Reification as the birth of metaphor

A. Sfard
The Hebrew University, Jeruzalem

> Basically, I'm not interested in doing research and I never have been. I'm interested in *understanding*, which is quite a different thing.
>
> David Blackwell, referring to his work as a mathematician (Albers & Alexanderson, 1985, p.19)

1. Introduction: the elusive experience of understanding

Reflections of my own experience of understanding helped but only to some extent. I could remember myself as a graduate mathematics student passing exams without difficulty but often feeling that the ease with which I was doing things was not enough to give me the sense of true understanding. Some time later I was happy to find out that even people who grew up to become well-known mathematicians were not altogether unfamiliar with this kind of experience. For example, Paul Halmos (1985) recalls in his "automatography" (p.47):

> ...I was a student, sometimes pretty good and sometimes less good. Symbols didn't bother me. I could juggle them quite well ...[but] I was stumped by the infinitesimal subtlety of epsilonic analysis. I could read analytic proofs, remember them if I made an effort,and reproduce them, sort of, but I didn't really know what was going on.

Halmos was fortunate enough to eventually find out what the 'real knowing' was all about (Albers & Alexanderson, 1985, p.123):

> ... one afternoon something happened. I remember standing at the blackboard in Room 213 of the mathematics building talking with Warren Ambrose and suddenly I understood epsilon. I understood what limits were, and all of that stuff that people were drilling in me became clear. I sat down that afternoon with the calculus textbook. All of that stuff that previously had not made any sense became obvious ...

Clearly, what people call 'true' understanding must involve something that goes beyond the operative ability of solving problems and of proving theorems. But although a person may have no difficulty with diagnosing the degree of his or her understanding, he or she does not find it equally easy to name the criteria according to which such assessment is made. Many works have already been written in which an attempt was made to understand what understanding is all about (for a comprehensive and insightful survey of these see Sierpinska, 1993). In the present paper I will try to make another little step toward capturing the gist of this elusive something that makes us feel we had grasped the essence of a concept, a relation or a proof.

Let me begin with a few words on the way in which I tackled the question. My quest for a better understanding of mathematical understanding went in two directions. First, I tried to find out what contemporary theories of meaning have to say on the subject. I soon discovered that, as far as the issue of understanding is concerned, current developments in the psychology of mathematics go hand in hand with some of the most significant recent advances in linguistics and in philosophy. The applicability of the latter to the field of mathematical education was already noted by some researchers (e.g. Doerfler, 1991; Presmeg, 1992). In this paper I will show how the idea of reification, the basic notion of the conceptual framework in which I have been working for quite a long time now, combines with the new theories of meaning and mind. I hope to make it clear that the theory of reification is perfectly in tune with the latest philosophical and linguistic developments, and that much may be gained by tightening the links between the different fields. Such marriage of ideas will be the central theme of the next section..

My second move was to approach the people who, so I believed, could provide me with a first-hand information about the experience of understanding mathematics. I turned to research mathematicians. While choosing mathematicians as my subjects I was fully aware that this decision had some pitfalls. To begin with, I knew that what would be found in my subjects would not have to be generally true. Whatever the actual difference between the 'professionals' and 'laymen', however, it was my deep conviction that mathematicians' reflections on their own thinking might provide me with insights the importance of which would go beyond the question of professionals' understanding. Another difficulty had to do with the method I chose for collecting the data. Introspection, being inherently subjective, is not necessarily the best way to obtain a reliable information. However, since I was interested in the inner sensations related to the process of understanding rather than in any visible behaviors, I could think about no better means of inquiry than my interlocutors' metacognitive skills.

2. What the non-objectivist theory of meaning has to say

Be the problem of mathematical comprehension as unique as it may be, the issue of understanding is certainly not limited to mathematics and it would thus be a mistake to deal with particularities of the special case in question without first referring to the existing, quite rapidly developing, general theories of meaning. The fact that I said 'meaning' rather than 'understanding' is not a slip of the tongue. As I will try to explain, my reasons for stressing the aspect of meaning go well beyond the obvious fact that meaning and understanding are intimately related. It is the relatively new approach to human thinking, imagination and comprehension promoted by such writers as Lakoff (1987) and Johnson (1987) which compels me to treat the question of understanding as almost equivalent to the question of meaning.

This last statement, ostensibly not very far from the centuries-old consensus, would, in fact, stir protests among those philosophers and linguists whom Johnson classifies as 'Objectivist'. Objectivism, in this case, is a generic name for all those schools and theories which, either implicitly or explicitly, endorse the assumption that meaning is primarily a characteristic of signs and concepts, that it is a kind of externally determined 'cargo' carried by symbols and sentences. This simple basic belief proved to be powerful enough to give rise to an all-embracing paradigm, paradigm so broad that almost all the past and many recent schools of thought fall within its boundaries.

The basic tenets of the Objectivist outlook may be summarized in a few sentences. According to Johnson's critical account, Objectivism is grounded in the view that meanings are 'disembodied', namely that they are received by a human mind rather than shaped by it. Accordingly, understanding is conceived as "grasping the meaning" and thus as a process which mediates between an individual mind and the universally experienced, absolute, ahistorical realm of facts and ideas. To put it in a different language, understanding consists in building links between symbols and a certain mind-independent reality. Further, Objectivism presupposes that all knowledge is of propositional nature, namely is "conceptually and propositionally expressible in literal terms that can correspond to objective aspects of reality" (Johnson, 1987, p.5). Finally, the Objectivist paradigm is intimately related to the representational view of mind (Putnam, 1988) according to which "To know is to represent accurately (in one's head) what is outside the mind" (Rorty, 1979, p.3).

