On geometry and discrete mathematics

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Samenvatting

De afgelopen jaren waren op het gebied van het wiskunde-onderwijs twee trends duidelijk waarneembaar, zoals met name bleek op de opeenvolgende ICME-conferenties. De eerste was een dalende interesse voor de meetkunde in veel landen, de andere was en is een stijgende belangstelling voor discrete wiskunde.
De auteur benadrukt het belang van zowel meetkunde- als discrete wiskundeonderwijs, en zo mogelijk in een vroeg stadium.
Het belang daarvan wordt aangetoond met een recente toepassing: de compact-disc.

Summary

Regular visitors of the ICME-conferences will be aware of two trends in math-education that seem to be present in many countries: a diminishing interest in the teaching of geometry and more and more interest in discrete mathematics. The author is strongly in favour of paying more interest to both subjects in the mathematics curriculum, and shows an application from recent years: the compact disc.

It is a great pleasure for me to express some thoughts on mathematical education on the occasion of the eightieth birthday of a great mathematician and educator. In a week or so it will be 35 years ago that I first heard a lecture by professor Freudenthal. The topic was *projective geometry* [2]. Let me tell you a didactical gem from this lecture which I have never forgotten. To explain the idea of axioms for points and lines (and to prepare for duality) Freudenthal took as an example the chesspiece "knight", which in Dutch is referred to as a "horse". The moves which a horse may make do not change if somebody breaks off the horse's head. So, it is not its appearance but the rules which it satisfies, which characterizes the horse.

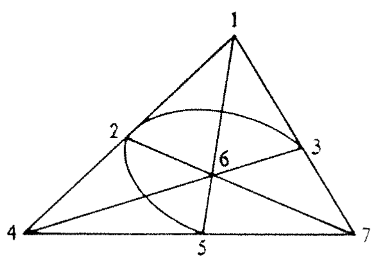

Fig. 1. The seven-point plane.

Figure 1 shows my favorite projective geometry. In these lectures on projective geometry I learned what a *field* is: an early occurrence of the idea of the geometric approach to algebra in teaching (cf. [4] p. 225). Despite this early initiation into geometry it was only via a long detour that I came to be partly a geometer myself (1983: editor of *Geometriae Dedicata*, a journal founded on the initiative of H. Freudenthal in 1972).

My first contact with the field of research (and discussion on mathematical education was at the colloquium. *"How to teach mathematics so as to be useful"* (cf. [9]) organized in Utrecht in August 1967 bij Freudenthal. Of course the title appealed to me very much indeed. The most important educational idea which I picked up at this meeting was based on the contribution by A. Révuz (cf. [9]). It is not surprising to me that Révuz is one of the people to whom Freudenthal extends particular thanks in the preface of 'Mathematics as an Educational Task" ([3] p. ix). The idea, which I have used with success in my *discrete mathematics* courses, is as follows.

Split the class into groups of two or three students. Each group discusses and tries to solve the problem sets which are handed out weekly. The main point of the scheme is giving problems for which the necessary methods and theorems have *not* been treated yet in class! Discrete mathematics is particularly suited for this idea since it has many problems which can be solved (often in a clumsy and much too long way) without much background knowledge. This method should be used much more often. It gives students the fun of discovery and it has as most important effect that, when a theorem is treated in class, several students immediately recognize its usefulness.

My first official contact with ICMI was again due to Freudenthal. Ten years ago he managed to talk me into preparing a contribution to ICME III. One of the souvenirs I have of this event is the following publication (cf. [6] Ch. IV):

大学および大学院における数学教育

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From this I quote the following statements on trends in mathematical education:
"The most significant change was the disappearance of geometry."
and
"A more recent trend is the introduction of a selection of topics from discrete mathematics in the early years."
At the same meeting M.F. Atiyah (cf. [1]) in his invited address came to the conclusion that "the virtual demise of Geometry in schools and universities . . . (is) . . . most unfortunate for a variety of reasons." It seems that it takes a long time for such observations to register. Four years later, at ICME IV, Freudenthal in his plenary address ([5] p. 1-7) observed " The mathematised spatial environment is geometry, the most neglected subject of mathematics teaching today." The case of discrete mathematics developed differently. At ICME IV the "Special Mathematics Topics" section had six contributions, three of which were of a combinatorial or algorithmic nature. There I had the opportunity to show the great educational possibilities of a course in *coding theory* (cf. [7] p. 299-303). Not only is the motivation provided by exciting things like satellite pictures, or more recently the compact disk, an advantage but even more useful is the fact that tools from very many parts of mathematics are necessary to learn this field. In fact, explaining the Reed-Muller codes which were used in the Mariner Mars mission (1969), I pointed out that "the increasing popularity of finite geometries is partly due to many applications in coding theory."

