Tegenwoordig worden in de wiskundeles materialen, plaatjes of modellen ingezet om het begrip van leerlingen van een concreet niveau naar een formeler en abstracter niveau te brengen. Nieuw is dit niet: in de achttiende en negentiende eeuw werden op diverse Nederlandse universiteiten al fysieke modellen gebruikt om het meetkundeonderwijs voor studenten aanschouwelijk te maken. De Spaanse didacticus **Irene Polo-Blanco**, in Nederland gepromoveerd op dit onderwerp, doet in het Engels verslag.

# Physical models for the learning of geometry

## Introduction

Mathematics teachers and researchers of didactics of mathematics are constantly looking for ways to improve the teaching of mathematics. Recent worldwide reseach supports the idea that student understanding is more effective when using manipulative material (Heddens; Canny, 1984; Fennema, 1973; Suydam, 1984). The important role that visualization plays in the case of geometry makes the use of concrete manipulatives particularly appropriate for its teaching. There is wide support that this use facilitates the construction of representations of geometric concepts in young children (Taylor et al., 2007; Grenier & Grenier, 1986; Gerhardt, 1973; Prigge, 1978). It has also been shown that manipulatives are an essential aid in learning geometry for older students as well, especially those at lower level of geometric knowledge (Clements & Battista, 1992; Fuys, Geddes & Tischler, 1988).

Other researchers, however, warn that concrete manipulatives are not sufficient to guarantee meaningful learning (Baroody, 1989) and point out the complexity of the matter. For example, students sometimes learn to use manipulatives only in an automatised manner or without recognizing mathematics in it, and although performing the correct steps, learn little from it. Clements and McMillen (1996) point out that teachers and students should avoid using manipulatives without careful thought, since they alone are not sufficient, but they must be used to actively engage children's thinking with teacher guidance.

Recent research also supports the use of virtual manipulatives, which have often replaced concrete ones in the teaching geometry due to the increasing use of computers. For instance, in Reimer & Moyer (2005) it is shown that working with virtual manipulatives helps children develop conceptual and procedural knowledge in geometry, and that these manipulatives are more motivating than paper and pencil tasks. Other studies (Sarama & Clements, 2004; Sarama et al., 2003) show that computer manipulatives help children of different ages learn various geometry concepts.

In this text we focus on a collection of about 150 physical models present at the University of Groningen. By physical models we mean here *concrete* models, often made of plaster or string, which represent certain geometric objects. The author's thesis (Polo-Blanco, 2007) reports the study of some of these models from both a historical and a mathematical perspective. Since this work was completed, the models have been the topic of several master theses of mathematics students in Groningen. One of these investigations resulted on finding the origins and mathematical meaning of a set of thread models on the Groningen collection, and has been reported in Top and Weitenberg (2011).

Most models at this university were sold by the German companies L. Brill and M. Schilling in the second half of the nineteenth century, and will be the focus of this text. We discuss the origin of the models, the motivation behind their construction, as well as other aspects like the mathematics behind them and their pedagogical use.

## Mathematical models in Germany and Klein's role

In Europe, the use of concrete mathematical models and dynamical instruments for higher education was already common during the seventeenth and eighteenth century, but received a new impulse in the nineteenth century (Maclaurin, 1720). We find an example of this at the polytechnic schools in Germany during the second half of the nineteenth century (Dyck, 1892; Fischer, 1986) where collections of mathematical models were constructed, for instance to simplify the visualization of some geometric objects (Polo-Blanco, 2007).

The great period of model building started in the 1870ss, when Ludwig Brill, brother of Alexander von Brill, began to reproduce and sell copies of some mathematical models. A firm under the name L. Brill was founded in 1880 for the production of models and was taken over in 1899 by Martin Schilling who renamed it. By 1904, Martin Schilling had already produced twentythree series of models. Schilling's 1911 catalogue (1911) describes forty series consisting of almost four hundred models and devices. But by 1932 Martin Schilling informed the mathematical institute at Göttingen that "in the last years, no new models appeared" (Schilling, 1911).