Objectivism, as conceived by Johnson, addresses the question of the way human beings create their understanding of the world rather than the question of the existence or the nature of this world.

The Objectivist paradigm is under growing criticism for the last several decades and is now being gradually abandoned by philosophers, linguists, and cognitive psychologists. Philosophers of science (Carnap, Kuhn, Feyrabend) may be those who gave the first decisive blow to the idea of "God's eye view" of reality. In cognitive psychology, a pressing reason for 'a thorough revision of our views on meaning and understanding is an obvious discrepancy between Objectivism and the broadly adopted constructivist approach to learning. The anti-Objectivist trend is strongly felt also in mathematics education:

> Given that mathematics educators almost universally accept that learning is a constructive process, it is doubtful if any take representational view literally and believe that learning is a process of immaculate perception (Cobb et al., 1992, p. 3)

One way to deal with this apparent dissonance is to reverse the Objectivist version of the relationship between meaning and understanding: while Objectivism views understanding as somehow secondary to, and dependent on, predetermined meanings, non-Objectivism implies that it is our understanding which fills signs and notions with their particular meaning. While Objectivists regard meaning as a matter of a relationship between symbols and a real world and thus as quite independent of human mind, the non-objectivist approach suggests that there is no meaning beyond that particular sense which is conferred in the symbols through our understanding.

In the view of this claim the question arises of the primary sources of our understanding. Moreover, if the meaning lies in the eyes of the beholder, should it be regarded as an entirely subjective matter? The way in which Lakoff and Johnson answer both these questions is where their truly original, imaginative contribution to the theory of meaning and understanding may be found. They devote much of their writings to a thorough description of a well-defined mechanism which turns even the most abstract of ideas into concepts filled with meaning. The possibility of shared, as-if-objective meanings stems from the fact that the functioning of this mechanism is subjected to certain universal laws.

Whereas the 'disembodiment' of meaning is the central motive of the Objectivist approach, Lakoff and Johnson took upon themselves 'to put body back into the mind'. The pivotal idea of their theory is that our bodily experience is the main, in fact the only, source of understanding. In this context, explains Johnson, 'experience' is to be understood in a very rich, broad sense as including basic perceptual, motor-program, emotional, historical, social, and linguistic experience. (Johnson, 1987, p.xvi). The physical and the experiential is the basis for even the purest, most sophisticated figments of our imagination. Moreover, even our reasoning is metaphorical in its nature. A careful look at the basic rules of reference will reveal that they have their roots in the physical experience of containment (being 'in' or 'out' of a certain space or set).

All this being said, the basic question now is how "the 'bodily' works up its way to the 'conceptual' and 'rational'." (ibid, p.xxi) In the centre of the answer given by Lakoff and Johnson stands a metaphor. Our conceptual system is a product of metaphorical projections of the bodily experience into the less concrete realm of ideas. Metaphor is understood here in a much broader sense than in traditional linguistics: it is not just a rhetoric form or a semantic gimmick. It is a mental construction which plays a constitutive role in structuring our experience and in shaping our imagination and reasoning. Basically, metaphor is a mapping from one conceptual domain into another. Thus, 'the essence of metaphor is understanding and experiencing one kind of thing in terms of another'.

Another question that requires thorough analysis concerns the mechanism of metaphorical projection. According to Lakoff and Johnson, the vehicle which carries conceptual structures from one domain to another is an embodied schema (known also as image schema).

Johnson defines embodied schemata as "structures of an activity by which we organize our experience in ways that we can comprehend. They are a primary means by which we construct or constitute order and are not mere passive receptacles into which experience is poured." (pp.29-30) Throughout our conscious existence we are engaged in the continuous activity of putting order into our manifold interaction with the world, in the never-ending attempt to make sense of the things we experience. "A schema is a recurrent pattern, shape, and regularity in, or of, these ongoing [ordering] activities". (p. 29) As such, the embodied schema is, first and foremost, dynamic in its nature; it epitomizes the process of organizing, of ordering, of pattern detection.

Unlike symbolic and linguistic expressions with which we communicate our knowledge to others and quite contrary to the Objectivist vision of knowledge, embodied schemata are usually non-propositional. This central characteristic is already reflected in the name given to these special mental constructs: they are image-like and embodied in the sense that they should be viewed as an analog reflection of the bodily experience rather than as factual statements we may wish to check for validity. The non-propositional nature of embodied schemata makes it difficult, sometimes impossible, to describe them in words. Only their entailments, the pieces of factual knowledge generated by the schemata, are amenable to verbal presentation. As to the schemata themselves, well, "while we may describe features of their structure propositionally using finite representations, we thereby lose our ability to explain their natural operation and transformations (ibid, p.23). We should keep this issue in mind while discussing the difficulty invariably experienced by mathematicians who try to communicate their highly abstract ideas to others.