At ICME V and more recently at the ICME symposium on "The Influence of Computers and Informatics on Mathematics and its Teaching" (Strasbourg 1985) it became obvious that discrete mathematics is on its way to becoming an important subject in the early years of mathematics curricula for students of mathematics and of computer science. The educational value of *self-discovery* in this area was pointed out above (also see [3] Ch. VI). In the U.S.A. the courses in discrete mathematics tend to be a hodge-podge of all kinds of knowledge, much of which (e.g. sets, groups, logic) belongs in other courses. Since I do not wish to give a talk with no real mathematics in it, I shall give an example of what I consider discrete mathematics to be. Furthermore I believe this example has educational value of a different nature. A *strongly regular graph* with parameters (v, k, λ, μ) (see figure 2) is a graph with v vertices, k edges on each vertex, such that any two vertices which are joined (resp. not joined) by an edge have exactly λ (resp. μ) common neighbours.

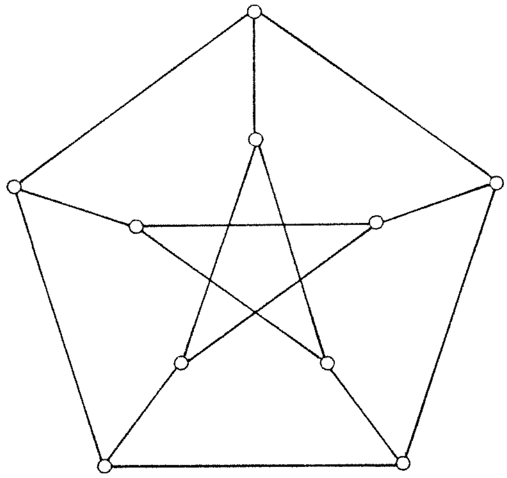

Fig. 2. The Petersen graph.

The example in figure 2 has $v = 10, k = 3, \lambda = 0, \mu = 1$. It is useful (but not surprising to the student) to show that *counting arguments* lead to restrictions on the parameters. E.g. Exercise: prove that $k\,(k-\lambda-1) = \mu\,(v-k-1)$. What *is* surprising (and indeed many students are fascinated by this part) is the following. Number the vertices from 1 to v. Define the matrix $A = (a_{ij})$ by $a_{ij} := 1$ if vertex i and vertex j are joined, and $a_{ij} := 0$ otherwise. The *eigenvalues* of A impose severe restrictions on the parameter sets (v, k, λ, μ) for which a strongly regular graph possibly exists. Here is a totally unexpected use of linear algebra. Just as in coding theory one finds in discrete mathematics examples showing how necessary it is to have many standard mathematical tools (algebra, probability theory, number theory, geometry (!)) at one's disposal.
Now, one could ask what use a computer scientist could have for finite geometries. Let me show a recent example (cf. [10]). Digital optical disks are a new storage medium which can store tremendous amounts of information. Such a disk has a thin reflecting coating of tellurium. To write on the disk, a laser is used to melt submicron pits in the tellurium at specified positions, changing those positions from "state 0" to "state 1". The disk is read with a weaker laser comparing reflectivity of positions. The drawback of this method is the *"write-once"* nature: the pits cannot be removed. Of course, we know this phenomenon from punched cards. It appears as if such a memory can only be used once. Suppose we wish to store one of the numbers 1 to 7 and wish to do this four consecutive times. Using a standard (write-once) binary memory we would need four sequences of three bits (one sequence for each usage), i.e. twelve positions of memory. However, it can be done with *seven* positions (cf. [8]). We return to figure 1.
Here are the first two rules:

Usage 1: Store number i by making a pit at point i of the plane;

Usage 2: To store number i:
- (a) do nothing if there is a pit at i;
- (b) make a pit at the *third* point of the line (i, j) if there is a pit at $j \neq i$.

The rules for reading the memory are obvious. The rules for usages 3 and 4 of course again depend on the state of the memory and on the *geometry*! It is probably more amusing for the reader to find such rules himself, so I leave it to him (or her), thus closing my talk on my themes of self-discovery, geometry and discrete mathematics.

References

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[10] Rivest, R.L. and A. Shamir, *How to Reuse a "Write-Once" Memory,* Information and Control *55* (1982), 1-9.