Maartens College

Math Enrichment Course

Fractals

Levels A and B

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Introduction
Math Enrichment Courses at Maartens College

What is Math Enrichment?
The goal of the Math Enrichment courses is to stimulate students' enthusiasm for mathematics, both for the weaker and brighter mathematics students. The courses are generally related to practical applications of mathematics and they illustrate how math is part of everyday life. Topics vary from codes to optical illusions to fractals to logic to the four-colour problem, etc. In this way we hope to inspire students and convince them that mathematics is a subject to be enjoyed and on top of that, that it has many practical uses. Especially for weaker students, an excursion into some areas of mathematics that they are not aware of can be stimulating, more so than doing another course on the fractions, for instance, that they do not understand. By experiencing success in another area of mathematics, the student can become more motivated and possibly more confident to tackle those areas with which he/she has difficulties.

Set-up at Maartens College
The Math Enrichment courses are part of the Small Groups Subjects (SGS) programme. Each student in the first two years of the bi-lingual classes (TTO1-2) or in the first three years of the International School (MYP1-3) takes part in this programme. The student is required to take one course per term (3 terms in the year, approximately 12 weeks per term). Of the three courses taken in a year, the student must choose one Math Enrichment course, one IT course and one from Drama, Debate or Current Events. The lessons are given to groups of 12 to 15 students who come from different classes and streams.
The Math Enrichment programme connects closely to the way mathematics is taught in the curriculum of the TTO and IS, where students receive regular opportunities for doing mathematical investigations and projects.

Student Reactions
A great majority of students is surprised to find out that 'this is also math'. In the course of time, the math Enrichment courses have become firmly embedded into the SGS programme. Generally students are enthusiastic about the courses. It has occurred that students' mathematical confidence has grown by finding out that there are mathematical topics he/she is able to do. Next to that, the extra challenge for real wiz-kids, for instance in the RSA course, has been greatly appreciated by both students and parents.

How to use these course descriptions
The Math Enrichment courses were developed from material on internet, mathematical books and sometimes magazines or newspapers. The description given is a framework that has been developed over the years for a series of about ten lessons for each course. In the booklets made for each course, the description is included together with material that has been used in the course. The descriptions leave plenty of room for flexibility, where you can delete some topics and add others. They are meant as a basic platform from which you can develop the course further. It is work in progress.

Course descriptions have been made for the following courses:

Level A:

- Computers and Binary Codes
- Cryptography
- Fractals
- Infinity
- Networks
- RSA Codes (probably the most challenging course)
- Topology

Level B

- Fibonacci numbers
- Fractals
- History of Codes
- Logic
- Map Colouring

Level C

- Islamic Art
- Tessellations

A Personal Note as Conclusion

I have enjoyed developing these courses and they have often brought mathematics to life for me as well as for the students. In that sense also, I was sometimes on one level with the students when I was trying to find my way in the development of a new course. I have found that students appreciate this type of interaction; there is a certain type of excitement that disappears when I give the course more often (even if the quality of the course does improve with time).

Each course always ends with a student evaluation of themselves and how they did in the course and an evaluation of the course itself. I have had many valuable ideas from students on how to improve the courses.

I hope that working with this material will bring you as much satisfaction as it has brought me. Good luck!

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Fractals (level A and B)

Note: The lessons have been written up for the level B course. For level A, I delete part of the lessons and do 'fractional dimension' instead (described at end of lesson series)

Lesson 1

- What is ME?
  - Relates Math to the world outside
  - Topics you do not cover in your normal Math lessons
  - Fun!
- Materials
  - Binder with section + plastic portfolios or
  - Notebook + folder for copies
- Assessment
  - Most important is class participation + curiosity
  - Also homework (sometimes, a bit) and sometimes a quiz to check your understanding.
  - Certificates at end of (IS) term (show)

- What is a fractal?
  - Show some pictures of fractals in nature: cauliflower, fern leaf, broccoli, crystals, shells. What do they have in common?
  - Two properties:
    - Self-similarity: a shape that repeats itself on a different scale
    - Pattern keeps repeating forever
  - These are the two basic properties of fractals
  - Can you think of some others:
    - In body: arteries, nervous system, skeleton
    - Trees, mountains, clouds. Show the hex tree fractal
    - Coastlines, lakes, rivers
    - There are also mathematical fractals like the Mandelbrot set or the Julia set. (Show pictures)
  - All kinds of natural things can be modelled by fractals: there is order in chaos!
  - Fractals are the basis of computer animations. Show how fractals are used to create the mountain and lake landscape.
- Design your own fractal.