If embodied schemata cannot be viewed as a mental counterpart of a system of factual statements, the question arises what are the cognitive means with which such schemata are handled. Here again, misled by our previous knowledge, we may easily slip into an oversimplified, distorted version. Mental images seem to be the natural alternative to the propositional structure. The idea that an embodied schema is, in fact, a mental image is even more convincing in view of the fact that both these cognitive structures have the same leading characteristics: they are analog and holistic. True, an embodied schema may be buttressed by a mental image, but there is a crucial difference between the two: whereas a mental image is always an image and is therefore full of details (that is why Johnson of something concrete calls it a 'rich image'), an embodied schema is general and malleable. It is but a skeleton with many variable parts which, being undetermined, cannot be visualized. This generality of an embodied schema is what gives it its structuring power and its capacity to encompass in one manageable mental construction a wide variety of our experiences. (In spite of an almost unbridgeable gap between Lakoff and Johnson's theory and the information-processing approach to cognition, one may be tempted to compare the idea of embodied schema to Minsky's (1975) concept of frame).

The initial question how bodily experience is metaphorically transmitted into a sphere of more abstract thinking has now got its answer: embodied schemata, originally built to put order into our physical experience, are 'borrowed' to give shape, structure, and meaning to our imagination.

The role of the metaphor (and metonymy) in mathematical and scientific thinking was already noted by many writers (see e.g. Pimm, 1987, 1990; Ortony, 1979). Lakoff and Johnson's theory, however, differs from all the previous works in two respects. First, it exceeds all the other approaches in the importance it ascribes to metaphors and to their impact on human thinking. Lakoff and Johnson's central thesis is that metaphors constitute the universe of abstract ideas, that they create it rather than reflect, that they are the source of our understanding, imagination and reasoning.

Second, the focus of the theory is on a special kind of metaphor, the metaphor the source of which is in our bodily experience. Thus, Lakoff and Johnson's central claim is that abstract ideas inherit the structure of the physical, bodily, perceptual, experience. In the next sections I will try to translate these ideas into the domain of mathematics. This special context will demonstrate with particular clarity that as far as our imagination is concerned, the mechanism of metaphorical projection is a doubly-edged sword.

On one hand, it is what brings the universe of abstract ideas into existence in the first place; on the other hand, however, 'the metaphors we live by' put obvious constraints on our imagination and understanding. Our comprehension and fantasy can only reach so far as the existing metaphorical structures would allow. Creative mathematicians, in order to make any progress, must often break beyond the demarcation line drawn by the bodily experience.

3. Objects in mathematician's mind: the metaphors that make a mathematical universe in the image of the physical world

To some of you, the idea that all mathematical abstractions are tightly connected to, and constrained by, the nature of our encounters with a physical reality would probably sound bizarre. Still, this is exactly the thesis I wish to promote in this paper. In advanced mathematics, at levels far removed from the physical reality, it may well be that the immediate source of a basic metaphor is another, lower-level mathematical structure.

Even so, and be the chain of metaphors as long as it may be, whatever is going on in our mind is primarily rooted in our body. The intelligibility of abstract objects stems from their being metaphorical projections of our bodily experience. It will be my goal in the following discussion to explicate the deep nature of the relationship between the abstract and the experiential and to show how the bodily aspects of our existence both enable and constrain our understanding. Mathematicians' accounts of their own quest after meaning will be the principal source of evidence. I'll confine myself to the material collected during three full-length (three hours and more each) semi-structured interviews with renowned mathematicians (one of them regarded as one of the most prominent in his domain in the world): a logician (let us call him ML), a set-theorist (ST) and a specialist on ergodic theory (ET). To make up for the small number of interviewees and the ill-balanced choice of disciplines (the study is only in its preliminary phase!), I will resort, here and there, to autobiographical writings and to the informal conversations I had with many other mathematicians.

Mathematical concepts with a human face: the metaphor of ontological object

Mathematical universe, populated by mathematical objects and animated by the manipulations which may be performed on these objects, can hardly be understood in any other way than as a metaphorical reflection of a physical world. Lakoff and Johnson (1980) explain the special strength of the 'metaphor of ontological object':

> Our experience of physical objects and substances provides a further basis for understanding... Understanding our experiences in terms of objects and substances allows us to pick out parts of our experience and treat them as discrete entities or substances of uniform kind.
> Once we can identify our experiences as entities or substances, we can refer to them, categorize them, group them, and quantify them and, by this means, reason about them. (p.25)

In the spirit of my earlier writings I will call this kind of metaphor structural. Listening to mathematicians talking about their ideas might be enough to realize that in mathematics, the structural metaphor is ubiquitous. To begin with, the language used in textbooks to describe the basic mathematical entities is clearly object-oriented: "A complex number is an ordered pair of...", 'A group is a set of elements together with binary operation such that...", "let's take a bounded region of an n-dimensional space..." . The names given to different mathematical entities and properties have clearly their roots in the world of material objects: a function may be increasing or decreasing, a field may be closed or open, a model or a theory may be saturated or stable. The fact that we use the word existence with reference to abstract objects (like in existential theorems) reflects in the most persuasive way the metaphorical nature of the world of abstract ideas. Greeno (1991) makes the metaphor of ontological object explicit when he compares understanding mathematics to "knowing one's way around in an environment and knowing how to use its resources." (p.175)

Metaphorical motifs appeared time and again in my conversations with mathematicians. In the answer to the question what happens in their minds when they feel that they have arrived at a deep understanding of a mathematical idea, they unanimously claimed that the basis of this unique feeling is not a manipulative power but an ability to "identify a structure that one is able to grasp somehow" (ST), or "to see an image" (ET), or to play with some unclear images of things" (ML). To put it in ET's words: "In those regions where I feel an expert, .. the concepts, the (mathematical) objects turned tangible for me." ST expressed his need of a metaphor explicitly (ST used the word 'metaphor' on his own accord; needless to say, I tried to formulate the questions to the interviewees in a theory-free language; at that particular stage it was not too difficult, as the idea of applying Lakoff and Johnson's framework to the analysis of mathematicians' understanding imposed itself on me as a result of the interviews):

> To understand a new concept I must create an appropriate metaphor. A personification. Or a spatial metaphor. A metaphor of structure. Only then I can answer questions, solve problems. I may even be able then to perform

> some manipulations on the concept. Only when I have the metaphor. Without the metaphor I just can't do it.

