Towards a new curriculum

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Samenvatting

In zijn bijdrage aan het OW & OC-symposium op 2 mei 1986 houdt Peter Hilton een pleidooi voor een vernieuwing van het wiskunde-curriculum.
Discrete wiskunde – met name 'handig tellen', 'recurrente betrekkingen' en het 'meetkundig aspect' daarbij – zou een belangrijke plaats in een dergelijk curriculum moeten krijgen. Het element van verrassing zou stimulerend kunnen werken.

Introduction

In a recent forum conducted by the editor, Warren Page, in an issue of the College Mathematics Journal [2], the present author raised the following question. What are the principal themes and core results in discrete mathematics to compare with those of the differential calculus? For all mathematicians would agree that the notions of instantaneous rate of change as a limit, of the derivative of x^n, of the linearity of the derivative, of the Leibniz rule, of the chain rule, and of the method of locating, distinguishing and calculating maxima and minima are central to the development of the calculus: but there do not seem to be obvious candidates for such a central role in a discrete mathematics course.
Many have responded to that challenge, notably Norman Biggs. Others have already described in some detail their plan for a discrete mathematics course [3], from which one might attempt to deduce their answer to the question. In this article, I give my own, tentative answer to a perhaps easier question; namely, what are certain key strategies and techniques in discrete mathematics? I do not believe I could myself attempt an answer to my original question without first confronting this easier question.
I make no attempt here to be comprehensive. In particular, I shall have little to say about the role of computers and about teaching the idea of a finite algorithm. I will be largely concerned with combinatorics and its relation to arithmetic, algebra, geometry and probability. I will present my point of view through what are, I hope, illuminating and interesting examples rather than through an attempted rationale for the adoption of certain general principles.
It may be argued that some (or all!) of my examples are unsuitable for presentation in a 'service' course to students. I respond, first, by pointing out that my examples are intended for consumption by the readers of this article, not by such students. However, I might also attempt a defence of the entirely mathematical nature of these examples by claiming that to rely on examples of interest to students who are supposed to be largely insensitive to mathematics itself is to pursue a will o' the wisp. I propose to discuss this (very different) question of pedagogical strategy at greater length elsewhere.

The strategies and techniques discussed in this article are *clever counting*, *recurrence relations*, and the *geometrical viewpoint*, in its most general sense. An important point to be brought out is that, while two mathematical arguments may be equally compelling from a logical point of view, one may be far superior in the understanding it conveys. It is essential that students (and mathematicians!) should understand *why* some statement is true – it is not enough simply to be rationally obliged to accept *that* it is true. Thus, for example, many assertions of a combinatorial nature involving binomial coefficients may be proved either algebraically or combinatorially; the combinatorial proof usually gives the deeper insight. An interesting example of an arithmetical statement admitting an elegant combinatorial proof is that which asserts that the product of r consecutive positive integers is divisible by r!. It is only necessary to remark that

$(k+1)(k+2)\ldots(k+r) / r! = \binom{k+r}{r}$, and hence is an integer. No simple algebraic proof seems to be available, although we may of course, proof the Pascal identity $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$ algebraically and thence prove that $\binom{n}{r}$ is an integer by induction on n. A further nice example of an arithmetical fact which admits an elegant combinatorial proof is provided by the following composite enunciation on special binomial coefficients:

$$\text{(i) } n \mid \binom{2n-2}{n-1}; \text{ (ii) } (2n-1) \mid \binom{2n-1}{n}; \text{ (iii) } (4n-2) \mid \binom{2n}{n}.$$

For the three quotients are equivalent forms of the n^{th} Catalan number, c_n, which is the number of ways of associating the sum of n objects. Indeed, from (i) and (iii) above one gets the very mysterious statement:

$$\text{If } n=0 \text{ or } 4 \bmod 6, \text{ then } 2(n+1)(2n-1) \mid \binom{2n}{n}.$$

The author is especially indebted to his colleague and friend Jean Pedersen for her role in developing the material of this article. Many of the examples are drawn from our book [5], and I am very grateful to her for allowing me to discuss these here in a rather different context. She also drew my attention to the rich vein of mathematical ideas inherent in Pascal's Triangle, and pointed out how the inspiration for mining this vein might be regarded as geometrical. That theme constitutes the content of [6].

