De relatie die het Freudenthal Instituut heeft met het Isfahan Mathematics House in Iran is eigenlijk al twaalf en een half jaar geleden begonnen. Toen ontmoetten Ali Rejali en **Martin Kindt** elkaar. Tijdens de feestelijkheden rond tien jaar IMH is deze lezing van Martin als videoboodschap vertoond. In het najaar zal Martin wel zien dat het IMH er heel anders uitziet...

# Re(j)alistic Maths

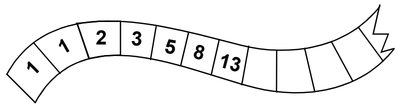
### on the occasion of ten years Isfahan Mathematics House

It was about two and a half years before the foundation of the Isfahan Mathematics House that I met Ali Rejali for the first time. It happened in Sevilla, a marvelous town in Spain, during the International Conference of Mathematics Education in 1996.

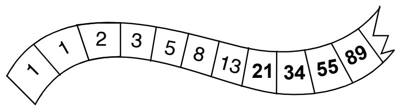
We both participated in the working group 'Curriculum Change' and Ali's presentation concerned the implementation of discrete mathematics in the curriculum of secondary schools in Iran. He appeared to be a fan of discrete mathematics, like me. I remember his terse statement: *'solving problems in number theory has a positive effect for strengthening students in reasoning and mathematical thinking*'. In addition we know that the concepts in discrete mathematics are in general more concrete and accessible than those in 'continuous mathematics'. Maybe this is the main reason why I agreed (and still do) strongly with Ali about the relevance of treating chapters of discrete mathematics in middle school and high school. Therefore I was very pleased with Ali's contribution in Sevilla.
At the end of the conference we had an interesting and inspiring discussion in our working group about the role of algebra in mathematics education. "Why should we teach so much algebra and how should we do this?" is still a very topical issue. I remember very well Ali's contribution to the discussion, where he proposed that '*algebra may serve as a tool to prove assertions about numbers*.' His message was to pay more attention to this aspect in mathematics education and it influenced my thinking about teaching algeba.

***Algebra with Fibonacci***
Some years ago I wrote an article for the magazine of the Freudenthal Institute (*Nieuwe Wiskrant*, December 2006) about some properties of the well known sequence which owes its name to Fibonacci, who introduced the Arabic number system in Europe in 1202 in his *Liber Abaci*.
In my article I made some didactical remarks about a possible start of algebra. and I will show you here my example, which I believe illustrates very well the spirit of Ali Rejali's statement in Sevilla.

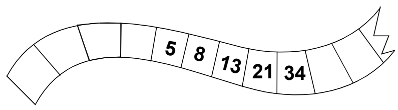
We all know the Fibonacci sequence:




You can present this sequence in the original context of the reproduction of an isolated group of rabbits, or similar, but that is not absolutely necessary. In my experience it is not difficult challenging young students (of say 11 or 12 years old), after a simple presentation of the number sequence, to guess the rule for extrapolation. After this rule is discovered – and I am sure this will happen rather soon – they can extrapolate step by step as far as they want.

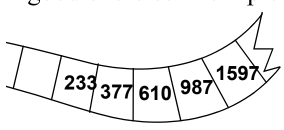



Now I will focus on subsequences of five consecutive Fibonacci numbers, for example: 5 , 8 , 13, 21 , 34.




**5 + 34 = 39 = 3 × 13**

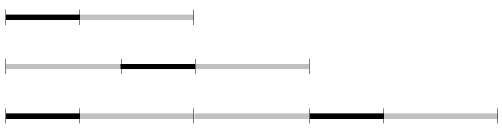
The sum of the first and the last number is three times the number in the middle. Perhaps that isn't surprising. But it is a real surprise that if you take five other consecutive terms, the same property is valid!
Of course it is a good idea to ask the students to investigate more of such subsequences, even with big numbers (which is also a good exercise in simple arithmetic).