He proceeded with a description which left no doubt as to the bodily origins of the metaphors he had in mind. First, there was a spatial metaphor:

> In the structure, there are spatial elements. Many of them. It's strange, but the truth is that also my student has noticed it ... great many spatial elements. And we are dealing here with the most abstract things one can think about! Things that have nothing to do with geometry, (that are) devoid of anything physical ... The way we think is always by means of something spatial ... Like in 'This concept is above this one' or 'Let's move along this axis or along the other one'. There are no axes in the problem, and still ...

Spatial thinking is not the only way to conceive structure. ST told me about yet another kind of metaphor which appears in his mathematical reasoning: a personification. "Perhaps the most obvious ontological metaphors are those where the physical object is further specified as being a person", observed Lakoff and Johnson (1980, p. 33). Hadamard (1949) was probably the first to notice that a mathematical concept may sometimes be imagined as having a "human face", "a physionomy which allows to think of it as a unique thing, however complicated it may be, just as we see a face of a man". ST gave an even more colorful description:

> There is, first and foremost, an element of personification in mathematical concepts... for example yesterday, I thought about some coordinates ... (I told myself) "this coordinate moves here and ... it commands this one to do this and that." There are elements of animation. It's not geometric in the sense of geometric pictures, but you see some people moving and talking to each other."

In a similar vein, ML remarked:

> When I think about a fat man, I see (in my mind's eye) a fat man. Saturated model seems to me quite like that, like a padded guy.

The way mathematicians refer to the mental constructs with which they pave their way toward understanding often brings to mind the concept of embodied schema, the carrier of a metaphor. For example, Hadamard's term 'cloudy imaginery' is more aptly interpreted as an evidence for an appearance of embodied schemata than as a reference to a simple visualization. Hadamard himself uses the word 'schema' to describe this particular mental construct: "...every mathematical research compels me to build.... a schema, which is always and must be of a *vague* character (my emphasis) so as not to be deceptive." (p.77). The "vague character" is the leading characteristic thanks to which the embodied schema acquires its generality and its unifying power. According to Johnson, this is exactly the feature which is lacking in a 'rich image', namely in a simple visualization.

Even though all my interviewees remarked many times that they frequently resort to visualization (compare Dreyfus, 1991), they also stressed that pictures, whether mental or in the form of real drawings, are only a part of the story. They support thinking, but they do not reflect it in its all dimensions. Using the term introduced by Doerfler, I would say that images of any kind are but concrete carriers for the embodied schemata. The pictures mathematicians use to draw on a paper or on a blackboard serve a double purpose: they are 'something to think with' and they function as a means of communication. In spite of the obvious limitations of a picture as an expression of generality, both aspects are extremely important.

All the mathematicians I talked to said they just could not think without making pictures. All of them drew different shapes when trying to explain to me certain mathematical theorems or conjectures.

What it means that a person is 'intimately' familiar with a mathematical concept and how such familiarity affects reasoning

The most natural way to assess one's understanding of a mathematical idea is to estimate the easiness with which he or she reasons and discovers new facts about it. On the face of it, mathematical reasoning is always based on a sequence of inferences which, in a systematic way, derive new facts from what is given and known. In fact, however, there seems to be another mode of thinking about mathematical concepts, a mode which has little to do with systematic deduction. This other mode is much more difficult to describe and to explain, but it is this special way of thinking which, according to many mathematicians, is the ultimate evidence of deep understanding.

Like all the others, ET said explicitly, and more than once, that the ability to construct a proof, or even to use it to construct another argument, does not suffice to give him a sense of the 'true' understanding. Here is one of the many remarks he made to that effect:

> I can understand a theorem or a proof on such level that I become convinced about its validity. I can understand a theorem sufficiently to reproduce it in a classroom. All this is still not a sufficient evidence for me that I really understood. There is another level, where I can take a proof of one theorem and prove another theorem with the help of the ideas presented in this proof. Even then, I may still claim that I didn't arrive at the true understanding of the proof.

ET went so far as to claim that he did not fully understand some of the proofs which were his own creations:

> There are things or theories that I developed myself and still, I don' t understand them as deeply as I would wish to.

From different remarks made by the interviewees it was quite clear that for them, one of the best indications of understanding is the capability to sense that something is true in an immediate manner, without having recourse to a formal proof. This ability to arrive at properties of mathematical objects in a direct way may well be what brought Gauss to make the following statement: "I have had my results for a long time; but I do not know yet how I am to arrive at them" (quoted by Lakatos, 1976, p.9).