Lesson 2

- Students work on their fractal and finish it for homework
  - It should be coloured in
  - They have to write a description of how their fractal is made

Lesson 3

- Gather desks in the centre of the room, students put their fractals on it.
  - Each student has a turn to present their fractal and tell how they have made it
  - Then discuss whether it is really a fractal (often students repeat patterns, but they forget to change the scale)

- Hand out worksheet on Sierpinski's triangle with bisection
    - Students start to draw the Sierpinski triangle
    - Explain the concept of iterations
- Homework:
    - Finish the drawing
    - Student looks on internet for a fractal that he/she will present and explain to the class. They are asked to:
        - Bring a print of the fractal for everyone to see
        - Take a fractal that they feel they <u>can</u> explain
        - Explain to the class how the fractal is made

Lesson 4

- Students present the fractals they have found.
- Look at Sierpinski's triangle for patterns by making a table. Students try to fill in the table in pairs first and they discuss what the pattern is. Let them try to figure out what the values would be at iteration 10, and for any iteration *n*. Then write on board and discuss together.

Iteration	Number of unshaded triangles	Area of the big triangle that is not shaded
0	1	1
1	3	3/4
2	9	9/16
3	27	27/64
4	81	81/256
n	3^n	$(3/4)^n$

- Note: Some students may not have worked with exponents yet, they need some extra explanation here!
- Discuss what happens to the amount of area that is unshaded as the iterations increase. It becomes smaller and smaller – does that make sense? Will it ever become zero?
- Students start on Sierpinski triangle with trisection. Finish the drawing for homework.

Lesson 5

- Same procedure as last lesson: Look at Sierpinski's triangle for patterns by making a table. Students try to fill in the table in pairs first and they discuss what the pattern is. Let them try to figure out what the values would be at iteration 10, and for any iteration *n*. Then write on board and discuss together.

Iteration	Number of unshaded triangles	Area of the big triangle that is not shaded
0	1	1
1	6	2/3
2	36	4/9
3	216	8/27
4	1296	16/81
n	6^n	$(2/3)^n$

- Discuss the differences between the two tables. In which one is the number of unshaded triangles growing faster? In which one the unshaded area? Does that make sense?

- Students start drawing a different type of fractal: the Koch snowflake. They receive the instruction sheet and start the drawing, finishing it up for homework

Lesson 6

- Check the drawing students have made for Koch's snowflake. Discuss how it will go on. Does it ever have to stop? Does it look like a snowflake?
- Investigation the perimeter of the snowflake:
    - Students make a table with headings: iteration number, number of sides, length of one side, perimeter
    - They will need to go back to their iterations to find the perimeters.
    - Students fill in the table in pairs (different pairs from last session?)
    - Students look for patterns and try to predict the perimeter of the tenth iteration and the n^{th} iteration.
    - Together, go over table and discuss:

Iteration	Number of sides	Length of side (cm)	Perimeter (cm)
0	3	9	27
1	12	3	36
2	48	1	48
3	192	1/3	64
4	768	1/9	85.33
5	3072	1/27	113.78
n	3×4^n	$9 \times (1/3)^n$	$27 \times (4/3)^n$

- Discuss what happens to the perimeter. Can it become infinitely big? How can an object that is enclosed still have an infinite perimeter??
- Students start on table for the area:
    - Make a table with the headings: iteration, number of triangles added, area of one added triangle, total added area, total area
    - One triangle counts as one square unit
    - Homework: try to fill in the table