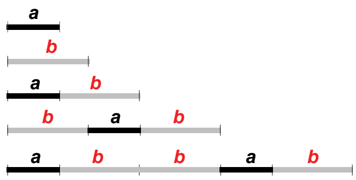

**233 + 1597 = 1830 = 3 × 610**

Obviously the question will arise: is it true that for *any* subsequence of five consecutive Fibonacci numbers the sum of the first and the last number equals three times the middle one?

Suppose the students are not familiar with algebra yet. Then you should help them a little, for instance by representing an arbitrary Fibonacci number by a (black) line segment
The next number in the sequence may be represented by a larger (grey) segment
Actually, you may make the greater segment a little bit more than 1.6 times the smaller one, but that's the secret of the teacher. Then ask the students to draw the next three numbers and the result may be:



Now look at the sum of the first (dark) segment and the fifth segment existing of two black and three grey ones. This sum contains three black and three grey segments, therefore it is exactly three times the third segment which consists of one black and one grey segment.
This is really a good proof!
You may say that it's in the style of the Alexandrian mathematicians in the era of Euclid. Those Greek mathematicians didn't use letters to express quantities, but that cannot restrain us from doing so.




Or

$a \qquad b \qquad a + b \qquad b + a + b \qquad a + b + b + a + b$

Or in a natural way of shortening:

$a \qquad b \qquad a + b \qquad a + 2b \qquad 2a + 3b$

Now our proof is equivalent with the identity:

$$a + (2a + 3b) = 3 \times (a + b)$$

This should be an adequate starting point to teach algebra in a style that I will call here:

*the Rejalistic Approach*

by which I mean teaching algebra as a tool to generalize from a finite number to an infinite number of calculations and to prove a general numerical property. In this way students will really experience the 'strength of algebra'.

I have to remark that the discovered property is not a privilege of the Fibonacci sequence; for instance it is also valid for the Lucas sequence:

1 , 3 , 4 , 7 , 11 , 18 , 29 , 47 , 76 , 123 , ...

as everyone can verify. In fact it is valid for all sequences of which each term is the sum of its two predecessors.

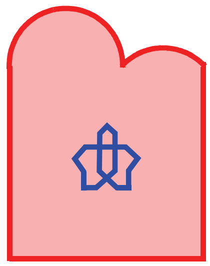
There are enough possible exercises to do in such a Fibonacci-like sequence, for instance:

– Take any subsequence of nine consecutive numbers. Then the sum of the first and the ninth number equals 7 times the number in the middle.
– The sum of any six consecutive numbers in the sequence is exactly 4 times the fifth one.
– ....

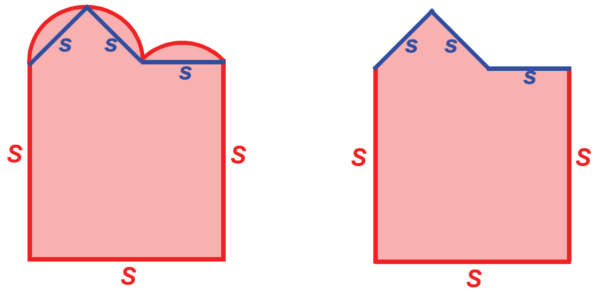
And wouldn't be a nice exercise for students to discover and prove their own interesting property of a Fibonacci-like sequence?

***A continued story with rectangles***

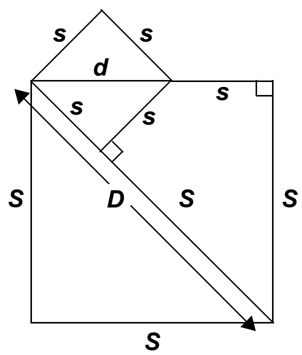
Now I want to visit the Isfahan Mathematics House.
I have not been there and it is hard for me to imagine the shape of the building. I tried to represent the house by a simple drawing and this is my result.