These models often represent objects from differential geometry, algebraic surfaces or instruments for physics. In Schilling's catalogue the models are given a name and a short mathematical explanation. In some cases this is accompanied by a drawing, as shown in figure 1.

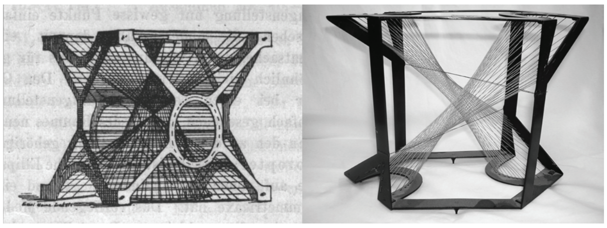


*fig. 1 Left: drawing of model nr. 1, Series XII: from Schilling (1911). Right: model nr. 1, Series XIII from the Groningen collection.*

## The case of the Clebsch diagonal model

During the second half of the nineteenth century, many models were constructed under the direction of Felix Klein. He developed his love for geometric models under the influence of Alfred Clebsch. The idea of model building reached many universities in Europe. Klein also brought the topic to the United States when he presented many of the models, including a model of the Clebsch diagonal surface, at the World Exhibition in Chicago in 1893. The Clebsch diagonal surface is defined by the equations

$$\begin{cases} x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 = 0 \end{cases}$$

and was discovered by Alfred Clebsch in 1871. It had been proved previously in 1849 by Arthur Cayley and George Salmon that every smooth cubic surface contains precisely twenty-seven lines over the complex numbers. On a Clebsch diagonal surface all the twenty-seven lines are also real. This means in particular that the twenty-seven lines can be represented in a model. Figure 2 shows such a models of this surface constructed in 2005 by the Spanish sculptor Cayetano Ramírez López.

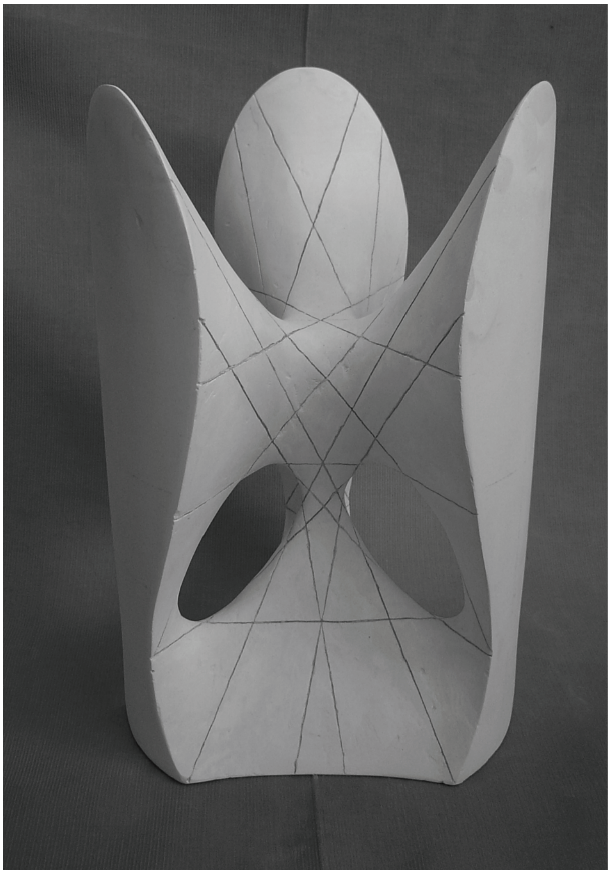


*fig. 2 Clebsch diagonal surface by C. Ramírez López. University of Groningen.*

The model might be appreciated by many for its beauty, but why should one use it to study and understand this surface? Wouldn't having a nice drawing or a computer dynamic representation be sufficient? We believe that having the physical model might help in understanding some of the properties of this surface, such as the complexity of the configuration of its lines. The twenty-seven lines intersect in a special manner, which was described by the Swiss mathematician Ludwig Schläfli in 1858 by defining the concept of *double six*. A double six is a set of twelve of the twenty-seven lines on a cubic surface:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}$$