Clever Counting

We discuss here the famous *Euler totient function* $\phi(n)$, the number of positive integers less than n and prime to n, or, alternatively, the number of residues mod n prime to n. Simply to compute $\phi(n)$ for successive values of n reveals no obvious pattern, thus:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\phi(n)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8	16	6	18	8

However, we may proceed systematically as follows. First, we compute $\phi(p^m)$, where p is prime. We observe (i) to be prime to p^m is just not to be divisible by p; (ii) it is obvious that there are p^{m-1} numbers $\leqslant p^m$ and divisible by p; (iii) thus there are $p^m - p^{m-1}$ numbers $<p^m$ and not divisible by p. We have shown that:

$$\phi(p^m) = p^{m-1}(p-1). \qquad (1)$$

Second, we establish a multiplicative law,

$$\phi(kl) = \phi(k)\phi(l), \text{ if } k,l \text{ are mutually prime.} \qquad (2)$$

To establish this, write the numbers from 1 to kl in a rectangular array:

$$A = \begin{matrix} 1 & 2 & \ldots & l \\ l+1 & l+2 & \ldots & 2l \\ 2l+1 & 2l+2 & \ldots & 3l \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ (k-1)l+1 & (k-1)l+2 & \ldots & kl \end{matrix}$$

We now argue (i) that any column of A consists *either* of numbers all of which are prime to l *or* of numbers none of which is prime to l; (ii) the last column contains all the residues mod k, since l is prime to k; (iii) hence *each* column contains all the residues mod k; (iv) a residue is prime to kl if and only if it is prime to both k and l.
Observation (i) establishes that there are $\phi(l)$ columns consisting of numbers prime to l, and then observation (iii) establishes that there are $\phi(k)\phi(l)$ entries in A which are prime to both k and l. Observation (iv) now establishes (2).
Armed with (1) and (2) we can calculate $\phi(n)$ easily. We find the prime factorization:

$$n = p_1^{m_1} p_2^{m_2} \ldots p_r^{m_r}$$

of n and deduce that:

$$\phi(n) = p_1^{m_1-1}(p_1-1)p_2^{m_2-1}(p_2-1)\ldots p_r^{m_r-1}(p_r-1).$$

Thus, for example,

$$12012 = 2^2.3.7.11.13$$

so: $\phi(12012) = 2.2.6.10.12 = 2880$

Notice the following strategies: (i) in calculating $\phi(p^m)$ we count the numbers we're *not* interested in first; (ii) we prove a structural result (2), which is more general but actually simpler than what we need, and which relates to the numbers we *are* interested in.

Another function whose calculation may proceed the same way is $\sigma(n)$, the sum of the positive integers which are factors of n. Again, a table is unrevealing:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\sigma(n)$	1	3	4	7	6	12	8	15	13	18	12	28	14	24	24	31	18	39	20	42

However, as we point out below, it is easy to show that:

$$\sigma(p^m) = \frac{p^{m+1}-1}{p-1} \qquad (3)$$

and

$$\sigma(kl) = \sigma(k)\sigma(l) \quad \text{if } k,l \text{ are mutually prime} \qquad (4)$$

Indeed, (3) follows from summing the progression $1+p+\ldots+p^m$, while (4) follows from the observation that, k and l being mutually prime, a factor of kl is *uniquely* expressible as the product of a factor of k and a factor of l.
The quantity $\sigma(n)$ is especially interesting in the search for *perfect* numbers, that is, numbers n which are equal to the sum of their proper factors: in our terminology, these are numbers n such that $\sigma(n) = 2n$. Notice that the function $\sigma^1(n)$ which gives the sum of the *proper* factors of n is not a good function to handle, since it does *not* satisfy the crucial multiplicative condition corresponding to (4), and hence cannot be so readily calculated. It is not difficult to see that the only *even* perfect numbers are the numbers $2^{p-1}(2^p-1)$, where 2^p-1 is a (Mersenne) prime; it is not known if there are any odd perfect numbers.