Lesson 7

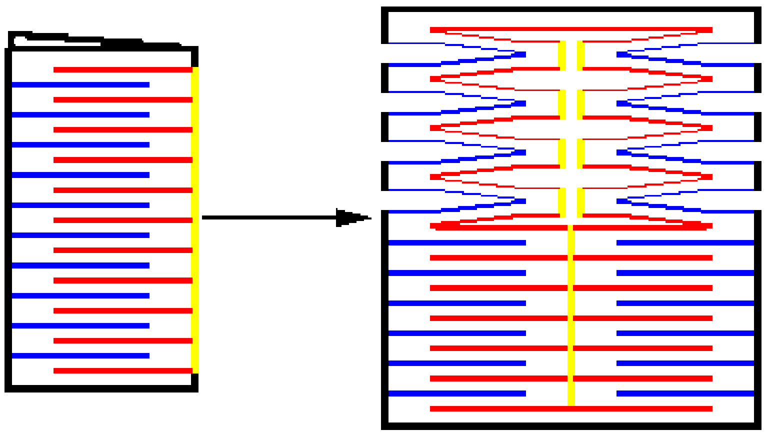
- Students receive one fourth of a sheet of A4 paper and are asked to make a hole in it so that they themselves can pass through. Let them try – discuss solution:

To start with, fold the paper in half. Next make a series of cuts in from the fold, making cuts roughly 1 cm apart, which stop roughly 0.5 cm from the edge.

Now make a series of cuts in from the other side. Make sure you cut between the first set of cuts (see diagram).

Finally, cut along the fold, but leave the two end segments intact.

If you unfold the paper, it should be two long zigzags. Pull it out and you will get a loop large enough to fit a person through.



The pattern of cuts and the resulting hole.

Discuss the relationship with the infinite perimeter of the Koch snowflake. How big a hole could you make in this paper using this process?

- Students check the tables they made for homework with a partner (a different one again?) and they try to find a pattern and, if possible, a formula for the total added area.
- Fill in table together and discuss patterns and results:

Iteration	Number of added triangles	Area of one added triangle	Total added area	Total area
0				81
1	3	9	27	108
2	12	1	12	120
3	48	1/9	5.33	125.33
4	192	1/81	3.37	127.7
10	786,432	1/43,046,721	0.0183	129.57
n	$3 \times 4^{n-1}$	$9 \times (1/9)^{n-1}$	$27 \times (4/9)^{n-1}$	

- The formulas can be a bit too difficult, but you can discuss the basic pattern: Do you think the area will also become infinitely big? Discuss that the boundary is the area of the circumscribed circle of the original triangle.
- No homework

Lesson 8

- Play the chaos game: remind students of what was said at the beginning that there is more order in chaos than we often see. We will do a game that seems chaotic and we will look for the pattern that comes out.
- Students need an overhead transparency each, with a marker, dice, a ruler or protractor and a small cut-out dot (punched out hole from hole puncher).
- The overhead transparency has three vertices of a triangle put in as three different coloured dots: green, red and blue. The triangle has the exact size of the one started out with for the Sierpinski triangle with bisection.
- Demonstrate on the board what students are expected to do:
    - Put the small cut-out dot anywhere on the overhead sheet as long as it is in the triangle
    - Throw a dice: 1or 2 is green, 3 or 4 is red and 5 or 6 is blue. Move you cut-out dot halfway to the point that has the colour of your dice.