where each line represented above only intersects the lines which are not on the same column, nor on the same row. That is, $a_i$ does not meet $a_j$ for $i \neq j$, $a_i$ does not meet $b_i$ and $a_i$ meets $b_j$ for $i \neq j$. This configuration is difficult to understand and to visualize just by looking at the definition, or even by using a computer representation. However, having the model in your hands might really help to get a feeling of this structure. In the model of the Clebsch diagonal surface, the six lines $a_i$ above have been drawn in green (light grey) and the six lines $b_i$ in red (dark grey). By moving the model around one can observe for instance that a green line never meets another green line and that it only intersects five of the red ones. Another property of the lines is that there are points where three lines of the cubic surface intersect (called *Eckard points*) which can also be found in the model by turning it around.

Figure 3 shows a fragment of Klein's lecture at the Chicago Exhibition where he discussed the construction of the model of this surface. His words also reflect the important pedagogical purpose he attributed to the models:

> In 1872 we considered, in Göttingen, the question as to the shape of surfaces of the third order. As a particular case, Clebsch at this time constructed his beautiful model of the *diagonal surface*, with 27 real lines, which I showed to you at the Exhibition. The equation of this surface may be written in the simple form
>
> $$\sum_1^5 x_i = 0, \quad \sum_1^5 x_i^3 = 0,$$
>
> which shows that the surface can be transformed into itself by the 120 permutations of the $x$'s.
>
> It may here be mentioned as a general rule, that in selecting a particular case for constructing a model the first prerequisite is regularity. By selecting a symmetrical form for the model, not only is the execution simplified, but what is of more importance, the model will be of such a character as to impress itself readily on the mind.

*fig. 3 Lecture by Klein at the Chicago Exhibition in 1893.*

## A Dutch perspective

Most Dutch universities keep some collection of Brill's or Schilling's models. We shall briefly describe some of them, namely the ones located at the Universities of Amsterdam, Leiden and Utrecht, and the condition in which they were kept at the time this investigation took place (between 2005 and 2007).

1. *University of Amsterdam*. The University of Amsterdam keeps the largest collection of mathematical models in the Netherlands. It consists of more than 180 models, most of them from the collections of Brill and Schilling. The University Museum of Amsterdam has made a detailed inventory of their collection. Most of the plaster models and many of the string ones have been restored. More detailed information on this collection can be found in the online catalogue[1] under *Bijzondere Collecties*.
2. *University of Leiden*. The Mathematical Institute of Leiden University keeps a collection of about 100 models distributed between the library and room number 210 of the Mathematical Institute. Most models are preserved in good condition, although literature, references or a local catalogue of the collection have not been found.
3. *University of Utrecht*. The Department of Mathematics in Utrecht keeps about twenty mathematical models. They are in a reasonably good condition. A few of them are on display in the library of the Department, while the rest are kept in a closed part of the library. Some of the broken models are in the University Museum.

One might wonder how universities in the Netherlands acquired a collection of models. How did these models become known in the first place? For example, were there advertisements from the Brill or Schilling companies in Dutch scientific journals of that period? A quick search through the volumes of the years 1875 to 1910 of the Dutch mathematical journal *Nieuw archief voor wiskunde* shows no advertisements concerning the models. However, several advertisements of Brill and Schilling's collections of models appeared in the renowned *American Journal of Mathematics* during the great period of model building (see figure 4). Collections of models were bought by many Universities in Europe, including some Dutch universities.