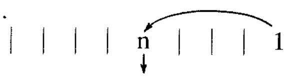
Recurrence Relations

We discuss in this section the question of *perfect mixups*, defined to be permutations (of finite sets) in which every element is moved. Let F_n be the number of perfect mixups of n objects, designated 1,2,...,n, and let P_n be the probability of a perfect mixup; thus:

$$P_n = F_n/n! \quad (5)$$

It is plain that $F_1 = 0$, $F_2 = 1$, $F_3 = 2$; for (2,1) is a perfect mixup of 1,2, and (2,3,1), (3,1,2) are perfect mixups of 1,2,3. Rather than seeking an expression in closed form for F_n (or P_n), we will aim to obtain a recurrence relation. To this end, let C_1 be the set of perfect mixups of n objects in which 1 appears in the n^{th} place. It is clear, by symmetry, that if $|C_1|$ is the number of objects in C_1, then $F_n = (n-1)|C_1|$; for any of $(n-1)$ objects can appear, in a perfect mixup, in the n^{th} place with equal likelihood. We separate C_1 into two disjoint sets, C'_1 and C''_1, as follows. A mixup (in C_1) goes into C'_1 if n appears in the first place, and into C''_1 if it does not. Now it should be clear that the mixups in C'_1 are in one-one correspondence with the perfect mixups of $(n-2)$ objects, so that $|C'_1| = F_{n-2}$. Further, given a perfect mixup in C''_1, we may put 1 where n occurs and thus obtain a perfect mixup of $(n-1)$ objects (see figure), so that:

$$|C''_1| = F_{n-1}.$$



We conclude that:

$$F_n = (n-1)(F_{n-1} + F_{n-2}),\ n \geq 3. \quad (6)$$

In view of (5) this is equivalent to:

$$P_n = \frac{n-1}{n} P_{n-1} + \frac{1}{n} P_{n-2},\ n \geq 3, \quad (7)$$

or

$$P_n - P_{n-1} = -\frac{1}{n}(P_{n-1} - P_{n-2}). \quad (8)$$

The recurrence relation (8) is very easy to handle, yielding by iteration:

$$P_n - P_{n-1} = \frac{(-1)^n}{n(n-1)\ldots 3}(P_2 - P_1).$$

But $P_2 = \frac{1}{2}$, $P_1 = 0$, so:

$$P_n - P_{n-1} = \frac{(-1)^n}{n!},\ n \geq 2. \quad (9)$$

It is interesting to observe that the recurrence relation (6) was obtained by a combinatorial argument, whereas the (simpler) relation (9) required algebraic manipulation and does not seem to correspond to any intuitive combinatorial fact. (This contrast between combinatorial and algebraic reasoning is seen, for example, in the Pascal identity $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$, which may be proved either by considering the selection of r objects from (n+1) objects, one of which is marked, so that $\binom{n}{r}$ is the number of selections *excluding* the marked object, while $\binom{n}{r-1}$ is the number *including* it; or by replacing $\binom{n}{r}$ by $\frac{n!}{r!(n-r)!}$ and manipulating, as suggested earlier. (We will discuss this aspect further in the next section.)

Reverting to (9) we quickly conclude that:

$$P_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^n}{n!}, \quad (10)$$

that is, the n^{th} convergent to $\frac{1}{e}$. Some interesting points emerge at this stage: (i) P_n oscillates about $\frac{1}{e}$, being alternately below and above it; (ii) P_n is fairly insensitive to the value of n – for example, $P_9 - P_8 = -0.0000027$ and P_9 differs from $\frac{1}{e}$ by less than 0.0000003. These facts do not seem to be intuitively evident.

Our result shows that if the cards from 2 decks are turned over simultaneously, one by one, the probability of the same card turning up in both decks at some stage is approximately $\frac{e-1}{e} = 0.63212$. The fact that the deck has 52 cards is largely irrelevant – the probability would be essentially the same for a deck of n cards for any $n \geq 9$.

Geometry and Combinatorics

It is our claim that geometry provides an essential link between discrete and continuous mathematics and that we most easily and most profitably conceptualize through geometrical configurations. We will give three examples to illustrate these points.