- Do this 10 times, then mark the place where your cut-out dot is with a point. Continue the procedure, but now mark dots on your sheet (small points) instead of moving the cut-out dot.
- Homework: finish to at least 30 dots

Lesson 9

- Bring overheads with iterations of Sierpinski triangle drawn on it.
- Students show their work and continue with putting in more points.
- Put one student's work on the overhead, do you see any pattern?
- Put the next student's work on top, matching exactly the vertices of the triangles. Do you see pattern?
- Continue until all sheets are on the overhead projector. Usually you will see a pattern – depending upon the size of the group.
- Put the Sierpinski grids on top of the pile, notice that as you go up iterations, the pattern of points keeps on fitting nicely in the shaded triangles. (Note: students who just put points anywhere are immediately found out in this exercise!)
- Discuss why the Sierpinski triangle comes out of this random game. See also http://math.bu.edu/DYSYS/chaos-game/node3.html#SECTION00030000000000000000

Lesson 10

Evaluation and students write self-reflections

Fractional dimension (for the A-level course)

This follows well after Koch's snowflake and takes about 3 lessons:

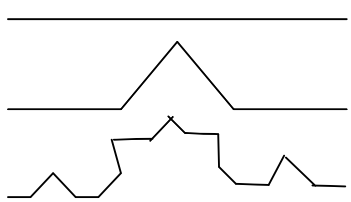
Lesson 1

- Take a part of the curve from Koch's snowflake:

Iteration 0 =

Iteration 1 =

Iteration 2 =



We have already discovered that the perimeter of Koch's snowflake is infinite as the number of iterations goes to infinity. What about the distance between points A and B, the endpoints of the line in iteration 0? It will also go to infinity. But are we still talking about a one-dimensional shape then? Can't you simply measure the length of a one-dimensional shape?

Take a look at different dimensions:

<u>One dimension</u>

A line: ___________________________

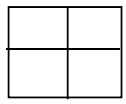
If you cut the line in half you get two pieces. If you cut it in thirds, you get three pieces. Etc.

Say r is the ratio you cut it in and N is the number of parts you get. Then, when: r = ½, N = 2 and when r = 1/3, N = 3. Etc.
Conclusion: We can see that Nr = 1. Always (in one dimension)

<u>Two dimensions</u>
Take a square:

If the sides are cut in half, how many pieces do you get?

Four pieces:



So when r = ½, N =4

If you cut it in thirds, how many pieces do you get?
When r = 1/3, N = 9

Conclusion: $Nr^2 = 1$

<u>Three dimensions</u>
Ask students to try this themselves first.
Then discuss using a cube and finding the relationship $Nr^3 = 1$
What pattern do you see? The dimension is the exponent of r.

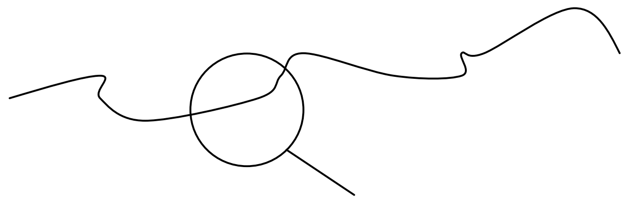
- Let's apply this to the Koch's snowflake curve:
  - When you divide the line in thirds you get 4 parts. So r = 1/3 and N = 4.
  - To find the dimension we now have to find the question mark in the formula: $Nr^? = 1$ or $4 \times (1/3)^? = 1$
  - To find the question mark, use trial and error on your calculator. Discuss why the number will be between 1 and 2. Also draw students' attention to the fact that as the exponent becomes higher, the outcome will actually be smaller.
  - To two decimal places, the value of the question mark is 1.26. This means that the Koch snowflake has a dimension of 1.26. A fractional dimension. Discuss how to interpret this.

- Students try to find the fractional dimension of Sierpinski's triangle with bisection and trisection. Note that when counting the number of shapes that result, we count only the unshaded triangles because they are copies of the one we started out with (the black triangles are simply 'holes' in the figure). Finish fopr homework.

