Visualizing Summation Formulas via L – Shape Geometry

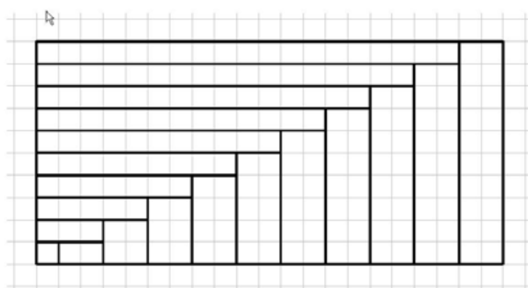
The purpose of this article is to let students represent six summation formulas of the form $\sum_{i=1}^{n} f(i) = g(n)$ geometrically as growing rectangles. In the notation $\sum_{i=1}^{n} f(i) = g(n)$, the term $f(i)$ stands for the subunit to be added at the i^{th} stage, and the term $g(n)$ is the corresponding product expression at this stage. Representationally, the left-hand-side of the expression $\sum_{i=1}^{n} f(i) = g(n)$ corresponds to the "Area as a sum" of the growing rectangle and the right-hand-side corresponds to the "Area as a product" of the growing rectangle for each task. Students may be provided 1 cm × 1 cm grid papers with colored pencils or 1 cm × 1 cm paper cutouts. 1 inch color tiles or cubes could be used as well.

For each task below, students may be asked to represent first the subunits $f(i)$ on the grid paper (or with the cubes or tiles) and then add "$f(i)$"s so that they generate a rectangle. The rectangle condition will help them establish the right-hand-side of the expression $\sum_{i=1}^{n} f(i) = g(n)$ for each task.

Task – 1: Summation of Counting Numbers $f(i) = i$.

One possibility is that students may come up with odd integers that are represented as rectangles of dimensions $1 \times i$ and even integers that are represented as rectangles of dimensions $\frac{i}{2} \times 2$ (see fig.1). The area of the growing rectangle as a product

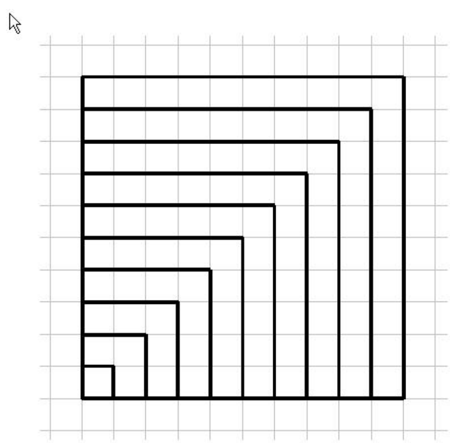
at the n^{th} stage $= \begin{cases} \frac{n}{2} \times (n+1), n \text{ is even.} \\ \frac{(n+1)}{2} \times n, n \text{ is odd.} \end{cases}$



$$f(i) = i, g(n) = \frac{n(n+1)}{2}$$

Task – 2: Summation of Odd Integers $f(i) = 2i - 1.$

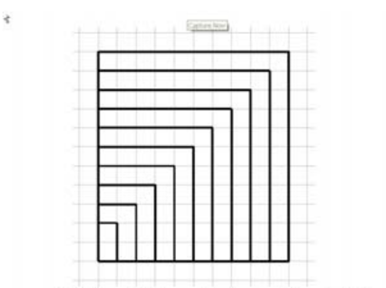
The instruction “Add them so that they generate a rectangle” would result in students’ representation of the odd integers as symmetric L – shape figures at each stage (see fig. 2). The area of the growing rectangle as a product at the n^{th} stage $= n \times n.$



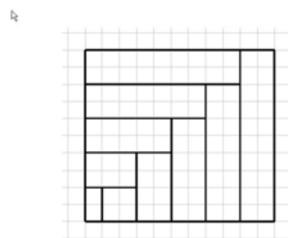
$$f(i) = 2i - 1, g(n) = n^2$$

Task – 3: Summing Even Integers $f(i) = 2i.$

In this task, students may be expected to generate multiple ways of representing the summation. One possibility is that even integers are represented as nonsymmetric L – shape figures (see fig. 3a), an extension of the odd integer patterns (see fig. 3b). Another possibility is the representation of the even integers as rectangles of dimensions $i \times 2$ (see figs. 3c). The area of the growing rectangle as a product at the n^{th} stage $= n \times (n+1)$ in both representations.



$f(i)=2i$, $g(n)=n(n+1)$

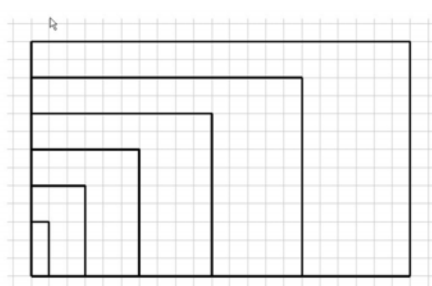


$f(i)=2i$, $g(n)=n(n+1)$

Task – 4: Summation of the Numbers of the Form $f(i) = 3i^2$.

One possibility is that each $f(i)$ is represented as a nonsymmetric L – shape as in figure 4. The area of the growing rectangle as a product at the n^{th} stage

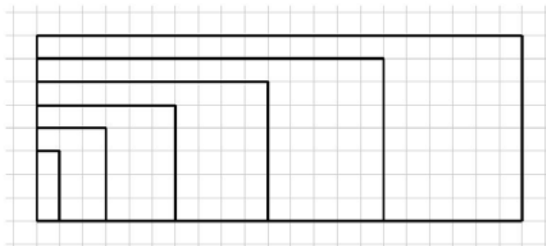
$= (2n+1) \times \left[\dfrac{n(n+1)}{2}\right]$.