A method for calculating π

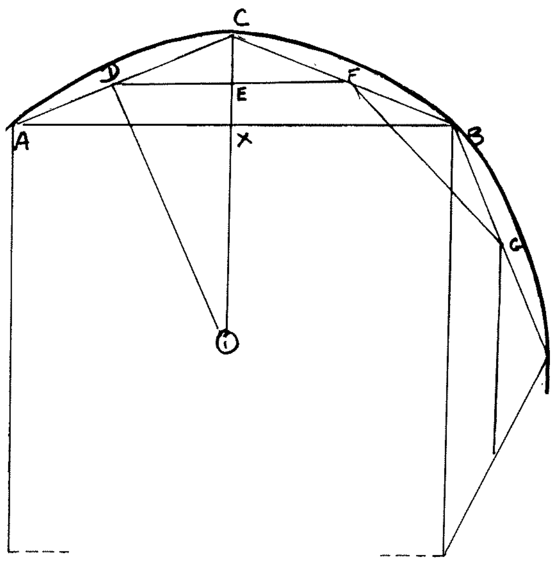


Figure 1

Consider figure 1. This is to be thought of as obtained as follows. We start with a regular 2^n – gon AB... (here n = 2) with centre 0. We draw the circumcircle and let C be the midpoint of the arc AB. If D, F are the midpoints of CA, CB, then it is easy to see that DF is the side of a regular 2^{n+1} – gon, also with centre 0, with the same perimeter as our original 2^n – gon.

Now let R_n, r_n be the circumradius and inradius of the 2^n – gon.

Then:

$$OC = R_n,\ OD = R_{n+1},\ OX = r_n,\ OE = r_{n+1}. \quad (11)$$

It is immediately obvious that:

$$r_{n+1} = \tfrac{1}{2}(r_n + R_n). \quad (12)$$

Also, since the triangles ODE, OCD are similar, we have $\frac{OD}{OC} = \frac{OE}{OD}$ or, by (11),

$$R_{n+1} = (r_{n+1}R_n)^{\frac{1}{2}} \quad (13)$$

Of course, $r_n < R_n$, so that (12), (13) describe r_{n+1}, R_{n+1} as arithmetic and geometric means and we have the pattern:

$$r_n \quad\quad r_{n+1} \quad\quad R_{n+1} \quad\quad R_n \quad (14)$$

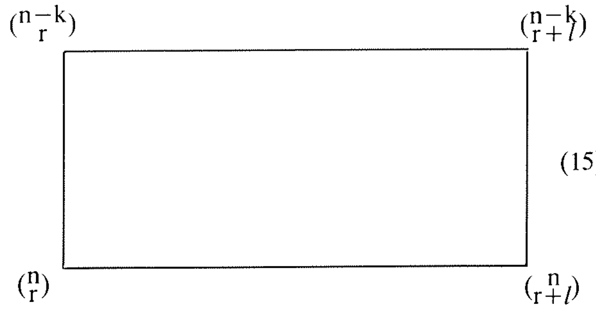
The picture (14) and the relations (12), (13) make it plain that the sequence $\{r_n\}$ increases to a limit l and the sequence $\{R_n\}$ decreases to the same limit l.
Let us assume that the perimeter of the 2^n – gon is 4. Then $R_2 = \frac{1}{\sqrt{2}}$, $r_2 = \frac{1}{2}$. In the limit we obtain a circle of circumference 4 and hence of radius $\frac{2}{\pi}$. Thus if we start with the values $R_2 = \frac{1}{\sqrt{2}}$, $r_2 = \frac{1}{2}$, and proceed to obtain R_n and r_n by iterated use of (12, 13), we will obtain sequences tending to $\frac{2}{\pi}$. This is, of course, a procedure very easily executed by a computer.

Patterns in Pascal's Triangle

n											
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1
r	0	1	2	3	4	5	6	7	8	9	10

Figure 2

Consider figure 2. This is Pascal's Triangle justified on the left. Thus the entry at depth n and r places to the right is the binomial coefficient $\binom{n}{r}$. Recognizing patterns in Pascal's Triangle in a splendidly creative exercise (see [6]). Here we show one pattern which shows up best in the justified version of the triangle. Let us consider a 'rectangle' of entries in Pascal's Triangle, with vertices labeled $\binom{n}{r}$, $\binom{n-k}{r}$, $\binom{n}{r+l}$, $\binom{n-k}{r+l}$; thus:



(15)

We say that the *weight*, $w = w(n,r;k,l)$, of this rectangle is the 'cross-ratio',

$$w = \binom{n}{r}\binom{n-k}{r+l}/\binom{n-k}{r}\binom{n}{r+l} \quad (16)$$

We obtain the *reflexion* of the rectangle (15) by interchanging the roles of k and l, and we may prove theorem 1.