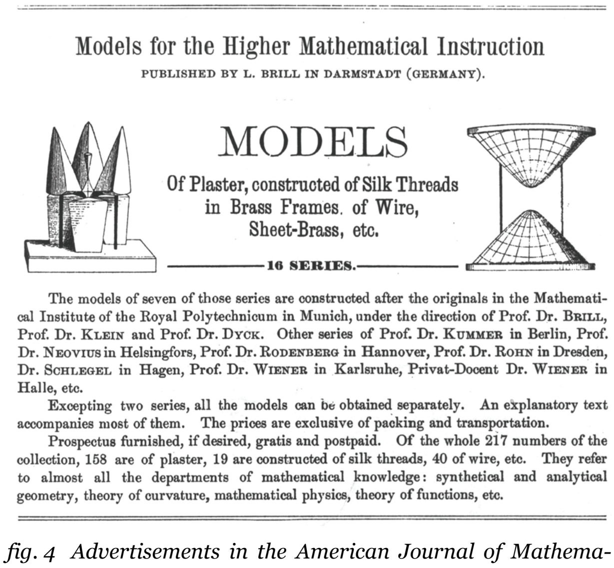



*fig. 4 Advertisements in the American Journal of Mathematics: July 1890.*

As it can be appreciated in the figure, models were advertised to be used for *Higher Mathematical Instruction*. Although the most important reason to buy these models seems to have been of a didactic nature, one of the few concrete pieces of evidence for this that we have been able to find, consists of the photograph shown below (figure 5). In this photograph, dating from 1938, Professor Van Dantzig is giving one of his lectures at Delft Polytechnic School. On his table there are two string models. One of them can be identified as model nr. 5 from Series IV in the Schilling catalogue and shows a hyperbolic paraboloid. On the blackboard one sees some of the equations defining this surface.

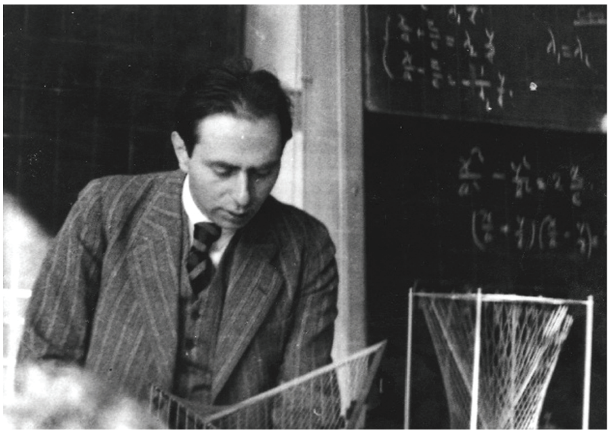


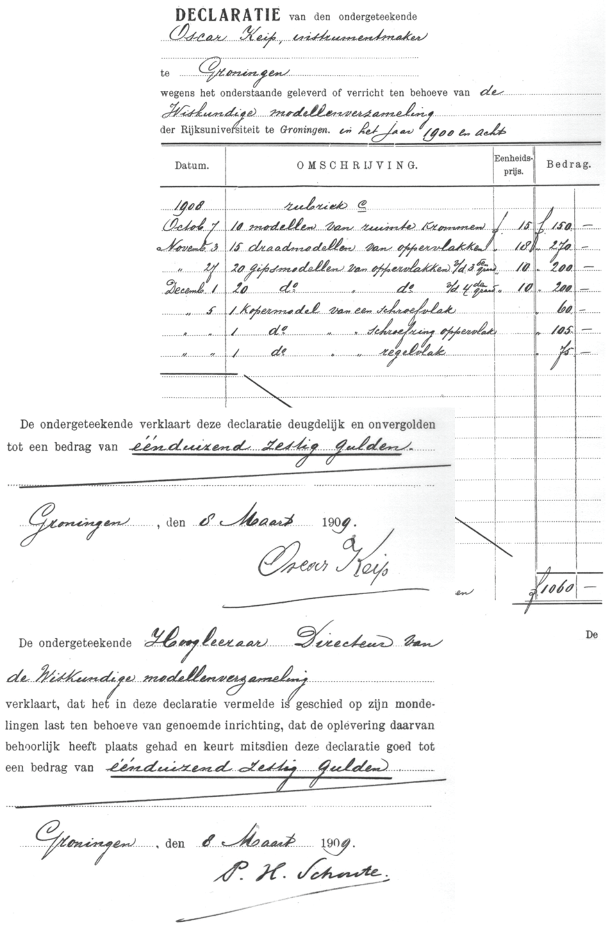
*fig. 5 Van Dantzig during one of his lectures at Delft Polytechnic in 1938.*[2]

The work of the author's thesis (Polo-Blanco, 2007) focused on the collection of models at the University of Groningen which we briefly describe next.

