomes of our study. Nevertheless, our findings showed that SE students were able to use IA. Providing evidence for this was the main goal of the study.

Notes

[1] For example, in a summary of Timmermans' (2005) thesis published on the NWO web site (Netherlands Organization for Scientific Research; retrieved from http://www.nwo.nl/nwohome.nsf/pages/NWOP_6HKFLE), it is stated: "Weak performing students in arithmetic […] attain better results with the traditional approach in which they learn to solve number problems in one particular way" (translated from Dutch by the authors of this paper).

Another example is based on Milo's (2003) thesis, of which the NWO web site (retrieved from http://www.nwo.nl/nwohome.nsf/pages/NWOP_5LEJNJ) also clearly states that "pupils at special schools for primary education can best learn arithmetic using one specific strategy."

[2] The ICT-based test was developed by the first two authors of this article and programmed by Barrie Kersbergen, a software developer at the Freudenthal Institute.

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