Theorem 1: The weight of a rectangle is invariant under reflexion.

$$w(n,r;k,l) = w(n,r;l,k).$$

For example, $w(10,2;4,3) = \binom{10}{2}\binom{6}{5}/\binom{6}{2}\binom{10}{5} = \frac{1}{14}$; and
$w(10,2;3,4) = \binom{10}{2}\binom{7}{6}/\binom{7}{2}\binom{10}{6} = \frac{1}{14}$.
Before proving theorem 1, let us enunciate another property of the weight, which the authors of [6] also conjectured as a result of observation and experiment. We regard $\binom{n}{r}$ as the pivotal vertex of the rectangle (15) and we consider the family of rectangles we obtain by sliding the pivotal vertex along the 45° axis parallel to the hypotenuse of Pascal's Triangle (we may call this the axis $n-r$ = constant).

Theorem 2: The weight of a rectangle is constant along the axis $n-r$ *= constant.*
For example, $w(10,2;4,3) = \frac{1}{14}$, while $w(11,3;4,3) = \binom{11}{3}\binom{7}{6}/\binom{7}{3}\binom{11}{6} = \frac{1}{14}$.
The proofs of theorems 1 and 2 are very easy using the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.
For then we quickly see that:

$$w(n,r;k,l) = \frac{(n-r-k)!(n-r-l)!}{(n-r)!(n-r-k-l)!}\ , \quad (17)$$

and the right side of (17) is visibly symmetric in k and l and, for fixed k,l, a function of $(n-r)$. Thus we use geometry to suggest a conjecture about combinatorial constructs, and algebra to prove it. It would then be reasonable to conjecture that the weight should itself have some combinatorial significance. Of course, there is an obvious *algebraic* generalization of these two theorems, in which we take an arbitrary function f of the non-negative integers and consider the 'Pascal f-triangle', in which the entry in position (n,r) is $\frac{f(n)}{f(r)f(n-r)}$.

An interesting example of this generalization is furnished by the *Gaussian polynomials* – also known as q-*analogues* of binomial coefficients.
Here $f(0)=1$, $f(n)=(1-q^n)(1-q^{n-1})\dots(1-q)$, $n \geq 1$, and

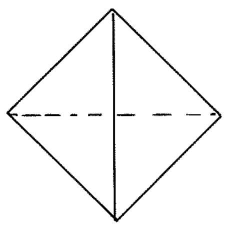
$$\left[{n \atop r}\right] = \frac{f(n)}{f(r)f(n-r)}, \quad 0 \leq r \leq n.$$

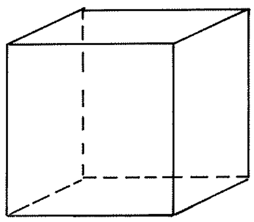
It is obvious that $\left[{n \atop r}\right]$ is a polynomial in q, but this follows from f(0) and the *Pascal* q-*identity*

$$q^r \left[{n \atop r}\right] + \left[{n \atop r-1}\right] = \left[{n+1 \atop r}\right], \quad 1 \leq r \leq n,$$

by induction on n. (Compare our earlier 'proof' that the binomial coefficients are integers!)
It is obvious, by L'Hôpital's Rule or the Pascal q-identity, that $\left[{n \atop r}\right] \to \binom{n}{r}$ as $q \to 1$; and it is further obvious that Theorems 1 and 2 remain true if $\binom{n}{r}$ is replaced by its q-analogue $\left[{n \atop r}\right]$.
It is worth mentioning that the Gaussian polynomials are important in the (combinatorial) theory of partitions.