## The collection at the University of Groningen

The collection of physical models at the University of Groningen consists of about 150 pieces and several questions concerning them were investigated. Some of these questions were: When where these models bought? Who decided to buy the models? What was their use? A search of the archives of the city of Groningen revealed the answer to the first two questions. In the financial records of the faculty, it was discovered that the Department of Mathematics received a yearly subvention of *f* 100,-- for buying mathematical models. Schoute spent this budget of 100 guilders between 1882 and 1912. A bill for one of these purchases is shown in figure 6. Schoute's successor Barrau continued, after Schoute's death in 1913, to buy models for another three years.

As for the use of the models in Groningen during that time, we wondered for instance whether the models were shown during lectures. Also, were the models appreciated for their artistic value? Where were the models kept? Were they on display? In trying to find information about this, the author interviewed seven alumni of the University of Groningen who studied mathematics between 1930 and 1960. From the interviews, one can draw some conclusions. Although it seems that J.C.H. Gerretsen (professor of geometry in Groningen between 1946 and 1977) showed models during some of his lectures, their didactical use does not seem to have been impressive enough as to remain in the mind of these alumni. However, a document found in the archives of the Faculty of Mathematics and Sciences shows that Gerretsen was at least interested in the maintenance of the mathematical models. This document[3] concerns the move of the office of Gerretsen and the Mathematical Library to the Mathematical Institute on the Reitdiepskade in 1957. It shows Gerretsen's plan for this new building. One of the seven items where Gerretsen describes the new structure of the building states that (translated from Dutch) *the room behind the lecture room is destined to become the room for the models.*



**DECLARATIE** van den ondergeteekende Oscar Keip, instrumentmaker

te Groningen

wegens het onderstaande geleverd of verricht ten behoeve van de Wiskundige modellenverzameling der Rijksuniversiteit te *Groningen.* in het jaar 1900 en acht

| Datum. | OMSCHRIJVING. | Eenheids-prijs. | Bedrag. |
|---|---|---|---|
| 1908 | rubriek C | | |
| Octob. 7 | 10 modellen van ruimte krommen | f. 15 | f. 150 — |
| Novemb. 3 | 15 draadmodellen van oppervlakken | 18 | 270 — |
| „ 27 | 20 gipsmodellen van oppervlakken 3de graad | 10 | 200 — |
| Decemb. 1 | 20 d° d° 4de graad | 10 | 200 — |
| „ 5 | 1 Kopermodel van een schroefvlak | | 60 — |
| „ „ | 1 d° „ „ schroefring oppervlak | | 105 — |
| „ „ | 1 d° „ „ regelvlak | | 75 — |
| | | | f 1060 — |

De ondergeteekende verklaart deze declaratie deugdelijk en onvergolden tot een bedrag van ééndduizend zestig gulden.

Groningen, den 8 Maart 1909.

Oscar Keip

De ondergeteekende Hoogleeraar Directeur van de Wiskundige modellenverzameling verklaart, dat het in deze declaratie vermelde is geschied op zijn mondelingen last ten behoeve van genoemde inrichting, dat de oplevering daarvan behoorlijk heeft plaats gehad en keurt mitsdien deze declaratie goed tot een bedrag van ééndduizend zestig gulden.

Groningen, den 8 Maart 1909.