$f(i)=3i^2$, $g(n)=(2\text{n}+1)\left[\dfrac{n(n+1)}{2}\right]$

Task – 5: Summation of the Numbers of the Form $f(i) = \dfrac{3}{2}i(i+1)$.

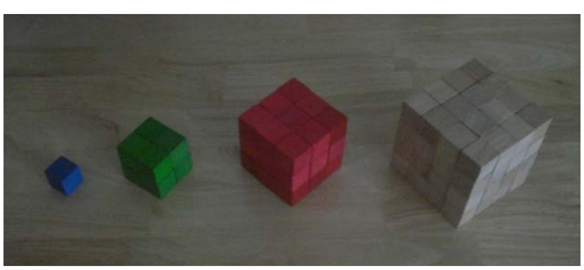
Students may be expected to represent each $f(i)$ as a nonsymmetric L – shape so that they generate a growing rectangle (see fig. 5). The area of the growing rectangle as a product at the n^{th} stage $= (2n+1) \times \left[\frac{n(n+1)}{2}\right]$.

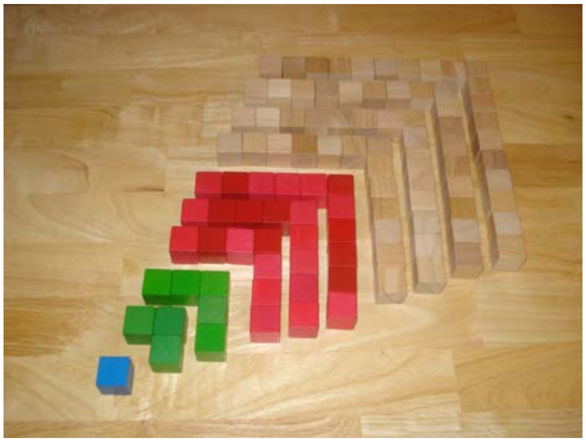


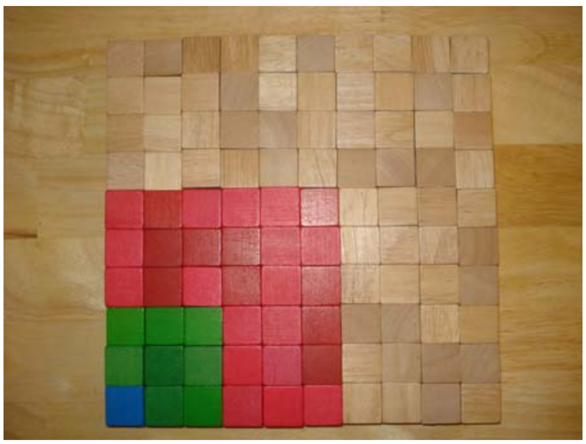
$$f(i) = \frac{3}{2} i(i+1), \, g(n) = (n+2)\left[\frac{n(n+1)}{2}\right]$$

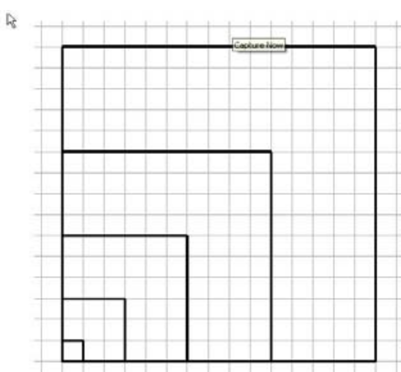
Task – 6: Summation of the Cubes $f(i) = i^3$.

One possibility relies on the fact that each i^3 (see fig. 6a) can be decomposed into exactly *i* consecutive odd integers that are symmetric L – shapes (see fig. 6b). Each perfect cube is represented as a big symmetric L – shape figure made of symmetric L – shape figures. When added together, they generate a growing square of side length $\frac{n(n+1)}{2}$ at the n^{th} stage (see figs. 6c-d). By using the result of Task – 1, students may obtain $\left[\frac{n(n+1)}{2}\right] \times \left[\frac{n(n+1)}{2}\right]$ as the area of the growing square as a product at the n^{th} stage.









$$f(i)=i^3,\ g(n)=\left[\frac{n(n+1)}{2}\right]^2$$

Suggestions: Students may be asked to draw a table that organizes information for each task. Table related to Task – 1 below is one of the possibilities. Students may also be asked to write down or talk about the patterns they see on their table.

Stage Number	Number of Tiles Added at Each Stage	Area of the Growing Rectangle as a Sum	Area of the Growing Rectangle as a Product	Total Area of the Rectangle
1	1	1	1×1	1
2	2	1 + 2	1×3	3
3	3	1 + 2 + 3	2×3	6
4	4	1 + 2 + 3 + 4	2×5	10
5	5	1 + 2 + 3 + 4 + 5	3×5	15
6	6	1 + 2 + 3 + 4 + 5 + 6	3×7	21
n	n	$1 + 2 + \ldots + n$	$\frac{n}{2} \times (n+1)$, if n is even $\frac{(n+1)}{2} \times n$, if n is odd	$\frac{n(n+1)}{2}$