'Having a result' without knowing how it was obtained is perhaps the most striking phenomenon in the work of a mathematician. All my interlocutors have experienced it in the past and they tried to describe it to me in many ways, sometimes quite ingenuous. ST used the expression 'intimate familiarity' to describe the feeling that accompanies the type of understanding which makes it possible to have the direct insight into properties of mathematical objects. The personification metaphor surfaced again when he tried to explain this special ability of predicting behaviors of abstract constructs:

> When do you feel that you really understood something? It is only when you are perfectly certain, without having to check, that the things must be exactly the way they are. It's like in the case of an intimate familiarity with a person. With such a person you often know what he or she is going to do without having to ask... The [abstract] things have a life of their own, but if you understand them, you make predictions and you are pretty sure that you will eventually find whatever you foresaw ... Like with a person whom you really know and understand, (the mathematical construct) will perform a certain operation or will react in a certain way to your action. This intimacy is exactly what I had in mind: you know what is going to happen without making any formal steps. Of course, like in the case of a human relationship, you may sometimes be wrong.

The following remark by Johnson (1987) renders well the essence of such an 'intimate' understanding:

> ... understanding is not only a matter of reflection, using finitary propositions, on some preexistent, determinate experience. Rather, understanding is the way we 'have a world', the way we experience our world as comprehensible reality ... our understanding is our mode of 'being in the world' . . .Our more

> abstract reflective acts of understanding (which may involve grasping of finitary propositions) are simply an extension of our understanding in this more basic sense of 'having a world'. (p. 102)

The intimate understanding we are talking about is best explained through a comparison to the way people comprehend basic aspects of the physical world.

The 'experiential' comprehension gives people an ability to anticipate behaviors of material objects without reflection. Indeed, when in a blink of an eye we jump to save a leaning glass of water from falling, it is not because we have recalled the law of gravity, confronted it with empirical data at hand and made an appropriate inference. Our understanding expresses itself in the ability to know what is going to happen without even being aware of the way in which the prediction was made. Having this kind of understanding renders our method of handling abstract ideas all the characteristics which, according to Fischbein (1987) are typical of intuitive thinking: our knowledge is self-evident, coercive, global, and extrapolative.

At this point, the central question is what are the sources of this overpowering feeling of obviousness and inevitability of properties and relations which have not been deductively derived from known facts. How can a mathematician anticipate 'behaviors' of abstract structures which have never been seen before? It seems quite obvious that this special mode of reasoning, let us call it a direct grasp, becomes only possible when a metaphor has been constructed to give concepts their meaning. It is an embodied schema projected from another area which brings the anticipatory insight. After all, this schema is built on mathematicians' previous experience and thus its inner logic and other properties are inherited from this earlier experience (it would be in point here to make a reference to Johnson-Laird's (1983) work on human reasoning; metaphors seem to play a role similar to that which the British psychologist ascribes to the constructs he calls mental models).

This 'hereditary' mechanism which underlies the construction of metaphors has, obviously, some disadvantages. First, because of the experiential origins of the hierarchical sequence of metaphors, the different constraints on our imagination, these basic side effects of embodiment, are carried like genetic traits from one generation of abstract concepts to another. Some confinements may have to be alleviated to make the movement toward the more abstract ideas possible; nevertheless much of them will be preserved along the way and will continue to delimit mathematical thought. Second disadvantage has to do with the principle on which the direct grasp is based.

Once the abstract objects emerge and their embodied schemata are constructed, our abstract reasoning becomes much like the reasoning induced by a sensory perception: it is holistic, immediate and, above all, it is based on analogy rather than on systematical logical inference. The central role of analogy in the direct-grasp reasoning was brought to my attention by current references to 'similarity to known facts' made by all my interviewees when they tried to account for their ability to 'foresee' behaviors of mathematical objects. The way ET described the mechanism behind his ability to predict mathematical facts is quite typical:

> when you ask me whether something is true or not, I can think about it a moment... find a similarity to something else... and I can give you an answer out of the sleeve. And all this when I have no inkling about a proof.

The way analogy shapes and curtails mathematical reasoning is presented schematically in fig. 1. The example is rather elementary but it illustrates well the as-if-synthetic nature of abstract reasoning based on a metaphor of an experiential, perceptual origin.

Before I end this section let me remark that the type of thinking and the kind of understanding I presented do not have to appear in all creative mathematicians to the same extent. All my interviewees claimed unanimously, and, of course, independently, that there exists more than one 'kind of mathematical mind'. Their remarks may be summarized in a claim that there is a full spectrum of possibilities, at the opposite ends of which stand two basic 'styles' of mathematical thinking, styles which may be described as operational and structural. The operational types have highly developed manipulative skills and use them as a principal means in their quest after meaning. Having a metaphor which makes a mathematical object in the image of a real thing is the dominant need of a structurally minded mathematician.

For the latter type of thinker, the manipulative skills, the ability to draw a systematic argument, are sometimes quite secondary. For example, this was certainly the case with the prominent mathematician S. Lefschetz who, according to Halmos (1985) "saw mathematics not as a logic but as pictures. His insights were great, but his 'proofs' were almost always wrong." (p.87) The structuralists are more capable of the direct-grasp understanding than those who think and understand in the operational way. This is probably why the belief that the structural thinking is superior to the operational was implicit in the opinions of the mathematicians I talked to.

Source of the metaphor: a balance between material objects
Target of the metaphor: an equation (an equality between two formulae)

Reasoning:

(1) A fact known from the source domain (material world): a balance between two objects is preserved when the same change in mass is carried on both of them. Let us present it symbolically as a proposition:

P(A,B, CoM)

where A and B are objects and CoM is a change in mass.