P. H. Schoute.

*fig. 6 Bill for the acquisition of mathematical models by G. Schoute in 1908.*[4]

However, it seems from these interviews that G. Schaake, Gerretsen's predecessor, was not interested in the didactic aspects of the models, and was probably not interested in the models at all. Some questions remain unanswered. What was the location of the collection of models before 1957? Lectures were given on the first floor of the Academy Building. According to de Jager, this was the case until 1957, when an Institute of Mathematics was formed at the Reitdiepskade. As the mathematics lectures were given in the Academy Building until 1957, one could conjecture that there was already a collection of models kept in this building when it caught on fire in 1906. Since the Groningen collection consists mostly of models with Schilling's stickers and not Brill's (like the collections of other Dutch universities), one may guess that a major part of the mathematical models was destroyed in the fire of the Academy Building, especially the models of the Brill collection, and that the big purchase of 1908 came from the wish to replace the previous collection.

Another part of the work in the author's thesis consisted on making a complete inventory of the Groningen collection and a restoration of the plaster models. As a result of the former, a web page was designed[5] where a photograph of each model appears, accompanied by a short mathematical explanation. As for the restoration of the plaster models, the sculptor Cayetano Ramírez López took care of that in 2005. The project, funded by the University Museum and the Department of Mathematics, consisted of a restoration and a cleaning of almost eighty plaster models. One of these models is shown in figure 7, both before and after the restoration. For more information the reader may consult the complete report on the restoration (Ramírez López, 2005).

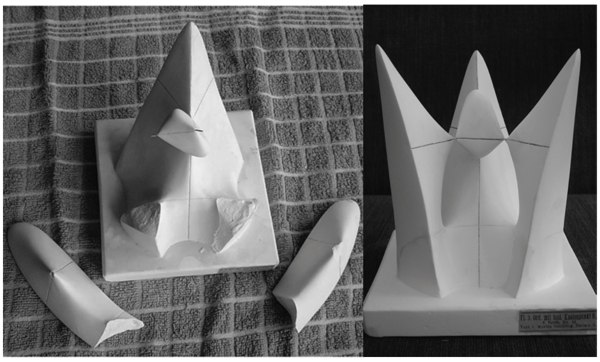


*fig. 7 Plaster model before and after the restoration.*

## Conclusion

The German collections of mathematical models were very popular among the Universities in Europe at the end of the nineteenth century, in particular at Dutch universities. Although the models were originally advertised as *Models for the higher mathematical instruction*, the account presented in this text shows that the pedagogical purpose of the models from the Groningen collection was almost forgotten already by the 1930s.

However, we find some evidence of how this old tradition can be resumed today. Research concerning the models in Groningen carried out as part of students master thesis is an example of this (Top & Weitenberg, 2011). Another one can be found in Bussi et al. (2010), where an account is given of the reproduction of many classical instruments as digital models using dynamic geometry software like Cabri, as well as their use as teaching material. But, as is pointed out:

> There is no claim that concrete models and dynamic instruments may be replaced by their digital copies with no loss. [...] a deep analysis of the changes (if any) in both didactical and cognitive processes when a concrete object is replaced by a digital copy is yet to be performed (Bussi et al., 2010).

We believe that both physical and computer manipulatives, as well as other meaningful representations, might help students to construct geometric concepts and discover some of their properties, and hope to encourage teachers to benefit from them by recapturing this classical tradition.

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## About Irene Polo

Irene Polo Blanco graduated in Mathematics in 2002 from the University of Groningen and from the University of the Basque Country (Spain). In 2007 she received her Ph.D. from the University of Groningen under the supervision of Jan van Maanen, Marius van der Put and Jaap Top. She currently has a teaching position at the University of Cantabria (Spain) in the group of Didactics of Mathematics. From May till August 2011 she obtained a Spanish research grant which she enjoyed working at the Freudenthal Institute in Utrecht.

## Noten

[1] http://opc.uva.nl.
[2] Photograph of Van Dantzig, personal archives G. Alberts (1938).
[3] Plan for the structure of the new Mathematical Institute building in Reitdiepskade, May 1957, Groningen Provincial Archive, Inv. Nr. 94.
[4] *Register of effected payments in 1908.* Groningen Provincial Archive, Inv. Nr. 487.
[5] http://www.math.rug.nl/models.

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