The formulae of Euler and Descartes for polyhedra
Let us understand, in the first instance, by a polyhedron a 2-dimensional rectilinear figure made up of *vertices*, *edges* and faces, which is homeomorphic to the sphere. It is thus a closed rectilinear surface and the faces which are polygons are put together by identifying a side of one with a side of another to form an edge of the polyhedron. Two simple, but important, examples are the *tetrahedron* and the *cube*:





Let V,E,F be the number of vertices, edges, and faces, respectively, of the polyhedron. Thus, for the tetrahedron, V = 4, E = 6, F = 4, while, for the cube, V = 8, E = 12, F = 6. The celebrated formula of Euler asserts that:

$$V - E + F = 2. \tag{18}$$

Now Descartes had, much earlier (of course), introduced the idea of the *angular defect* at a vertex of a convex polyhedron and the *total angular defect*, Δ, of the polyhedron, being the sum of the angular defects at each of the vertices. If v is a vertex of the polyhedron, then the angular defect at v is the difference $2\pi-$ (sum of the face angles of the polyhedron at the vertex v); notice that Euclid had shown that this quantity is always positive for a convex polyhedron. For the regular tetrahedron the angular defect at each vertex is π, and for the cube it is $\frac{\pi}{2}$. Descartes proved the remarkable result that, for *any* convex polyhedron,

$$\Delta = 4\pi. \tag{19}$$

In fact, Euler did not know Descartes' result, and reached his conclusion (18) quite independently. Polya [7] showed that the two statements (18), (19) are equivalent; using Polya's argument, the author and Jean Pedersen proved a more general result relating to any closed rectilinear surface. Thus we do not require that the surface be homeomorphic to a sphere, and we do not require that it be convex (we simply allow negative angular defects). Our conclusion was [4]:

Theorem 3: For any closed rectilinear surface,
$\Delta = 2\pi(V - E + F)$.

Proof
We proceed by 'clever counting'. We defined Δ by counting by vertices; but we could equally well count by faces. If we count the sum of *all* the face angles by vertices we see easily, from the definition of the angular defect at a vertex, that this sum A is $2\pi V - \Delta$,

$$A = 2\pi V - \Delta. \tag{20}$$

Now let us count by faces. We suppose that there are F_j faces which are j-gons. Then $F = \Sigma F_j$, and $\Sigma j F_j = S$, the total number of sides. Since each edge on our surface is obtained by putting together two sides, we see that S = 2E, so that:

$$\Sigma F_j = F, \quad \Sigma j F_j = 2E. \tag{21}$$

Finally, we observe that the sum of the angles of a j-gon is $(j-2)\pi$. Thus the sum of all the face angles, counting by faces, is given by:

$$A = \Sigma F_j\,(j\text{-}2)\pi,$$

or, using (21),

$$A = 2\pi E - 2\pi F. \tag{22}$$

The theorem follows immediately from a comparison of (20) and (22).

It should be noted that Euler's formula does not require that the polyhedron be rectilinear, so long as it is broken up into faces with adjacent faces sharing a common edge; faces and edges may be curved. Thus Euler's formula has to do with *combinatorial* notions (in fact, as we know, it really has to do merely with *topological* notions). On the other hand, Descartes' notion of angular deficiency is genuinely geometrical in nature, and could not be applied to a polyhedron which had curved faces. Thus theorem 3 is remarkable in linking a combinatorial concept with a geometrical concept.
If one wished to extend the idea of angular deficiency to the case of a curved surface, or, indeed, even to higher dimensions, one would find oneself led to the celebrated Gauss-Bonnet theorem. A beautiful account of this development is given in [1].

Conclusion

We hope that these examples illustrate both the diversity of method and application in combinatorics and also the possibility of some kind of systematization. This very diversity should be a source of added interest in presenting a course centered on combinatorics, even if it makes the organization of such a course more difficult.

Related to the diversity is another feature which we may notice from our examples, and which was not deliberately inserted, let alone fostered. There is an element of *surprise* in all these examples which should be stimulating to students. It has to be admitted that it is not easy to incorporate such an element into a basic calculus course! Indeed, the present author has often argued for the early introduction of the theory of functions of a complex variable into the curriculum in order to inject some excitement into the proceedings. Perhaps combinatorics has the potential to provide this vital ingredient of a successful learning experience.

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