(2) Metaphorical projection:
- formulae (F, E) are (represent) objects (F=A, E=B)
- equation is (an expression of) a balance between objects
- an operation on a formulae (OoF) is a change in mass of a material object (CoM),

namely OoF = CoM

(3) Inference:

$$\frac{\text{P(A, B, CoM)}}{\text{F=A, E=B, OoF=CoM}} \quad ==> \quad \text{P(F, E, OoF)}$$

namely: An equation (equality between two formulae) is preserved if the same operation is performed on both its sides

Fig. 1: Reasoning on equations based on the metaphor 'equality is balance'

To sum up the things that have been said in this section, metaphors impinge upon mathematical reasoning in a very special way: with the emergence of an embodied schema, thought processes may loose their purely analytical character. The metaphorical projections introduce quasi-synthetical elements into mathematical reasoning. New mathematical truths are no longer discovered through a systematic inference from axioms and definitions (are they ever discovered in this way?!); rather, they impose themselves upon a mathematician directly as obvious properties of a mathematical reality. When the abstract construct is supported by an image schema, the perception of its salient characteristics may become much like our perception of the properties of physical bodies: it is immediate, it is holistic, and it is not mediated by a long chain of inferences. It is this ability to grasp ideas in the direct quasi-synthetic way which, according to the mathematicians I talked to, gives them the feeling of a 'true' understanding. Even though there exist many different kinds of mathematical minds, the phenomenon of a direct grasp is probably known to this extent or another to the majority of creative mathematicians.

Platonism not only on weekdays: sense of obviousness, objectivity and inevitability

In the language introduced by David Hume, the upshot of the last section is that our knowledge about a mathematical realm is not always achieved just by investigating 'relations of ideas'. Quite often, a new truth is discovered (yes, discovered) as a 'matter of fact'. In the eyes of a person who feels that he or she 'really' understood an abstract idea, mathematical truth bears a synthetic rather than analytic character.

'The typical working mathematician is a Platonist on weekdays and a formalist or Sundays' claim Davis and Hersh (1981, p. 321). From what was said in the previous section it becomes clear that this 'practical' Platonism is not a matter of deliberate choice, of insufficient sophistication or of a lack of mathematical (or philosophical) maturity. It is because of the very nature of our imagination, because of our embodied way of thinking about even the most abstract of ideas, that we spontaneously behave and feel like Platonists. Our imagination and reasoning are limited by our sensual experience, and even if we can make a deliberate sortie beyond the constraint of the physical-world lenses, such move, being consciously imposed, may only be temporary. When not forced (by reason) to renounce Platonism on behalf of, say, formalism, our mind will immediately go back to its 'natural' state, the state of Platonic belief in the independent existence of the mathematical objects, the nature and properties of which are not a matter of human decisions.

Along the history, no new mathematical construct has gained full recognition until mathematicians could feel that, to put it in Davis and Hersh's words, it was as real for them as 'the Rock of Gibraltar or Halley's comet'. To arrive at such feeling, it was not enough to understand the inner logic of a definition and to recognize its consistency with all the other mathematical facts. What was necessary was an appropriate metaphor, a metaphor which would show that, in fact, the new idea did not violate the basic laws of the abstract universe. In the Platonic world of ideas, the term 'basic laws' has a very special meaning and signifies more than laws of logic. In the realm of material objects, all the events are determined by the laws of nature.

Phenomena such as a free fall of a stone thrown from a window are inevitable. Our feeling that the abstract universe is governed by similarly uncompromising, deterministic laws is inherent in the metaphorical way we construct the system of ideas.

A very interesting contribution to my insight into the relationship between understanding a concept and the belief in its objective existence was provided by ET. ET declared that being a religious person he fully adopts the Platonic views and he then stated that mathematical concepts which he understands well are conceived by him as referring to objects as real as "a leaf falling from a tree in a forest". For example, he thinks he knows well what an infinite set is and this feeling of understanding also means that he does "not doubt the existence of the actual infinity". On the other hand, ET does have doubts about real numbers, or rather about the set of all the subsets of the latter. What bothers him is the independence of the continuum hypothesis from the accepted axiomatic systems.

> Until a few years ago I was prepared to declare that our problem with the continuum hypothesis is that we did not formulate (understand] our system in the right way -- the way which would make it possible to decide in this way or another. I could not put up with the independence of the continuum hypothesis because it was my deep conviction that the set in question either exists or not.

ET's doubt stemmed, obviously, from the unclear status of a certain set with regard to the nature of its existence. His objections aptly substantiate the confining nature of metaphorical projections. The undecidability of the claim about the existence of an object, any object, either concrete or abstract, defies our basic experientially based intuitions: it implies that it would be legitimate to assume the existence of a number greater than x and smaller than x, and it would be equally admissible to presuppose it non-existence. But in our perceptual world, governed by the principle of tertium non datur, objects either exist or not and the question 'to be or not to be' can only be answered in one way, either 'yes' or 'no' Moreover, it is not up to us to decide about the answer. Incidentally, this example sheds much light on the difficulty mathematicians once had with accepting the idea of non-Euclidean geometries. All this shows the other edge of the metaphorical sword, namely the constraint which bodily experience puts on our imagination.