As well as the nice symbol of the IMH, you can see the composition of a square, a semicircle and a segment of a circle. In two steps I transform it to a Dutch house:




Then I will take a look inside the last figure:




So you see two squares, a big one and a smaller one, with sides respectively *S* and *s*, and with diagonals *D* and *d*.
It is not difficult to prove that the quadrangle on the right

side is a symmetric one, in fact it's a kite .
From this follows: $D = S + s$.
Moreover $S = s + d$ and evidently: $\frac{D}{S} = \frac{d}{s}$ .
Now I will consider the ratio

$$Q = \frac{S+D}{S} = \frac{s+d}{s}$$

This ratio is between 2 and 3, for $D$ is greater than $S$, but smaller than $2S$. But the most interesting fact is that the difference of $Q$ with 2 equals the inverse of $Q$.

$$Q = \frac{S+D}{S} = \frac{S+S+s}{S} = 2+\frac{s}{S} = 2+\frac{s}{s+d} = 2+\frac{1}{Q}$$

Now let $Q$ bite his own tail:

$$Q = 2+\frac{1}{Q} = 2+\cfrac{1}{2+\cfrac{1}{Q}}$$

And once again:

$$Q = 2+\frac{1}{Q} = 2+\cfrac{1}{2+\cfrac{1}{Q}} = 2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{Q}}}$$

Etcetera.
So we have:

$$Q = 2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\dots}}}}}} \quad \textit{ad infinitum}$$

This *continued fraction* is a never ending story analogous to a 'Dream' of the Argentine author Jorge Luis Borges (from his collection *La Cifra*):

> In a deserted place in Iran stands a stone tower, not too high, with neither door nor windows. In the only room inside (in the form of a circle, with an earthen floor) stand a wooden table and a bench. In that circular cell, a man who looks like me writes a long poem in characters that I do not understand about a man in another circular cell who writes a poem about a man in another circular cell who... The process goes on without end and none can read what the men write.

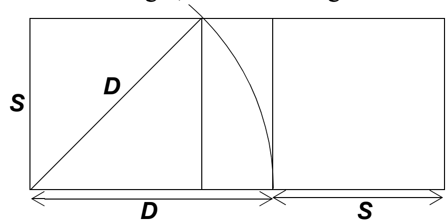
Back to the world of mathematics.
From the Euclidean algorithm on division one can demonstrate that any ratio of two positive integers can be uniquely represented by a *finite* continued fraction consisting of integers $a_0, a_1, a_2, \dots, a_n$ (the *partial quotients*):

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots \; \cfrac{1}{a_{n-1}+\cfrac{1}{a_n}}}}}$$

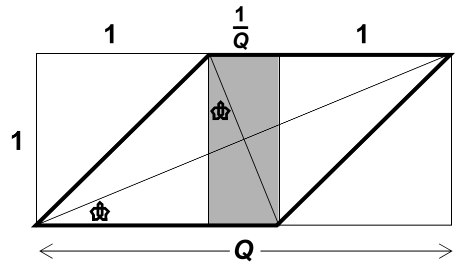
with $a_0 \geq 0, a_1, a_2, \dots, a_{n-1} > 0, a_n > 1$

Because the continued fraction presenting $Q$ is *infinite*, we certainly know that $Q$ cannot be equal to the ratio of two natural numbers. So to say: $Q$ is an 'irrational ratio'. In other (Greek) terms: the line segments $S + D$ and $S$ are inconmesurable.

The legend tells us that the existence of inconmesurable line segments caused a big shock in the Pythagorean school 2500 years ago. Perhaps the first discovery of this fact took place in a geometrical context. For instance by studying a rectangle of wich length and width are in the proportion of $Q : 1$. Using two squares, it is easy to construct such a rectangle, as the next figure shows.