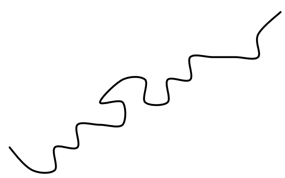
Lesson 2
- Check and discuss homework:
  - Sierpinski's triangle with bisection: r = ½ and N = 3. In $Nr^? = 1$, ? = 1.59 so that is the dimension of Sierpinski's triangle with bisection

    - Sierpinski's triangle with triisection: $r = 1/3$ and $N = 6$. In $Nr^{?} = 1$, $? = 1.63$ so that is the dimension of Sierpinski's triangle with bisection

- Why is fractional dimension useful? Think of measuring a coastline. Look at it from a distance, it might look like this:



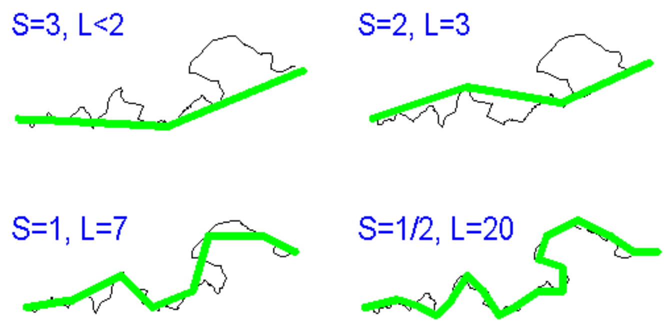
  If you take a closer look in the loop, however, the part that looks smooth might look like this:



  Then if we take a small part out of that one, again we will have more squiggles if we look at it more closely.
  We can continue like this until we reach the level of grains of sand, and if we look at those under a microscope, we will see that they to are not smooth.
  So what do we measure when we measure a coastline?
  Isn't a coastline actually infinitely long, just like the line in Koch's snowflake?

  Mathematicians have developed ways to find the fractional dimension of coastlines. (See article in the file)
  These fractional dimensions will always be between 1 and 2. Why?

- In class we are going to determine the fractional dimension of the coastline of France between Brest and Cherbourg (in Normandy)
    - Hand out maps. Students will need rulers to measure with
    - Use lengths of 8 cm to measure the coastline and count how many times the 8 cm ruler will fit in. (You will cut off a lot squiggles and sometimes go straight through the sea)
    - Then do the same with a 4 cm ruler
    - Then with a 2cm ruler
    - Then per cm.
    - The pictures at right give some idea of the intention of the exercise, where S in this case stand for the ruler size and L for the number of lengths that fit in
    - Students make a table with two columns: size of ruler and number of times the ruler fits in. Finish for homework.



Lesson 3

- Check tables made for homework
- Can you find r and N from your tables for this coastline?
- Let students think in pairs. Everyone writes an answer with sentence of explanation before going on.
- Discuss:
  - r = ½ because the ruler is halved each time
  - N is about 2.5, because the number of rulers that fit in is multiplied by 2.5 to obtain the new number of rulers that fit in when the rulers are halved in size.
  - Table (approximations)

Ruler size	Number of rulers
8cm	3.8
4cm	9.6
2 cm	22
1 cm	54

  - Students try to find the fractional dimension: $2.5 \times (1/2)^{?} = 1$.
  - The fractional dimension comes out to 1.32

What does this mean? If the fractional dimension is closer to 2 than to 1 – the coastline is more rugged.

Other possibilities:

- Continue worksheets with Sierpinski's carpet and the hexagonal gasket (especially for groups that find it difficult to find the patterns). You can make the same types of tables, with following results:
  - Sierpinski's carpet n^{th} iteration is 8^n unshaded squares and $(8/9)^n$ is the unshaded area
  - Hexagonal gasket n^{th} iteration is 6^n unshaded squares and $(2/3)^n$ is the unshaded area (just like Sierpinski with trisection)

  In the lesson series in which I used all of these, the students did complain that it was a bit boring.
- Jurassic park fractal
  This can be found on http://math.rice.edu/~lanius/frac/jurra.html (a print is in the file). It's an interesting fractal. I have tried it out with different classes, where each student would make a piece of the fractal and we put them all together into one big Jurassic park fractal. It is not easy to see how to put all the pieces together, and students have to be very precise in their drawing otherwise it will not fit together.
- Koch's Anti-snowflake
  There is a worksheet in the file. Can be an interesting addition after Koch's snowflake, again exploring perimeter and area of the iterations
- 3D fractals: I had design sheets once to make 3D fractals, but I lost them. If you can find any: the students enjoyed making them!