4. Reification as a birth of a metaphor and a crucial step towards understanding

If the meaning of abstract concepts is created through the construction of appropriate metaphors, then metaphors, or figurative projections from the tangible world into the universe of ideas, are the basis of understanding. As I already observed in the former Sections, (see also Sfard, 1987, 1991, 1992) the leading type of sense-rendering metaphor in mathematics is metaphor of an ontological object. In this last section I will deal very briefly with the intricate question of the way such metaphor is created and with the inherent difficulties which hinder this process. I have already discussed these issues quite thoroughly elsewhere. In the present analysis I will try to take advantage of Lakoff and Johnson's theory to both underline certain points I made in the past and to shed a new light on some previous neglected aspects. Out of necessity, I will not go deeply into the subject; in this closing section I will do no more than identify issues for further discussion.

As I once noted (Sfard 1991), on the face of it there is no reason why we should talk about such impalpable "things" like numbers, functions, sets, groups, and Banach spaces. A closer look at mathematics would reveal that what really counts are processes which we perform mentally first on physical objects (e.g. counting, measuring), and then, at a higher level, on these primary processes themselves. The fact, however, that the world of abstract mathematical ideas is made in the image of the physical reality is in full conformity with Lakoff and Johnson's theory. It is our bodily experience which compels us to think about processes as performed on certain objects and as producing objects. The name reification was given to the act of creation of the appropriate abstract entities (some other writers, e.g. Dubinsky, 1991, use the term encapsulation in a similar way). I may now put it in slightly different words and say that reification is the birth of a structural metaphor, a metaphor of an ontological object.

The basic claim underlying the above ideas is that from the developmental point of view, operational conceptions precede structural, namely that a familiarity with a process is a basis for reification. Using Lakoff and Johnson's ideas I may now broaden the picture and say that, more often than not, reification is a transition from an operational to a structural embodied schema. The classification of schemata into operational and structural requires much more explanation than may be given in this short closing section. (Doerfler (1992) and Presmeg (1992) make some slightly different distinctions). Hoping that the ideas are more or less self-explanatory, I will confine myself to a few basic points. An operational schema brings into the domain of abstractions a metaphor of doing, of operating on certain objects to obtain certain other objects. As such, it is a schema of action. The structural embodied schema, on the other hand, conveys a completely different ontological message, a message of a permanent, object-like construct which may be acted upon to produce other constructs. The advantage of the latter type of schema over the former is its being more integrative, more economical and manipulable, more amenable to the holistic treatment (or parallel processing.) Visual imagery is its integral component.

In the light of mathematicians' testimonies, the general rule of the developmental precedence of operational conceptions over structural has its exceptions. Mathematicians do not necessarily follow this process-object path. These adepts of abstract thinking, well trained in conjuring new abstract entities out of other abstract entities, may often reach for the metaphor of an ontological object directly, without worrying about the underlying processes. It is certainly the way ST thinks and understands mathematics:

> When I have a new concept, I need a human metaphor. Personification of the concept. Or a spatial metaphor. A new metaphor of a structure. Only when I have it I can answer questions, solve problems, perform manipulations. I can do all this only after I have the metaphor.

Let me stress once more that ST used the word 'metaphor' on his own accord and that he heard about the work of Lakoff and Johnson for the first time only after the interview. Notwithstanding his idiosyncrasies, ST suggested (again, on his own accord) that operational-structural periodicity can be detected in many historical processes, such as the development of algebra.

As I observed many times in the past, reification, whether it precedes or follows the construction of an operational schema, is often achieved only after a strained effort, if at all . The present treatment of the issue of understanding sheds a new light on the inherent difficulty of reification. The frequent problem with new abstract ideas is that they have no counterpart in the physical world or, worse than that, that they may openly contradict our experiential knowledge. Obviously, in the latter case no metaphor is available to support these abstractions. For example, the concept of transfinite numbers violates the fundamental, experientially established principle "a part is less than a whole". This discrepancy between the abstract and the experiential bothered Cantor, the founder of the idea of a transfinite number, to such extent that he wrote to Dedekind asking for his help in dealing with the thing he himself "could see, but could not believe." In fact, the very idea of reification contradicts our bodily experience: we are talking here about creation of something out of nothing. Or about treating a process as its own product. There is nothing like that in the world of tangible entities, where an object is an 'added value' of an action, where processes and objects are separate, ontologically different entities which cannot be substituted one for the other. Our whole nature rebels against the ostensibly parallel idea of, say, regarding a receipt for a cake as the cake itself.

The last remark I wish to make concerns the discontinuous, almost chaotic nature of reification and, more generally, of the process of understanding. A pertinent illustration of what I have in mind here may be found in the excerpt from Halmos' autobiography quoted in the introduction to this paper. Numerous testimonies by mathematicians, including all my interviewees, confirm Hadamard's thesis that sudden illuminations like the one which brought Halmos the 'understanding of epsilon' are "absolutely general and common to every student of research" (Hadamard, 1949). All my interlocutors remarked many times that the process of understanding is full of singularities and sudden jumps. It seems quite likely that the jumps are the result of reification, namely that they mark a birth of a structural metaphor which renders the concept its 'physiognomy, and thereby makes it meaningful. The following quotation is a typical autobiographical story which aptly illustrates this point.

> After struggling for years, the insights eventually came to me that made it all fall into a place. It all hung together in an incredible way -- every loose end had its natural location. (Pollik tells here the story of his most important works on concentration of signals he did with two other mathematicians; quoted in Albers and Alexanderson, 1985, p.243).