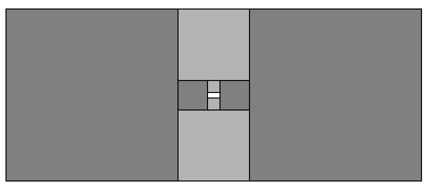

With one diagonal of each square I can make a parallellogram, which obviously is a rhombe. The diagonals of a rhombe are perpendicular, so in this case they make equal angles with the horizontal and the vertical lines (angles marked with the IMH-symbol):




From this equality it follows that the grey small rectangle is similar to the big one! So the proportion of length and width of the small rectangle is also $Q : 1$ and again we may conclude:

$$Q = 2+\frac{1}{Q}$$

Applying the recursive similarity we may construct this figure which can be infinitely extended.



The white rectangle in the centre is similar to the original one. Within this small rectangle two small dark grey squares and a new smaller white rectangle (which again is similar to the original one) will fit; inside the last white rectangle two grey squares and again a smaller white rectangle which is similar to the original one will fit. And so on, without coming to an end.

Nowadays we would use algebra to find the recursive formula:

$$Q = \sqrt{2}+1 = 2+(\sqrt{2}-1) = 2+\frac{1}{\sqrt{2}+1} = 2+\frac{1}{Q}$$

Mathematics is the discipline of generalization, and from this point of view I will replace two squares by $n$ squares. What will happen?

Well, the recursion formula will be $Q = n+\frac{1}{Q}$ (here again $Q$ represents the proportion of length and width). Below, I solve these equations with the classical method of Al Khwarizmi:

$$Q = n+\frac{1}{Q} \longrightarrow Q^2 = nQ+1$$

$$\downarrow$$

$$(Q-\tfrac{1}{2}n)^2 = 1+\tfrac{1}{4}n^2$$

$$\downarrow$$

$$Q = \tfrac{1}{2}n+\tfrac{1}{2}\sqrt{4+n^2}$$

And the conclusion is:

$$\tfrac{1}{2}n+\tfrac{1}{2}\sqrt{4+n^2} = n+\cfrac{1}{n+\cfrac{1}{n+\cfrac{1}{n+\cfrac{1}{n+\cfrac{1}{\dots}}}}}$$

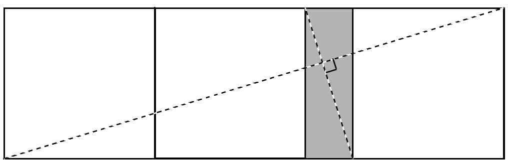
So the pure periodic continued fractions with a period of length 1 correspond with the numbers of the form: ,

$$\tfrac{1}{2}n+\tfrac{1}{2}\sqrt{4+n^2}$$

with $n = 1, 2, 3, \ldots$

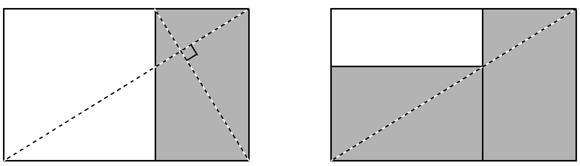
For instance, the case $n = 3$ allows me to construct a rectangle consisting of three equal squares and a small rectangle which is similar to the original rectangle

$$Q = \tfrac{3}{2}+\tfrac{1}{2}\sqrt{13}$$



The case $n = 1$ results in the famous golden ratio.

$$Q = \tfrac{1}{2}+\tfrac{1}{2}\sqrt{5}$$

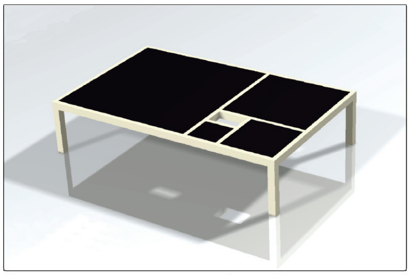


The golden rectangle exists of one square and one smaller golden rectangle, which is coloured grey. If I rotate this grey rectangle 90°, it will fit in the white square.