It is remarkable how "physical" the language is used by Pollak in the above description. He talks about abstract ideas as if they were material bodies: "it all fall into place" (an expression used also by Halmos: "It all clicked and fell together"!), "it hung together", and "every loose end had its natural location". There can be little doubt that this is a story of a sudden emergence of a metaphor of ontological object.

The issue of the discontinuities in the process of understanding seems to be of utmost importance and, at the same time, it does not yield itself easily to investigation. Freudenthal (1978), who agrees that "what matters in learning process are discontinuities" (p. 165), is nevertheless quite sceptical as to the possibility of empirical research:

> Discontinuities can only be discovered in continuous observation, but even for teachers and educational researchers it will not be easy to observe these essentials in the learning process: the discontinuities.

Thus, the thorough study which this 'big-bang' phenomenon certainly deserves will have to be preceded by methodological preparations.

5. Morals for experts and for novices

In keeping with Lakoff and Johnson's theory on one hand, and with my own work on the other hand, I tried to show in this paper that a metaphor of an ontological object, even though ostensibly only optional in mathematical thinking, is in fact indispensable for the kind of understanding people are prepared to call 'deep' or 'true'. By quoting mathematicians who talked about their own ways of constructing meaning, I explained how in this process our bodily experience enters the realm of abstract ideas both to create it and to confine it. Reification, a transition from an operational to a structural mode of thinking, is a basic phenomenon in the formation of a mathematical concept. Here I tried to demonstrate that reification is, in fact, a birth of a metaphor which brings a mathematical object into existence and thereby deepens our understanding. The constraint that our perceptually acquired knowledge puts on our imagination makes reification inherently difficult.

One conclusion from all that has been said here is that we can educate and broaden our mathematical universe by loosening real-world constraint on our imagination bit by bit, and by gradually paving the way from the mundane to the "never heard of" with an elaborate chain of more and more abstract metaphors. Each layer in the hierarchical edifice of mathematical ideas is a new step in our struggle for a freedom from the body-exerted restrictions, and for a better understanding of the world of abstraction.

In this paper I confined myself to mathematicians and to their special ways of struggling for understanding. An important question is to what extent the observations about experts apply also to novices, to school and college students. I have no choice but to leave this question open. I would not finish this paper, however, without formulating some tentative implications for learning and teaching.

The study of mathematicians' ways of thinking brings an important and probably quite universal message about the nature and conditions of understanding. The role of the structural metaphor in this process cannot be overestimated. Even though the idea may be conveyed in many different disguises, the literature abounds in findings and arguments which support the claim that the natural tendency for structural thinking is typical not only for mathematicians but also for more able students (see e.g. Krutetskii, 1976). Thus, the immediate implication is that as teachers, we should foster the structural thinking and help 'novices' construct their own structural metaphors. The natural question follows: How can we induce the process which brings the structural metaphor into being? A lot have been said about the inherent difficulty of reification. Studies have shown that even most sincere efforts to bring the appropriate metaphor about would often be rewarded with only a limited success (see e.g. Sfard 1992). Because of the tight relationship between structural metaphors and the issue of visualization it seems that today's wide accessibility of computer graphics opens promising didactic possibilities. When opting for this new path, however, we should remember that too concrete a 'carrier', more than metaphors themselves, may be a doubly-edged sword.

References

Albers, D.J. & G.L. Alexanderson (1985). *Mathematical people - profiles and interviews*. Chicago: Contemporary Books.

Cobb, P., E. Yackel & T. Wood (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education, 23*, 1, 2-33.

Davis, P.J. & R. Hersh (1981). *The mathematical experience*. London: Penguin Books.

Doerfler, W. (1991). Meaning: image schemata and protocols. In F.Furinghetti (Ed.), *Proceedings of PME 15, Vol.1*, (pp.17-33), Assisi (Italy).

Dreyfus, T. (1991). On the status of visual reasoning in mathematics and mathematics education. In F.Furinghetti (Ed.), *Proceedings of the Fifteenth PME Conference, Vol. 1*, (pp.32-48), Assisi, Italy.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D.Tall (Ed.), *Advanced Mathematical Thinking* (pp.95-123), Dordrecht: Kluwer Academic Press.

Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht: Reidel.

Freudenthal (1978). *Weeding and sowing, preface to a science of mathematical education*. Dordrecht: Reidel.

Greeno, J.G. (1991). Number sense as a situated knowing in a conceptual domain. *Journal of Research on Mathematics Education, 22*, 3, 170-218.

Hadamard, J.S. (1949). *The Psychology of invention in the mathematics field*. Princeton: Princeton University Press.

Halmos, P.(1985), *I want to be an mathematician - an automathography in three parts*. MAA Spectrum.

Johnson, M. (1987). *The Body in the mind: The bodily basis of meaning, imagination and reason*. Chicago: The University of Chicago Press.

Johnson-Laird, P.N. (1983). *Mental models: Towards a cognitive science of language, inference and consciousness*. Cambridge, Mass.: Harvard University Press.

Krutetskii, V.A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: University of Chicago Press.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge: Cambridge University Press.

Lakoff, (1987). *Women, fire and dangerous things: What categories reveal about the mind*. Chicago: The University of Chicago Press.

Lakoff and Johnson, (1980). *The metaphors we live by*. Chicago: The University of Chicago Press.