You may demonstrate this for instance with two credit cards and a ruler. The diagonal of the horizontal card will pass through the right vertex of the vertical one. Indeed: the shape of a credit card is that of a golden rectangle.

As we saw in the case $n = 2$ it is an infinite process making squares in the remaining golden rectangles.

This was the idea behind the construction of a table by the Dutch designer Nauta. The tabletop consists of four squares and a hole in the shape of a golden rectangle.



***Back to Fibonacci***

Nauta called his table the 'Fibonacci table' and I will show why this is not a crazy name. Look at the continued fraction corresponding with the golden ratio. It's really the nicest fraction you can imagine:

$$\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5} = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\dots}}}}}$$

Calculating the 'partial fractions' (or *convergents*) of this continued fraction, we get:

$$1+\frac{1}{1} = \mathbf{\frac{2}{1}}$$

$$1+\cfrac{1}{1+\cfrac{1}{1}} = \mathbf{\frac{3}{2}}$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}} = \mathbf{\frac{5}{3}}$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}} = \mathbf{\frac{8}{5}}$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}} = \frac{\mathbf{13}}{\mathbf{8}}$$

Notice the numerators and denominators of each convergent. It seems that in each case they are succesive numbers of the Fibonacci sequence! The proof that this goes on and on and on .... is a typical example of mathematical induction.

Compare the ratios of pairs of consecutive numbers in the sequence: 1, 1, 2, 3, 5, 8, ...

The first ratio is $\frac{1}{1} = 1$ .

For any two consecutive pairs we have this scheme:

$$a \quad b \quad a+b$$

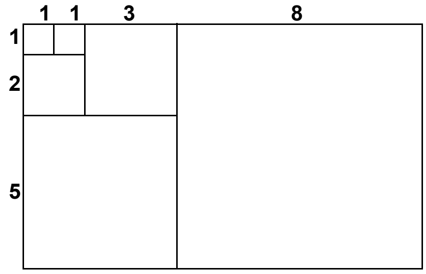
$$Q = \frac{b}{a} \qquad Q^* = \frac{a+b}{b} = \frac{a}{b} + 1$$

and therefore

$$Q^* = 1 + \frac{1}{Q}$$

This recursion formula guarantees that the conjecture about the partial fractions of the 'golden continued fraction' is true. Once more this is a demonstration of powerful algebra!

Another conclusion may be: if rectangles are built by pasting squares, starting with a square of 1 by 1, you get rectangles of which the ratio of length and width tends to the golden ratio.



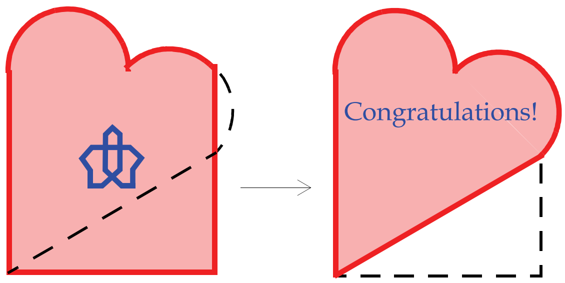

Now the circle is closed and I come to the end of my talk.

***Congratulations***

I really admire the idea behind the Isfahan Mathematics House as an inspiring meeting-place of mathematics teachers and students and I want to congratulate the House with her first decade. Let there be many new ones! Like a continued process of which the end is invisible.

Moreover I hope that the starting cooperation and the exchange of ideas between your Mathematics House and our Freudenthal Institute will be very fruitful in the near future. I am very sorry that I could not be in Isfahan on your 'birthday party', but I expect to visit Isfahan and to work with Iranian teachers, later in this year.

Finally, I will transform my drawing of the Isfahan Mathematics House once more:





*Martin Kindt,*
*Freudenthal Instituut , Utrecht*


Zie ook de website van het IMH: www.mathhouse.org